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Analytic geometry of three dimensions

(Chapter No. 8)

Co-ordinate system:

We select a point O for the origin, x, y & z are the directed distances along x-axis, y-axis & z-axis respectively, then we define co-ordinates of pt P as

Distance b/w two points in space

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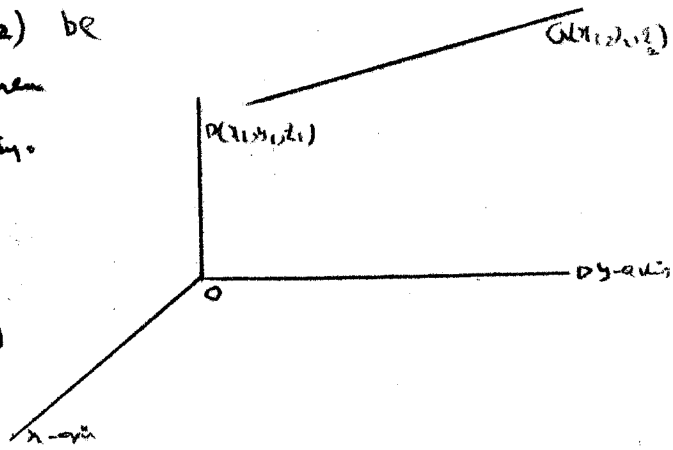
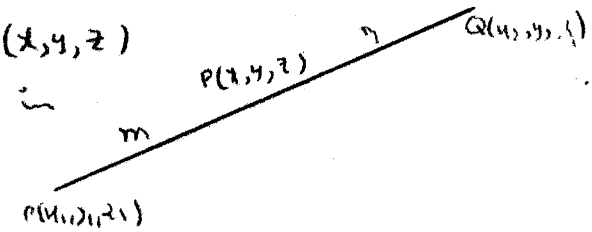
Let $P(x_1,y_1,z_1)$ & $Q(x_2,y_2,z_2)$ be two points in space then the distance b/w the pts. $P(x_1,y_1,z_1)$ & $Q(x_2,y_2,z_2)$



$$|PQ| = (x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)$$

Point dividing a line seg. in a given ratio

The co-ords of a point $P(x,y,z)$ dividing a line segment in a given ratio $m:n$ are



$$P(x,y,z) = \left(\frac{nx_1+mx_2}{m+n}, \frac{ny_1+my_2}{m+n}, \frac{nz_1+mz_2}{m+n}\right)$$

B.S.C

CALCULUS. 8

12.40

Exercise No. 8.1

P & Q are the opposite vertices of a parallelepiped having its faces parallel to the co-ord. planes. Find the co-ords. of the other vertices & sketch the parallelepiped.

Q1 $P(-1,1,2)$, $Q(2,3,5)$

Sol: Complete the parallelepiped with faces parallel to the co-ord planes & PQ as a diagonal.

The co-ords. of the other vertices are as

$A(2,1,2)$, $B(-1,3,2)$

$C(-1,1,5)$, $R(2,3,2)$

$S(-1,3,5)$, $T(2,1,5)$

Q2 $P(2,-1,-3)$, $Q(4,0,-1)$

Sol:

Complete the parallelepiped with faces $\parallel$ to the co-ord planes & PQ as a diagonal.

Then the co-ords of other vertices are

$A(4,-1,-3)$, $B(2,0,-3)$

$C(2,-1,-1)$, $R(4,0,-3)$

$S(2,0,-1)$, $T(4,-1,-1)$

Q3 $P(2,5,-3)$, $Q(-4,2,1)$

Sol. Complete the parallelopiped with faces $\parallel$ to the Co-ord. planes & PQ as a diagonal.

Then the Co-ords. of the remaining vertices are

$A(-4,5,-3)$, $B(2,2,-3)$, $C(2,5,1)$

$R(-4,2,-3)$, $S(2,2,1)$, $T(-4,5,1)$.

Show that the three given pts. are either the vertices of a triangle or the vertices of an isosceles Δ or both

Q4 $A(1,5,0)$, $B(6,6,4)$, $C(0,9,5)$

Sol. Consider a ΔABC with vertices as given pts.

Now $|AB| = \sqrt{(6-1)^2+(6-5)^2+(4-0)^2}$

$= \sqrt{25+1+16}$

$= \sqrt{42}$

$|BC| = \sqrt{(0-6)^2+(9-6)^2+(5-4)^2}$

$= \sqrt{36+9+1}$

$= \sqrt{46}$

$|AC| = \sqrt{(1-0)^2+(5-9)^2+(0-5)^2}$

$$|AC| = \sqrt{1+16+25}$$
$$= \sqrt{42}$$

Since $|AB| = |AC|$

Hence given Δ is isosceles.

Q5 $A(4,9,4)$, $B(0,11,2)$, $C(1,0,1)$

Soln Consider a ΔABC with vertices as given pts.

Now
$$|AB| = \sqrt{(0-4)^2+(11-9)^2+(2-4)^2}$$
$$= \sqrt{16+4+4}$$
$$= \sqrt{24}$$
$$= 2\sqrt{6}$$

$$|BC| = \sqrt{(1-0)^2+(0-11)^2+(1-2)^2}$$
$$= \sqrt{1+121+1}$$
$$= \sqrt{123}$$

$$|AC| = \sqrt{(1-4)^2+(0-9)^2+(1-4)^2}$$
$$= \sqrt{9+81+9}$$
$$= \sqrt{99}$$
$$= 3\sqrt{11}$$

Now $|AB|^2 + |AC|^2 = 24+99 = 123 = (\sqrt{123})^2 = |BC|^2$

Thus ΔABC is a right triangle with right angle at A.

Q.6 $A(1,0,2)$, $B(4,3,2)$, $C(0,7,6)$

Sol. Consider a ΔABC with vertices as given pts.

Now

$$|AB| = \sqrt{(4-1)^2+(3-0)^2+(2-2)^2}$$

$$= \sqrt{9+9+0}$$

$$= \sqrt{18}$$

$$= 3\sqrt{2}$$

$$|BC| = \sqrt{(0-4)^2+(7-3)^2+(6-2)^2}$$

$$= \sqrt{16+16+16}$$

$$= \sqrt{48}$$

$$= 4\sqrt{3}$$

$$|AC| = \sqrt{(1-0)^2+(0-7)^2+(2-6)^2}$$

$$= \sqrt{1+49+16}$$

$$= \sqrt{66}$$

Now $|AB|^2+|BC|^2 = 18+48$

$$= 66$$

$$= (\sqrt{66})^2$$

$$= |AC|^2$$

Hence ΔABC is a right triangle with right angle at B.

Q7 $A(2,3,4)$, $B(8,-1,2)$, $C(-4,1,0)$

Sol.

Consider a ΔABC with vertices as given pts.

Now

$$|AB| = \sqrt{(8-2)^2+(-1-3)^2+(2-4)^2}$$

$$= \sqrt{36+16+4}$$

$$= \sqrt{56}$$

$$|BC| = \sqrt{(-4-8)^2+(1+1)^2+(0-2)^2}$$

$$= \sqrt{144+4+4}$$

$$= \sqrt{152}$$

$$|AC| = \sqrt{(2+4)^2+(3-1)^2+(4-0)^2}$$

$$= \sqrt{36+4+16}$$

$$= \sqrt{56}$$

Since $|AB| = |AC|$

Hence given ΔABC is an isosceles Δ.

Q8 Show that the pts. $(1,6,1)$, $(1,3,4)$, $(4,3,1)$ & $(0,2,0)$ are the vertices of a regular tetrahedron.

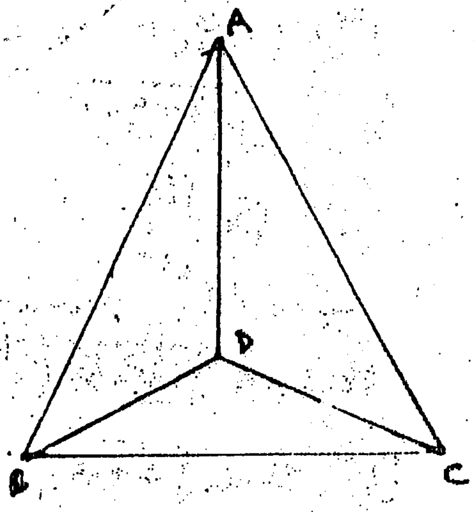
Sol.

Given pts. are $A = (1,6,1)$, $B = (1,3,4)$, $C = (4,3,1)$ & $D = (0,2,0)$

To show that given pts. are vertices of a regular tetrahedron, we have to show that $|AB| = |AC| = |AD| = |BC| = |CD| = |BD|$

Now

$$|AB| = \sqrt{(1-1)^2+(3-6)^2+(4-1)^2}$$
$$= \sqrt{0+9+9}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$



$$|AC| = \sqrt{(4-1)^2+(3-6)^2+(1-1)^2}$$
$$= \sqrt{9+9+0}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

$$|AD| = \sqrt{(0-1)^2+(2-6)^2+(0-1)^2}$$
$$= \sqrt{1+16+1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

$$|BC| = \sqrt{(4-1)^2+(3-3)^2+(1-4)^2}$$
$$= \sqrt{9+0+9}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

$$|CD| = \sqrt{(0-4)^2+(2-3)^2+(0-1)^2}$$
$$= \sqrt{16+1+1}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

$$|BD| = \sqrt{(0-1)^2+(2-3)^2+(0-4)^2}$$
$$= \sqrt{1+1+16}$$
$$= \sqrt{18}$$
$$= 3\sqrt{2}$$

Since $|AB| = |AC| = |AD| = |BC| = |CD| = |BD|$

Hence the given pts. are the vertices of a regular tetrahedron.

Q9 Show that the pts. $(3,-1,3)$, $(1,-1,2)$, $(2,1,0)$ & $(4,1,1)$ are the vertices of a rectangle.

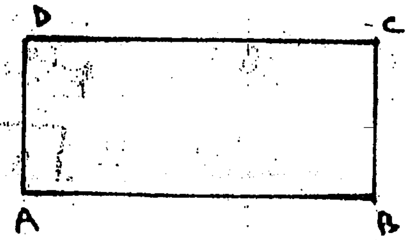
Sol.

Suppose given pts. are

$A = (3,-1,3)$, $B = (1,-1,2)$, $C = (2,1,0)$, $D = (4,1,1)$

The given pts. will form a rectangle if

$$|AB| = |CD| \quad \& \quad |BC| = |AD|$$
$$\& \quad \angle A = 90^\circ$$



Now

$$|AB| = \sqrt{(1-3)^2+(-1+1)^2+(2-3)^2}$$
$$= \sqrt{4+0+1}$$
$$= \sqrt{5}$$

$$|CD| = \sqrt{(4-2)^2+(1-1)^2+(1-0)^2}$$
$$= \sqrt{4+0+1} = \sqrt{5}$$

Now

$$|BC| = \sqrt{(2-1)^2+(1+1)^2+(0-2)^2}$$

$$= \sqrt{1+4+4}$$

$$= \sqrt{9}$$

$$= 3$$

& $$|AD| = \sqrt{(4-3)^2+(1+1)^2+(1-3)^2}$$

$$= \sqrt{1+4+4}$$

$$= \sqrt{9}$$

$$= 3$$

Hence $|AB| = |CD|$ & $|BC| = |AD|$.

Now we prove $\angle A = 90°$

Consider

$$|AB|^2 + |AD|^2 = 5 + 9$$

$$= 14$$

Since $$|BD| = \sqrt{(4-1)^2+(1+1)^2+(1-2)^2}$$

$$= \sqrt{9+4+1}$$

$$= \sqrt{14}$$

So $|BD|^2 = 14$

Put in above eq.

$$|AB|^2 + |AD|^2 = |BD|^2$$

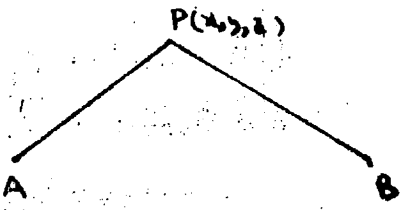
So $\angle A = 90°$. Hence given pts. are vertices of a rectangle.

Q10 under what Conditions on x, y & z is the pt. $P(x,y,z)$ equidistant from the pts. $(3,-1,4)$ & $(-1,5,0)$?

Sol. Suppose the given pts. are $A(3,-1,4)$ & $B(-1,5,0)$

Here $P(x,y,z)$ is any pt.

According to given Condition



$$|PA| = |PB|$$

$$\Rightarrow \sqrt{(x-3)^2+(y+1)^2+(z-4)^2} = \sqrt{(x+1)^2+(y-5)^2+(z-0)^2}$$

Sq. both sides

$$(x-3)^2+(y+1)^2+(z-4)^2 = (x+1)^2+(y-5)^2+(z-0)^2$$

$$\cancel{x^2}-6x+9+\cancel{y^2}+2y+1+\cancel{z^2}-8z+16 = \cancel{x^2}+2x+1+\cancel{y^2}-10y+25+\cancel{z^2}$$

$$-6x+2y-8z+2\cancel{6} = 2x-10y+2\cancel{6}$$

$$-6x+2y-8z-2x+10y = 0$$

$$-8x+12y-8z = 0$$

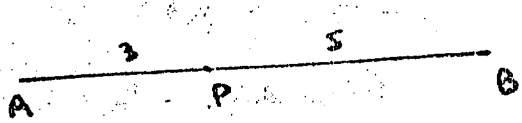
Dividing both sides by -4

$\boxed{2x-3y+2z=0}$ is the req. Condition.

Q11 Find the Co-ords. of the pt. dividing the join of the pts. $(-3,1,4)$ & $(5,-1,1)$ in the ratio $3:5$.

Sol.

Sol. Given pts. are $A(-3,1,4)$ & $B(5,-1,6)$.
Let $P(x,y,z)$ be the req. pt. dividing the line AB in ratio $3:5$.
We know that the Co-ords. of the pt. dividing the join of (x_1,y_1,z_1) & (x_2,y_2,z_2) in the ratio $m_1:m_2$ are



$$\left(\frac{m_2x_1+m_1x_2}{m_1+m_2}, \frac{m_2y_1+m_1y_2}{m_1+m_2}, \frac{m_2z_1+m_1z_2}{m_1+m_2}\right)$$

Hence Co-ords. of pt. P are

$$P\left(\frac{5(-3)+3.5}{3+5}, \frac{3(-1)+5(1)}{3+5}, \frac{3(6)+5(4)}{3+5}\right)$$

$$=P\left(\frac{-15+15}{8}, \frac{-3+5}{8}, \frac{18+20}{8}\right)$$

$$=P\left(0, 1, \frac{19}{4}\right)$$

Q12 Find the ratio in which the yz-plane divides the segment joining the pts. $(-2,4,7)$ & $(3,-5,8)$.

Sol. Given pts. are $A(-2,4,7)$ & $B(3,-5,8)$.
Let the yz-plane divides the join of the given pts. in the ratio $m_1:m_2$.
Now the x-Co-ord. of the pt. dividing the join of given pts. is

$$x = \frac{3m_1+(-2)m_2}{m_1+m_2} = \frac{3m_1-2m_2}{m_1+m_2}$$

Since this pt. lies on yz-plane

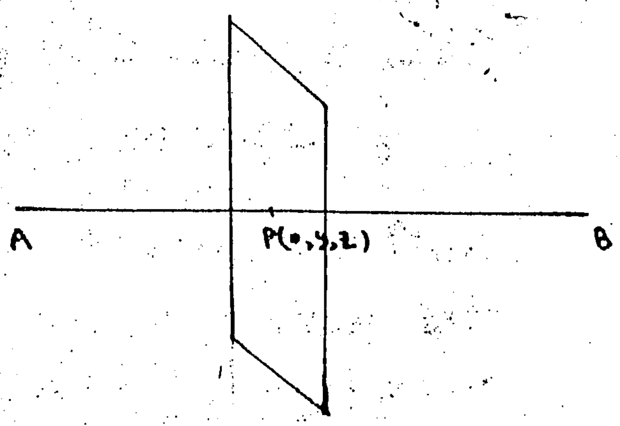
So $x = 0$

$$\Rightarrow \frac{3m_1 - 2m_2}{m_1 + m_2} = 0$$

$$\Rightarrow 3m_1 - 2m_2 = 0$$

$$3m_1 = 2m_2$$

$$\frac{m_1}{m_2} = \frac{2}{3}$$



or $m_1 : m_2 = 2 : 3$ is req. ratio

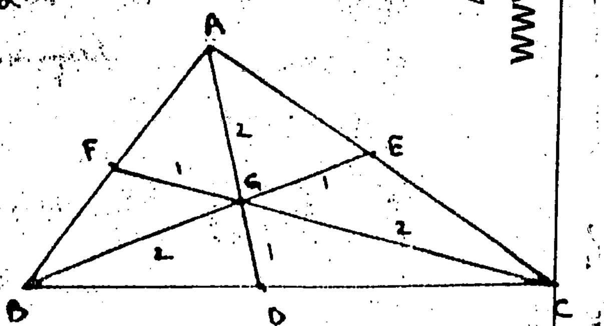
Q13 Show that the centroid of the triangle whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Sol.

Let the given vertices of the triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ & $C(x_3, y_3, z_3)$

Suppose D, E, F are the mid pts. of the sides BC, AC & AB respectively.



Now Co-ords. of D are

$$D\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

Suppose G is the centroid of ΔABC, then Co-ords. of pt. G dividing AD in ratio $2:1$ are

$$G\left(\frac{1\cdot x_1 + 2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1\cdot y_1 + 2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1\cdot z_1 + 2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

or $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Now the Co-ords. of pts. E & F are

$$E\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right) \text{ \& } F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Now the Co-ords. of the Centroid G dividing BE in the ratio 2 : 1 are

$$G\left(\frac{1.x_2+2\left(\frac{x_1+x_3}{2}\right)}{1+2}, \frac{1.y_2+2\left(\frac{y_1+y_3}{2}\right)}{1+2}, \frac{1.z_2+2\left(\frac{z_1+z_3}{2}\right)}{1+2}\right)$$

$$= G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Similarly the Co-ords. of Centroid G dividing CF in ratio 2 : 1 are

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Hence Co-ords. of Centroid G are

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Q14 Find the Centroid of tetrahedron whose vertices are (x_i, y_i, z_i); $i = 1, 2, 3, 4$.

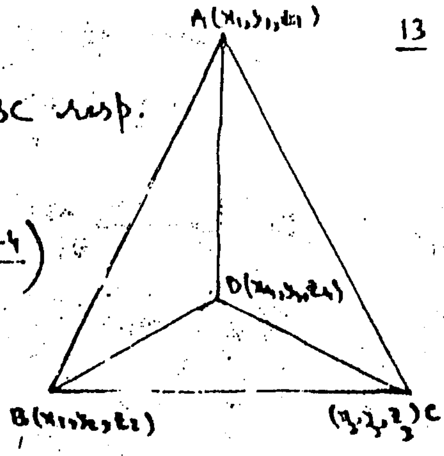
Sol-
let the vertices of the tetrahedron are $A = (x_1, y_1, z_1)$

$B = (x_2, y_2, z_2)$

$C = (x_3, y_3, z_3)$

$D = (x_4, y_4, z_4)$

Let E, F, G, H are the centroids of the triangles BCD, ACD, ABD & ABC resp.



Then their co-ords. are

$$E = \left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}, \frac{z_2+z_3+z_4}{3}\right)$$

$$F = \left(\frac{x_1+x_3+x_4}{3}, \frac{y_1+y_3+y_4}{3}, \frac{z_1+z_3+z_4}{3}\right)$$

$$G = \left(\frac{x_1+x_2+x_4}{3}, \frac{y_1+y_2+y_4}{3}, \frac{z_1+z_2+z_4}{3}\right)$$

$$H = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Now co-ords. of centroid dividing the line AE in ratio 3:1 are

$$\left(\frac{1\cdot x_1+3\left(\frac{x_2+x_3+x_4}{3}\right)}{1+3}, \frac{1\cdot y_1+3\left(\frac{y_2+y_3+y_4}{3}\right)}{1+3}, \frac{1\cdot z_1+3\left(\frac{z_2+z_3+z_4}{3}\right)}{1+3}\right)$$

$$= \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

Now co-ords. of centroid dividing the line BF in ratio 3:1 are

$$\left(\frac{1\cdot x_2+3\left(\frac{x_1+x_3+x_4}{3}\right)}{1+3}, \frac{1\cdot y_2+3\left(\frac{y_1+y_3+y_4}{3}\right)}{1+3}, \frac{1\cdot z_2+3\left(\frac{z_1+z_3+z_4}{3}\right)}{1+3}\right)$$

$$= \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

Similarly we can prove that co-ords. of centroid in Case of CG' & DH are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$

So Co-ords. of Centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$$