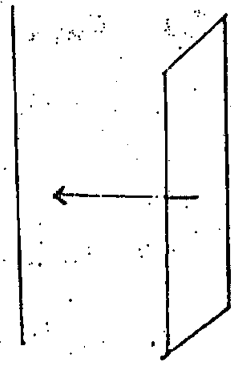
Exercise 8.5

Q1 Show that the st. line $\frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3}$ is parallel to the plane $4x+2y+2z=9$

Sol:- Given line & plane are

$$\left.\begin{array}{l} \frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3} \\ \& \quad 4x+2y+2z=9 \end{array}\right]$$



D.rs. of the line are $2, -7, 3$

d.rs. of normal to plane are $4, 2, 2$

Now $2(4)+(-7)(2)+3(2)$

$= 8-14+6$

$= -6+6$

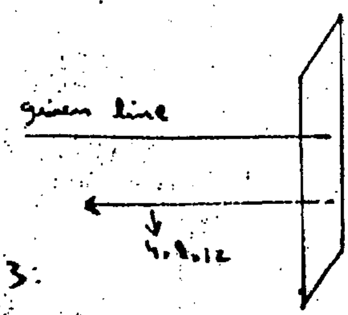
$= 0$

Hence the given line & plane are parallel.

Q2 Show that the st. line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is $\perp$ to the plane $4x+8y+12z+19=0$

Sol:- Given line & plane are

$$\left.\begin{array}{l} \frac{x}{1} = \frac{y}{2} = \frac{z}{3}, \\ \& \quad 4x+8y+12z+19=0 \end{array}\right]$$



Now d.rs. of given line are $1, 2, 3$.

& d.rs. of normal to plane are $4, 8, 12$

Since $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$

or $\frac{1}{4} = \frac{1}{4} = \frac{1}{4}$, since d.rs. are proportional

So given line & plane are perpendicular

Q3 Find the condition that the st. line

$x = mz + a$, $y = nz + b$ may lie in the plane

$$Ax + By + Cz + D = 0$$

Sol. Given line is

$$x = mz + a \quad , \quad y = nz + b$$

Its symmetric form is

$$\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = t \text{ (say)}$$

or $$\left.\begin{aligned} x &= a + mt \\ y &= b + nt \\ z &= t \end{aligned}\right\}$$

Let the given line lies on the plane then each pt. of line should lie on plane. Hence the pt. $(a+mt, b+nt, t)$ should lie on plane, so

$$A(a+mt) + B(b+nt) + Ct + D = 0$$

$$(Aa + Bb + D) + t(Am + Bn + C) = 0$$

This eq. must be satisfied for every value of t

$\Rightarrow$ $\left.\begin{aligned} &Aa + Bb + D = 0 \\ \& \ &Am + Bn + C = 0 \end{aligned}\right\}$ which are req. condition for given line to lie in the given plane

Q4 Determine the pt. if any common to the st. line $x = 1+t,\ y = t,\ z = -1$ & the plane $x+y+z = 3$.

Sol.- Given line & plane are

$$x = 1+t,\ y = t,\ z = -1+t$$

$$\& \quad x+y+z = 3$$

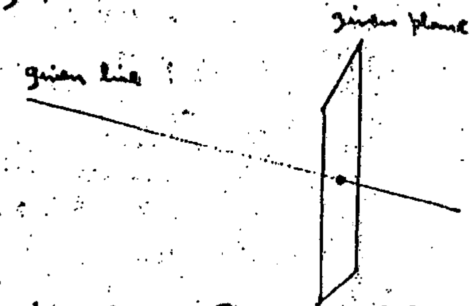
Any pt. on the given line is $(1+t, t, -1+t)$

If this pt. lies on plane $x+y+z = 3$

then $1+t+t-1+t = 3$

$$3t = 3$$

$$\boxed{t = 1}$$



So $(1+1, 1, -1+1)$ or $(2, 1, 0)$ is the common pt. of line & plane.

Q5 Find the eq. of the plane through the pt (x_1, y_1, z_1) & through the st. line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$

Sol.- Given line is

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$$

If A, B, C are the d.r.s. of the normal to req. plane then eq. of plane through given line is

$$A(x-a) + B(y-b) + C(z-c) = 0 \quad \text{——} (1)$$

where $Al + Bm + Cn = 0 \quad \text{——} (2)$

Since (x_1, y_1, z_1) lies on plane ①

so $A(x_1-a)+B(y_1-b)+C(z_1-c)=0$ —— ③

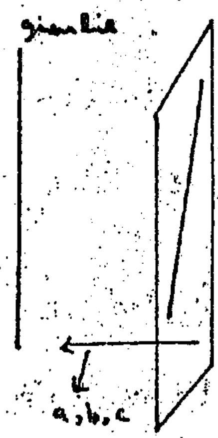
Eliminating A, B, C from ①, ② + ③, we have

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ x_1-a & y_1-b & z_1-c \end{vmatrix} = 0$$

which is the eq. of req. plane

Q6 Find the eq. of the plane passing through the st. line $x+2z=4$, $y-z=8$ & parallel to the st. line $\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}$

Sol. Given line is



$$\left.\begin{aligned} x+2z &= 4 \\ y-z &= 8 \end{aligned}\right]$$

$$\Rightarrow \left.\begin{aligned} \frac{x-4}{2} &= z \\ \& \quad y-8 &= z \end{aligned}\right]$$

So $\frac{x-4}{2} = \frac{y-8}{1} = \frac{z}{1}$ is the given line in symmetric form.

If a, b, c are d.rs. of the normal to req. plane then eq. of plane through given line is

$$a(x-4)+b(y-8)+cz=0 \quad \text{—— ①}$$

where $2a+b+c=0$ —— ②

As the plane ① is $\parallel$ to line $\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}$

So $2a + 3b + 4c = 0$ —— ③

Eliminating a, b, c from ①, ② & ③

$$\begin{vmatrix} x-4 & y-8 & z \\ -2 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

expanding from R_1

$$(x-4)(4-3) - (y-8)(-8-2) + z(-6-2) = 0$$

$$(x-4) + 10(y-8) - 8z = 0$$

$$x + 10y - 8z - 4 - 80 = 0$$

$\boxed{x + 10y - 8z - 84 = 0}$ is req. eq.

Q7 Find the eq. of the plane passing through the pt. (α, β, γ) & $\parallel$ to each of the st. lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \ \& \ \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

Sol. Given lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

$$\& \ \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

If a, b, c are d.rs. of normal to req. plane then eq. of plane through (α, β, γ) is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \quad \text{—— ①}$$

As plane ① is parallel to the given lines so

$$a l_1 + b m_1 + c n_1 = 0 \quad \text{——— } ②$$

$$a l_2 + b m_2 + c n_2 = 0 \quad \text{——— } ③$$

Eliminating a, b, c from ①, ② & ③

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $\Sigma (x-\alpha)(m_1 n_2 - m_2 n_1) = 0$ is req. eq.

Q8 Find the eq. of the plane through the st. line $ax+by+cz+d = 0 = a'x+b'y+c'z+d'$ & parallel to the st. line $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$

Sol. Eq. of a plane through the st. line $ax+by+cz+d = 0 = a'x+b'y+c'z+d'$ is

$$(ax+by+cz+d) + K(a'x+b'y+c'z+d') = 0$$

or $(a+Ka')x + (b+Kb')y + (c+Kc')z + Kd' = 0$

Since this plane is parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

So $l(a+Ka') + m(b+Kb') + n(c+Kc') = 0$

$$al + a'lK + bm + b'mK + nc + c'nK = 0$$

$$K(a'l + b'm + c'n) = -al - bm - cn$$

$$K = -\frac{al+bm+cn}{a'l+b'm+c'n}$$

Put in above eq. of plane

$$(ax+by+cz+d) - \frac{al+bm+cn}{a'l+b'm+c'n}(a'x+b'y+c'z+d')=0$$

$$(ax+by+cz+d)(a'l+b'm+c'n) - (al+bm+cn)(a'x+b'y+c'z+d') = 0$$

$$(ax+by+cz+d)(a'l+b'm+c'n) = (al+bm+cn)(a'x+b'y+c'z+d')$$
$$=0$$

is req. eq

Q9 Prove that the st. lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \;\&\; \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

are Coplanar.

Sol:- Given st. lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

$$\& \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

we know that the st. lines

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$$

$$\& \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$ are Coplanar if

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Now we check this Condition for given lines

take
$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

Expanding from R_1

$$= 1(15-16) - 1(10-12) + 1(8-9)$$

$$= -1 - (-2) - 1$$

$$= -1 + 2 - 1$$

$$= 0$$

Hence given st. lines are Coplanar.

Q10 Prove that the st. lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \quad \& \quad \frac{x-4}{1} = \frac{y+3}{-7} = \frac{z+1}{7}$$

intersect. Also find their pt. of intersection & the plane through them.

Sol. Given lines are

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = t \text{ ——— } (1)$$

$$\& \quad \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = s \text{ ——— } (2)$$

Now parametric eqs. of lines (1) & (2) are

$$\left.\begin{aligned} x &= 1+2t \\ y &= -1-3t \\ z &= -10+8t \end{aligned}\right\} \text{—} (1) \quad \& \quad \left.\begin{aligned} x &= 4+s \\ y &= -3-4s \\ z &= -1+7s \end{aligned}\right\} \text{——} (2)$$

Any pt. on line (1) is $(1+2t, -1-3t, -10+8t)$

Any pt. on line (2) is $(4+s, -3-4s, -1+7s)$

Let the two lines intersect at (x_0, y_0, z_0)

So this will satisfy (1) & (2)

$$\left.\begin{aligned} x_0 &= 1+2t \\ y_0 &= -1-3t \\ z_0 &= -10+8t \end{aligned}\right\} \quad \& \quad \left.\begin{aligned} x_0 &= 4+s \\ y_0 &= -3-4s \\ z_0 &= -1+7s \end{aligned}\right\}$$

Hence
$$\left.\begin{aligned} 1+2t &= 4+s \\ -1-3t &= -3-4s \\ -10+8t &= -1+7s \end{aligned}\right\}$$

or
$$\begin{aligned} 2t-s &= 3 \text{ ——— } (1) \\ 3t-4s &= 2 \text{ ——— } (2) \\ 8t-7s &= 9 \text{ ——— } (3) \end{aligned}$$

Multiplying (1) by 4

$$\begin{aligned} 8t-\cancel{4s} &= 12 \text{ ——— } (1) \\ -3t\cancel{+4s} &= -2 \text{ ——— } (2) \\ \hline 5t &= 10 \end{aligned}$$

$\boxed{t=2}$

Put in ①

$2(2) - s = 3$

$4 - s = 3$

$s = 4-3$

$\boxed{s=1}$

we see that these values of t & s satisfy ③

Hence given lines intersect & their pt. of intersection is $(x_0, y_0, z_0) = (5, -7, 6)$.

Now we find plane through these lines.

A plane through these lines should contain the pt. of intersection of these lines.

If a, b, c are d.rs. of normal to plane then eq. of plane through $(5, -7, 6)$ is

$$a(x-5) + b(y+7) + c(z-6) = 0 \quad \text{———} \quad \mathrm{I}$$

Since this plane contains both lines so

$$2a - 3b + 8c = 0 \quad \text{———} \quad \mathrm{II}$$

$$a - 4b + 7c = 0 \quad \text{———} \quad \mathrm{III}$$

Eliminating a, b, c from I, II & III we have

$$\begin{vmatrix} x-5 & y+7 & z-6 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

{Expanding from R_1

1346

$$(x-5)(-21+32)-(y+7)(14-8)+(z-6)(-8+3)=0$$ 117

$$(x-5)(11)-(y+7)(6)+(z-6)(-5)=0$$

$$11x-6y-5z-55-42+30=0$$

$$11x-6y-5z-67=0$$ is req. eq.

Prof. Shakeel Azhar
Department Of Mathematics
Govt. Shalimar College Baghbanpura
Lahore