Exercise No. 8.4

Q1 Prove that the planes $4x+4y-5z=12$, $8x+12y-13z=32$ intersect & eqs. of their line of intersection can be written in the form

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$$

Sol. Given planes are

$$4x+4y-5z=12 \quad \text{——— (1)}$$

$$8x+12y-13z=32 \quad \text{——— (2)}$$

d.rs. of the normal to plane (1) are $4, 4, -5$

d.rs. of the normal to plane (2) are $8, 12, -13$

Since d.rs. of both normals are not proportional

Hence both normals are not parallel

So given planes are not parallel

~~Hence given~~ planes ~~intersect~~.

Now we find eq. of their line of intersection.

For this put $z=0$ in (1) & (2)

$$\left.\begin{aligned} 4x+4y-12&=0 \\ 8x+12y-32&=0 \end{aligned}\right\}$$

$$\frac{x}{-128+144}=\frac{-y}{-128+96}=\frac{1}{48-32}$$

8.

$$\frac{x}{16} = \frac{y}{32} = \frac{1}{16}$$

$$\boxed{x = 1,\ y = 2}$$

So $(1,2,0)$ is a pt. on line of intersection

Again Put $x = 0$ in ① & ②

$$\left.\begin{aligned} 4y - 5z - 12 &= 0 \\ 12y - 13z - 32 &= 0 \end{aligned}\right\}$$

$$\frac{y}{160-156} = \frac{-z}{-128+144} = \frac{1}{-52+60}$$

$$\frac{y}{4} = \frac{z}{-16} = \frac{1}{8}$$

$$\Rightarrow \boxed{y = \frac{1}{2},\ z = 2}$$

So $(0, \frac{1}{2}, 2)$ is another pt. on line of intersection

Hence the req eq. of line through $(1,2,0)$ & $(0, \frac{1}{2}, -2)$

is $\dfrac{x-1}{0-1} = \dfrac{y-2}{\frac{1}{2}-2} = \dfrac{z-0}{-2-0}$

$$\frac{x-1}{-1} = \frac{y-2}{-\frac{3}{2}} = \frac{z}{-2}$$

$$\frac{x-1}{-2} = \frac{y-2}{-3} = \frac{z}{-4}$$

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4} \text{ is req eq.}$$

Q2 Find symmetric form for the line 81

$x+y+z+1=0=4x+y-2z+2$

Sol. Given eq. of line is

$$x+y+z+1=0 \quad \text{——} ①$$

$$4x+y-2z+2=0 \quad \text{——} ②$$

Put $z=0$ in ① & ②

$$\left.\begin{aligned} x+y+1&=0\\ 4x+y+2&=0\end{aligned}\right]$$

$$\frac{x}{2-1}=\frac{-y}{2-4}=\frac{1}{1-4}$$

$$\frac{x}{1}=\frac{y}{2}=\frac{1}{-3}$$

$$\Rightarrow x=\frac{-1}{3},\ y=-\frac{2}{3}$$

So $A\left(-\frac{1}{3},-\frac{2}{3},0\right)$ is a pt. on the given line.

Now Put $x=0$ in ① & ②

$$\left.\begin{aligned} y+z+1&=0\\ y-2z+2&=0\end{aligned}\right]$$

$$\frac{y}{2+2}=\frac{-z}{2-1}=\frac{1}{-2-1}$$

$$\frac{y}{4}=\frac{z}{-1}=\frac{1}{-3}$$

$$\Rightarrow y=-4/3,\ z=\frac{1}{3}$$

So $B\left(0,-\frac{4}{3},\frac{1}{3}\right)$ is another pt. on given line

Then the eq. of the line through $A\left(-\frac{1}{3}, -\frac{2}{3}, 0\right)$ [82] & $B\left(0, -\frac{4}{3}, \frac{1}{3}\right)$ is

$$\frac{x+\frac{1}{3}}{0+\frac{1}{3}} = \frac{y+\frac{2}{3}}{-\frac{4}{3}+\frac{2}{3}} = \frac{z-0}{\frac{1}{3}-0}$$

or $$\frac{x+\frac{1}{3}}{\frac{1}{3}} = \frac{y+\frac{2}{3}}{-\frac{2}{3}} = \frac{z}{\frac{1}{3}}$$

or $$\frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z}{1}$$

is req. eq. of line in symmetric form.

Q3 Show that the lines $L: x+2y-z-7=0=y+z-2x-6$ & $M: 3x+6y-3z-8=0=2x-y-z$ are parallel.

~~Sol~~ Given lines are

$L: x+2y-z-7=0=-2x+y+z-6$

$M: 3x+6y-3z-8=0=2x-y-z$

~~Let l_1, m_1, n_1~~ are the d.cs. of ~~the line L, since~~ it lies on both planes so by condition of perpendicularity

$$\left.\begin{aligned} l_1+2m_1-n_1 &= 0 \\ -2l_1+m_1+n_1 &= 0 \end{aligned}\right\} \text{———— L}$$

$$\frac{l_1}{2+1} = \frac{-m_1}{1-2} = \frac{n_1}{1+4} \text{———— M}$$

$$\frac{l_1}{3} = \frac{m_1}{1} = \frac{n_1}{5}$$

So d.rs. of L are $\lambda_1, \mu_1, \nu_1 = 3, 1, 5$.

Now

let l_2, m_2, n_2 be the d.cs. of line M. Since it lies on both planes so by Condition of perpendicularity

$$\left.\begin{aligned} 3l_2 + 6m_2 - 3n_2 &= 0 \\ 2l_2 - m_2 - n_2 &= 0 \end{aligned}\right\}$$

$$\frac{l_2}{-6-3} = \frac{-m_2}{-3+6} = \frac{n_2}{-3-12}$$

$$\frac{l_2}{-9} = \frac{m_2}{-3} = \frac{n_2}{-15}$$

So d.rs. of M are $\lambda_2, \mu_2, \nu_2 = -9, -3, -15$

Since $\frac{\lambda_1}{\lambda_2} = \frac{\mu_1}{\mu_2} = \frac{\nu_1}{\nu_2}$

So the two given lines are parallel.

<u>Q4</u> Show that the lines

$L: x+2y-1 = 0 = 2y-z-1$ &

$M: x-y-1 = 0 = x-2z-3$ are perpendicular

<u>Sol.</u> Given lines are

$L: x+2y-1 = 0 = 2y-z-1$

& $M: x-y-1 = 0 = x-2z-3$

let l_1, m_1, n_1 be the d.cs. of line L. Since it lies on both planes so by Condition of perpendicularity

$$\left.\begin{aligned} l_1 + 2m_1 + 0n_1 &= 0 \\ 0l_1 + 2m_1 - n_1 &= 0 \end{aligned}\right\}$$

$$\frac{l_1}{-2-0} = \frac{-m_1}{-1-0} = \frac{n_1}{2-0}$$

$$\frac{l_1}{-2} = \frac{m_1}{1} = \frac{n_1}{2}$$

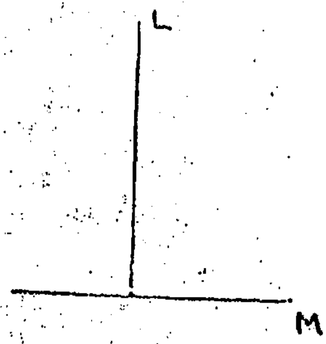
$\Rightarrow$ d.rs. of L are $\lambda_1, \mu_1, \nu_1 = -2, 1, 2$

Now let l_2, m_2, n_2 be the d.cs. of line M. Since it lies on both planes so by condition of perpendicularity.

$$\left.\begin{aligned} l_2 - m_2 + 0n_2 &= 0 \\ l_2 + 0m_2 - 2n_2 &= 0 \end{aligned}\right\}$$

$$\frac{l_2}{2-0} = \frac{-m_2}{-2-0} = \frac{n_2}{0+1}$$

$$\frac{l_2}{2} = \frac{m_2}{2} = \frac{n_2}{1}$$



So d.rs. of line M are $\lambda_2, \mu_2, \nu_2 = 2, 2, 1$

Now $\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = (-2)(2) + (1)(2) + (2)(1)$

$= -4 + 2 + 2$

$= 0$

Hence L & M are perp. to each other.

Q5 Find the eq. of the st. line through the pt. $(1, 2, 3)$ & parallel to the line

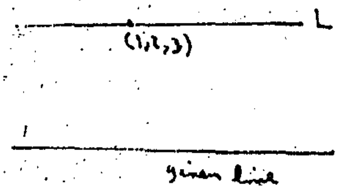
$$x - y + 2z - 5 = 0 = 3x + y + z + 6$$

Sol. Given eq. of line is

$$x-y+2z-5=0=3x+y+z+6$$ 85

Let l, m, n be the d.cs. of this line. Since it lies on both planes so by condition of perpendicularity

$$\left.\begin{aligned} l-m+2n &= 0 \\ 3l+m+n &= 0 \end{aligned}\right\}$$



$$\frac{l}{-1-2} = \frac{-m}{1-6} = \frac{n}{1+3}$$

$$\frac{l}{-3} = \frac{m}{5} = \frac{n}{4}$$

So d.rs. of given line are $\lambda, \mu, \nu = -3, 5, 4$

Let L be the req. line.

Since $L \parallel$ given line

So d.rs. of req. line L are $-3, 5, 4$

Hence req. eq. of L through $(1,2,3)$ & having d.rs. $-3, 5, 4$ is.

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

Q6 Find the eqs. of the planes through the st. line $x+y-z=0=2x-y+3z-5$ & perpendicular to the Co-ord. planes.

Sol. Given line is

$$x+y-z=0=2x-y+3z-5$$

Then the eq. of plane through given line is

$$(x+y-z)+k(2x-y+3z-5)=0$$

or $(1+2K)x+(1-K)y+(3K-1)z-5K=0$ —— ①

d.rs. of the normal to plane ① are

$1+2K,\ 1-K,\ 3K-1$

Case (i)

Since plane ① is perp. to yz-plane whose eq. is

$x=0$

So $(1+2K)(1)=0$

$1+2K=0$

$2K=-1$

$\boxed{K=-\frac{1}{2}}$

Put in ①

$$(1+2(-1/2))x+\left(1+\frac{1}{2}\right)y+(3(-1/2)-1)z-5(-1/2)=0$$

$$(1-1)x+(3/2)y+\left(-\frac{3}{2}-1\right)z+\frac{5}{2}=0$$

$$\frac{3}{2}y-\frac{5}{2}z+\frac{5}{2}=0$$

$\boxed{3y-5z+5=0}$ is req. eq.

Case (ii) Since plane ① is perp. to xz-plane whose eq. is $y=0$

So $(1-K)\cdot 1=0$

$\boxed{K=1}$

Put in ①

$$(1+2)x+(1-1)y+(3-1)z-5=0$$

or $\boxed{3x+2z-5=0}$

Case (iii)

Since plane ① is $\perp$ to xy-plane whose eq. is

$$z = 0$$

So $(3k-1).1 = 0$

$$3k-1 = 0$$

$$3k = 1$$

$$\boxed{k = \frac{1}{3}}$$

Put in ①

$$[1+2(1/3)]x + (1-\tfrac{1}{3})y + (1-1)z - 5(1/3) = 0$$

$$\frac{5}{3}x + \frac{2}{3}y - \frac{5}{3} = 0$$

or $\boxed{5x + 2y - 5 = 0}$ is req. eq.

Q7 Find eq. of the plane containing the line $x = 2t,\ y = 3t,\ z = 4t$ & the intersection of the planes $x+y+z = 0$ & $2y - z = 0$

Sol: The eq. of a plane through the intersection of planes $x+y+z = 0$ & $2y-z = 0$ is

$$(x+y+z) + k(2y-z) = 0$$

or $x + (1+2k)y + (1-k)z = 0$ ——— ①

Here d.rs. of normal to plane ① are $1, 1+2k, 1-k$

Since this plane contains the line $x = 2t, y = 3t, z = 4t$. whose d.rs. are $2, 3, 4$

So by condition of perpendicularity

$$1(2)+(1+2K)(3)+(1-K)(4)=0$$

$$2+3+6K+4-4K=0$$

$$9+2K=0$$

$$2K=-9$$

$$\boxed{K=-\frac{9}{2}}$$

Put in above eq.

$$(x+y+z)-\frac{9}{2}(2y-z)=0$$

$$2x+2y+2z-18y+9z=0$$

$$2x-16y+11z=0 \text{ is req. eq.}$$

<u>Q8</u> write an eq. of the family of planes having x-intercept 3, y intercept 2 & a non zero z-intercept. Find the member of the family which is perpendicular to the plane $3x-2y+z-4=0$

<u>Sol:-</u> Suppose that the non zero z-int. is c

then the eq. of the req. family of planes is

$$\frac{x}{5}+\frac{y}{2}+\frac{z}{c}=1 \quad \text{——} (1) \quad \text{where } c \text{ is parameter}$$

Here d.rs. of normal to planes are $\frac{1}{5}, \frac{1}{2}, \frac{1}{c}$

If a member of family (1) is perpendicular to plane $3x-2y+z-4=0$

then $3\left(\frac{1}{5}\right)-2\left(\frac{1}{2}\right)+1\left(\frac{1}{c}\right)=0$

$$\frac{3}{5} - 1 + \frac{1}{c} = 0$$

$$-\frac{2}{5} + \frac{1}{c} = 0$$

$$\frac{1}{c} = \frac{2}{5}$$

$$\boxed{c = \frac{5}{2}}$$

Put in ①

$$\frac{x}{5} + \frac{y}{2} + \frac{z}{\frac{5}{2}} = 1$$

or $\boxed{\frac{x}{5} + \frac{y}{2} + \frac{2}{5}z = 1}$ is req. member of family

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Q9 Find eq. of the plane passing through the pt. $(2,-3,1)$ & containing the line $x-3 = 2y = 3z-1$

Sol:- Given line is

$$x-3 = 2y = 3z-1$$

It can be written as

$$x-2y-3 = 0 = 2y-3z+1$$

Then the eq. of plane containing this line is

$$(x-2y-3) + k(2y-3z+1) = 0$$

Since it passes through $(2,-3,1)$ so

$$[2-2(-3)-3] + k[2(-3)-3(1)+1] = 0$$

$$(2+6-3) + k(-6-3+1) = 0$$

$$5 + k(-8) = 0$$

$$8K = 5$$

$$\boxed{K = \frac{5}{8}}$$

Put in above eq.

$$(x-2y-3) + \frac{5}{8}(2y-3z+1) = 0$$

$$8x-16y-24+10y-15z+5=0$$

$$\boxed{8x-6y-15z-19=0}$$ is req. eq.

Q10 Find eq. of the plane through the line of intersection of the planes $2x-y+3z=0$ & $x+2y-2z-3=0$ and at unit distance from origin.

Sol Eq. of a plane through the line of intersection of given planes is

$$(2x-y+3z)+K(x+2y-2z-3)=0$$

or $(2+K)x+(2K-1)y+(3-2K)z-3K=0$

Since this plane is at unit distance from O

So $$\frac{|(2+K)(0)+(2K-1)(0)+(3-2K)(0)-3K|}{\sqrt{(2+K)^2+(2K-1)^2+(3-2K)^2}} = 1$$

$$\Rightarrow \quad 3K = \sqrt{(2+K)^2+(2K-1)^2+(3-2K)^2}$$

Sq. both sides

$$9K^2 = 4+\cancel{4K}+K^2+4K^2-\cancel{4K}+1+9-12K+4K^2$$

$$\frac{9}{k^2} = \frac{9}{k^2} - 12k + 14$$

$$0 = -12k + 14$$

$$12k = 14$$

$$\boxed{k = \frac{7}{6}}$$

Put in above eq.

$$(2x - y + 3z) + \frac{7}{6}(x + 2y - 2z - 3) = 0$$

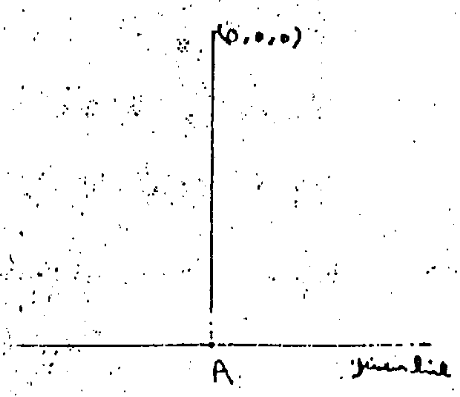
$$12x - 6y + 18z + 7x + 14y - 14z - 21 = 0$$

$19x + 8y + 4z - 21 = 0$ is req. eq.

Q11 Find eq. of the perpendicular from the origin to the line $x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$. Also find the Co-ords. of the foot of perpendicular.

Sol. Given eq. of line is

$$\left.\begin{aligned} x + 2y + 3z + 4 &= 0 \\ 2x + 3y + 4z + 5 &= 0 \end{aligned}\right]$$



First we find its symmetric form.

Let l, m, n be the d.c. of given line.

Since it lies on both planes

so by condition of perpendicularity

1.2

$$\Rightarrow \left.\begin{array}{l} l+2m+3n=0 \\ 2l+3m+4n=0 \end{array}\right]$$

$$\frac{l}{8-9} = \frac{-m}{4-6} = \frac{n}{3-4}$$

$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$$\text{or } \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

So d.rs. of given line are $1, -2, 1$

To find a pt. on line Put $z = 0$

$$\left.\begin{array}{l} x+2y+4=0 \\ 2x+3y+5=0 \end{array}\right]$$

$$\frac{x}{10-12} = \frac{-y}{5-8} = \frac{1}{3-4}$$

$$\frac{x}{-2} = \frac{y}{3} = \frac{1}{-1}$$

$$\Rightarrow x = 2, \ y = -3$$

So $(2, -3, 0)$ is a pt. on the given line.

Hence the symmetric form of given line is

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = t \text{ (say)} \quad \text{——— (1)}$$

$$\Rightarrow \left.\begin{array}{l} x = 2+t \\ y = -3-2t \\ z = t \end{array}\right\}$$

Any pt. on this line is $(2+t, -3-2t, t)$

Let A be the foot of perpendicular from origin to (1) given line then Co-ords of pt. A are

$$A(2+t, -3-2t, t)$$

Since line OA is perp. to line (1)

So $1(2+t)-2(-3-2t)+1(t) = 0$

$$2+t+6+4t+t = 0$$

$$6t+8 = 0$$

$$3t+4 = 0$$

$$\boxed{t = -\frac{4}{3}}$$

So Co-ords. of foot of ⊥ are $A\left(2-\frac{4}{3}, -3+\frac{8}{3}, -\frac{4}{3}\right)$

$$= A\left(\frac{2}{3}, \frac{-1}{3}, -\frac{4}{3}\right)$$

Now the req. perp. distance $= |OA|$

$$= \sqrt{\left(\frac{2}{3}-0\right)^2+\left(-\frac{1}{3}-0\right)^2+\left(-\frac{4}{3}-0\right)^2}$$

$$= \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{16}{9}}$$

$$= \sqrt{\frac{4+1+16}{9}}$$

$$= \sqrt{\frac{21}{9}}$$

$$= \sqrt{\frac{7}{3}} \text{ ——— Ans.}$$

Q12 A Variable plane is at a Const. distance p from origin & meets the axes in A, B, C. Through A, B & C, planes are drawn parallel to the Co-ord. planes. Show that the locus of their pt. of intersection is given by $x^{-2}+y^{-2}+z^{-2}=p^{-2}$ 1.4

Sol.

Let the eq. of plane is

$$lx+my+nz=p \quad \text{———} \quad (1)$$

where l, m, n are the d.cs. of normal to plane (1)

Now eq. (1) Can be written as

$$\frac{lx}{p}+\frac{my}{p}+\frac{nz}{p}=1$$

$$\text{or} \quad \frac{x}{\frac{p}{l}}+\frac{y}{\frac{p}{m}}+\frac{z}{\frac{p}{n}}=1$$

which is intercept form of plane (1)

Hence Co-ords. of pts. A, B & C are

$$A\left(\frac{p}{l},0,0\right), B\left(0,\frac{p}{m},0\right) \;\&\; C\left(0,0,\frac{p}{n}\right)$$

Now eq. of a plane through pt. $A\left(\frac{p}{l},0,0\right)$ & $\parallel$ to yz-plane is $x=\frac{p}{l}$

eq. of plane through pt. $B\left(0,\frac{p}{m},0\right)$ & $\parallel$ to xz-plane is $y=p/m$

Similarly eq. of plane through $C\,(0,0,p/n)$ & $\parallel$ to xy-plane is $z=p/n$

Clearly the pt. of intersection of given planes is

$$(x, y, z) = \left(\frac{p}{l}, \frac{p}{m}, \frac{p}{n}\right)$$

$$\Rightarrow x = \frac{p}{l}, \quad y = \frac{p}{m}, \quad z = \frac{p}{n}$$

$$\Rightarrow \left.\begin{aligned} l &= \frac{p}{x} \\ m &= \frac{p}{y} \\ n &= \frac{p}{z} \end{aligned}\right\}$$

Since $l^2 + m^2 + n^2 = 1$

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 1$$

or $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$

or $\boxed{x^{-2} + y^{-2} + z^{-2} = p^{-2}}$ is req. locus.

Q13 Let A, B, C be the pts. as in problem 12, Prove that the locus of the centroid of the tetrahedron OABC is

$$x^{-2} + y^{-2} + z^{-2} = 16p^{-2}, \text{ O being origin.}$$

Sol:- Since the Co-ords. of pts. A, B & C are

$A\left(\frac{p}{l}, 0, 0\right)$, $B\left(0, \frac{p}{m}, 0\right)$ & $C\left(0, 0, p/n\right)$.

Now Co-ords. of pt. O are $O(0,0,0)$

Hence the Co-ords. of the vertices of tetrahedron 1.6
OABC are $O(0,0,0)$, $A(p/l,0,0)$, $B(0,p/m,0)$, $C(0,0,\frac{p}{n})$

Now Centroid of Tetrahedron is $\left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right)$

$$\Rightarrow \left.\begin{aligned} x &= \frac{p}{4l} \\ y &= \frac{p}{4m} \\ z &= \frac{p}{4n} \end{aligned}\right\}$$

$$\text{or} \quad \left.\begin{aligned} l &= \frac{p}{4x} \\ m &= \frac{p}{4y} \\ n &= \frac{p}{4z} \end{aligned}\right]$$

$$\because \quad l^2 + m^2 + n^2 = 1$$

$$\frac{p^2}{16x^2} + \frac{p^2}{16y^2} + \frac{p^2}{16z^2} = 1$$

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 16$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}$$

$$\text{or} \quad \boxed{x^{-2} + y^{-2} + z^{-2} = 16p^{-2}}$$

which is req. locus.