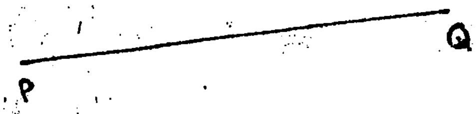
(Exercise 8.2)

In each of the problems 1—4, find the parametric eqs., direction ratios, direction cosines & measure of the direction angles of the st. line through P & Q

Q1 $P(1,-2,0)$ & $Q(5,-10,1)$

Sol. Given pts. are $P(1,-2,0)$



& $Q(5,-10,1)$

then eq. of line through $P(5,-10,1)$ & $Q(1,-2,0)$ is

$$\frac{x-1}{5-1} = \frac{y+2}{-10+2} = \frac{z-0}{1-0}$$

or $$\frac{x-1}{4} = \frac{y+2}{-8} = \frac{z}{1} = t$$

$$\Rightarrow \left.\begin{aligned} x &= 1+4t \\ y &= -2-8t \\ z &= t \end{aligned}\right\} \text{ are parametric eqs. of given line}$$

D.rs. of line PQ are $4, -8, 1$

Here $\sqrt{4^2+(-8)^2+(1)^2} = \sqrt{16+64+1} = \sqrt{81} = 9$

So d.cs. of line through P & Q are

$$\left.\begin{aligned} l &= \frac{4}{9} \\ m &= -\frac{8}{9} \\ n &= \frac{1}{9} \end{aligned}\right\}$$

let α, β, γ be the direction angles of line PQ then

$$\left.\begin{aligned} \cos\alpha &= \frac{4}{9} \\ \cos\beta &= -\frac{8}{9} \\ \cos\gamma &= \frac{1}{9} \end{aligned}\right\}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{4}{9}\right) = 63^\circ 37'$$

$$\beta = \cos^{-1}\left(-8/9\right) = 152^\circ 44'$$

$$\gamma = \cos^{-1}\left(\frac{1}{9}\right) = 83^\circ 37'$$

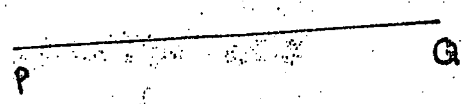
Q2 $P(6,5,-3)$ & $Q(4,1,1)$

Sol:- Given pts. are

$P(6,5,-3)$ & $Q(4,1,1)$

Then eq. of line through $P(6,5,-3)$ & $Q(4,1,1)$ is

$$\frac{x-6}{6-4} = \frac{y-5}{5-1} = \frac{z+3}{-3-1}$$



$$\frac{x-6}{2} = \frac{y-5}{4} = \frac{z+3}{-4}$$

$$\frac{x-6}{1} = \frac{y-5}{2} = \frac{z+3}{-2} = t$$

$$\Rightarrow \left.\begin{aligned} x &= 6+t \\ y &= 5+2t \\ z &= -3-2t \end{aligned}\right\}$$ are parametric eqs. for given line.

Now d.rs. of line PQ are $1, 2, -2$

Now $\sqrt{(1)^2+(2)^2+(-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

So d.Cs. of line PQ are

$$\left.\begin{aligned} l &= \frac{1}{3} \\ m &= \frac{2}{3} \\ n &= -\frac{2}{3} \end{aligned}\right\}$$

let α, β, γ be the direction angles of line PQ

then $$\left.\begin{aligned} \cos\alpha &= \frac{1}{3} \\ \cos\beta &= \frac{2}{3} \\ \cos\gamma &= -\frac{2}{3} \end{aligned}\right\}$$

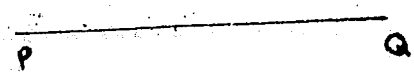
$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{3}\right) = 70^\circ 32'$$

$$\beta = \cos^{-1}\left(\frac{2}{3}\right) = 48^\circ 11'$$

$$\gamma = \cos^{-1}\left(-\frac{2}{3}\right) = 131^\circ 49'$$

Q3 $P(1,-5,1)$ & $Q(4,-5,4)$

Sol:- Given pts. are



$P(1,-5,1)$ & $Q(4,-5,4)$

Then eq. of line through P & Q is

$$\frac{x-1}{4-1} = \frac{y+5}{-5+5} = \frac{z-1}{4-1}$$

$$\frac{x-1}{3} = \frac{y+5}{0} = \frac{z-1}{3}$$

$$\frac{x-1}{1} = \frac{y+5}{0} = \frac{z-1}{1} = t$$

$$\Rightarrow \left.\begin{aligned} x &= 1+t \\ y &= -5+0t \\ z &= 1+t \end{aligned}\right\}$$ are parametric eqs. of line PQ.

Now d.rs. of PQ are $1, 0, 1$

$$\sqrt{(1)^2+(0)^2+(1)^2} = \sqrt{1+0+1} = \sqrt{2}$$

So d.cs. of PQ are

$$\left.\begin{aligned} l &= \frac{1}{\sqrt{2}} \\ m &= 0 \\ n &= \frac{1}{\sqrt{2}} \end{aligned}\right\}$$

let α, β, γ be the direction angles of line PQ

then

$$\left.\begin{aligned} \cos\alpha &= \frac{1}{\sqrt{2}} \\ \cos\beta &= 0 \\ \cos\gamma &= \frac{1}{\sqrt{2}} \end{aligned}\right\}$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$$

$$\beta = \cos^{-1}(0) = 90^\circ$$

$$\beta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$$

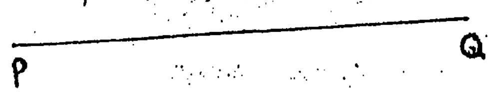
Q4 $P(3,5,7)$ & $Q(6,-8,10)$

Sol:- Given pts. are $P(3,5,7)$ & $Q(6,-8,10)$.

Then eq. of line PQ is

$$\frac{x-3}{6-3} = \frac{y-5}{-8-5} = \frac{z-7}{10-7}$$

$$\frac{x-3}{3} = \frac{y-5}{-13} = \frac{z-7}{3} = t$$



$$\Rightarrow \left.\begin{aligned} x &= 3+3t \\ y &= 5-13t \\ z &= 7+3t \end{aligned}\right]$$ are parametric eqs. of given line

Now d.rs. of PQ are $3, -13, 3$

$$\sqrt{(3)^2+(-13)^2+(3)^2} = \sqrt{9+169+9} = \sqrt{187}$$

Then the d.cs. of line PQ are

$$\left.\begin{aligned} l &= \frac{3}{\sqrt{187}} \\ m &= -\frac{13}{\sqrt{187}} \\ n &= \frac{3}{\sqrt{187}} \end{aligned}\right]$$

let α, β, γ be the direction angles of line PQ then

$$\left.\begin{aligned} \cos\alpha &= \frac{3}{\sqrt{187}} \\ \cos\beta &= -\frac{13}{\sqrt{187}} \\ \cos\gamma &= \frac{3}{\sqrt{187}} \end{aligned}\right)$$

$$\Rightarrow \alpha = \cos^{-1}\left(\frac{3}{\sqrt{187}}\right) = 77^\circ 19'$$

$\beta = \cos^{-1}\left(-\frac{13}{\sqrt{187}}\right) = 159^\circ 19'$

$\gamma = \cos^{-1}\left(\frac{3}{\sqrt{187}}\right) = 77^\circ 19'$

Q5 Find the direction cosines of Co-ord. axes.

Sol: We want to find the eqs. of x-axis, y-axis & z-axis.

Since x-axis makes angles $0^\circ, 90^\circ, 90^\circ$ with co-ord. axes.

So d.cs. of x-axis are $\cos 0^\circ, \cos 90^\circ, \cos 90^\circ$

or $1, 0, 0$

Since y-axis makes angles $90^\circ, 0^\circ, 90^\circ$ with co-ord. axes, then

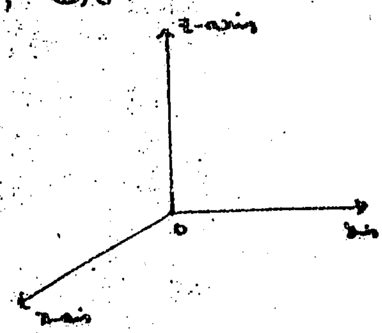
d.cs. of y-axis are $\cos 90^\circ, \cos 0^\circ, \cos 90^\circ$

or $0, 1, 0$

Since z-axis makes angles $90^\circ, 90^\circ, 0^\circ$ with co-ord. axes, so d.cs. of z-axis are

$\cos 90^\circ, \cos 90^\circ, \cos 0^\circ$

or $0, 0, 1$

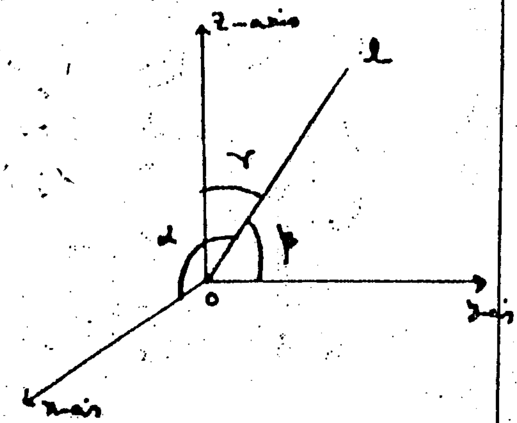


Q6 Prove that if, the direction angles of a st. line are α, β & γ then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

Sol.-

We know that if direction angles of a line are α, β, γ then d.cs. are $\cos\alpha, \cos\beta, \cos\gamma$

Also we know



$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$1-\sin^2\alpha + 1-\sin^2\beta + 1-\sin^2\gamma = 1$$

$$3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$$

$$-(\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = -2$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

Q7 If measures of two of the direction angles of a st. line are 45° & 60°, find measures of the third direction angle.

Sol.

Let α, β, γ be the direction angles of line

then $\alpha = 45^\circ$

$\beta = 60^\circ$

$\gamma = ?$

$\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos^2 45^\circ + \cos^2 60^\circ + \cos^2\gamma = 1$

$\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

$\cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4}$

$= \frac{4-2-1}{4}$

$= \frac{1}{4}$

$\cos^2\gamma = \frac{1}{4}$

$\cos\gamma = \frac{1}{2}$

$\boxed{\gamma = 60^\circ}$

Q8 The d.cs. of two st. lines are given by the eqs. $l+m+n=0$, $l^2+m^2-n^2=0$
Find the measure of the angle b/w them.

Soln
Direction Cosines of two lines are given by

$l+m+n=0$ ———— (1)

$l^2+m^2-n^2=0$ ———— (2)

From (1) $n = -(l+m)$ ———— (3)

Put in (2)

$l^2+m^2-\left[-(l+m)\right]^2 = 0$

$$l^2 + m^2 - (l^2 + m^2 + 2lm) = 0$$

$$\cancel{l^2} + \cancel{m^2} - \cancel{l^2} - \cancel{m^2} - 2lm = 0$$

$$-2lm = 0$$

$$lm = 0$$

$l = 0$

Put in ③

$n = -m$

or $\frac{n}{1} = \frac{m}{-1}$

So

$\frac{l}{0} = \frac{m}{-1} = \frac{n}{1} = \frac{1}{\sqrt{2}}$

$\Rightarrow l_1 = 0$

$m_1 = -\frac{1}{\sqrt{2}}$

$n_1 = \frac{1}{\sqrt{2}}$ are req. d.cs.

$m = 0$

Put in ③

$n = -l$

$\frac{n}{1} = \frac{l}{-1}$

So $\frac{l}{-1} = \frac{m}{0} = \frac{n}{1} = \frac{1}{\sqrt{2}}$

$\Rightarrow l_2 = \frac{-1}{\sqrt{2}}$

$m_2 = 0$

$n_2 = \frac{1}{\sqrt{2}}$ are req. d.cs.

Let θ be the angle b/w given lines then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$= (0)\left(\frac{-1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 0 + 0 + \frac{1}{2}$$

$$\cos\theta = \frac{1}{2}$$

$$\Rightarrow \boxed{\theta = 60^\circ}$$

Q9 The direction Cosines of two st. lines are given by the eqs. $l+m+n=0$ & $lm+2ln-mn=0$ Find the measure of the angle b/w them. 24

Sol:- The d.cs. of given lines are given by eqs.

$$l+m+n=0 \quad \text{——— (1)}$$

$$2lm+2ln-mn=0 \quad \text{——— (2)}$$

from (1) $\quad n=-(l+m)$ ——— (3)

Put in (2)

$$2lm+2l\left(-(l+m)\right)-m\left(-(l+m)\right)=0$$

$$2lm-2l(l+m)+m(l+m)=0$$

$$2\cancel{lm}-2l^2-2\cancel{lm}+lm+m^2=0$$

$$-2l^2+lm+m^2=0$$

$$2l^2-lm-m^2=0$$

$$2l^2-2lm+lm-m^2=0$$

$$2l(l-m)+m(l-m)=0$$

$$(l-m)(2l+m)=0$$

$l-m=0$

or $l=m \Rightarrow \frac{l}{1}=\frac{m}{1}$

Put in (3)

$n=-2l$

$\frac{n}{-2}=\frac{l}{1}$

$2l+m=0$

or $m=-2l \Rightarrow \frac{l}{1}=\frac{m}{-2}$

Put in (3)

$n=l$

or $\frac{l}{1}=\frac{n}{1}$

So $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2} = \frac{1}{\sqrt{6}}$

$\Rightarrow l_1 = \frac{1}{\sqrt{6}}$

$m_1 = \frac{1}{\sqrt{6}}$

$n_1 = -\frac{2}{\sqrt{6}}$

are d.cs. for l_1

or $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{1}{\sqrt{6}}$

$\Rightarrow l_2 = \frac{1}{\sqrt{6}}$

$m_2 = -\frac{2}{\sqrt{6}}$

$n_2 = \frac{1}{\sqrt{6}}$

are d.cs. for l_2

let θ be the angle b/w given lines then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$= \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(-\frac{2}{\sqrt{6}}\right) + \left(-\frac{2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)$

$= \frac{1}{6} - \frac{2}{6} - \frac{2}{6}$

$= \frac{1-2-2}{6}$

$= -\frac{3}{6}$

$\cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

$\boxed{\theta = 60^\circ}$

Q10 Find eqs. of the st. lines L & M in symmetric forms. Determine whether the pairs of lines intersect. Find the pt. of intersection if it exists. Prob (10–12)

Q1. L : through $A(2,1,3)$, $B(-1,2,-4)$

M : through $P(5,1,-2)$, $Q(0,4,3)$

Sol.

The eq. of the line through $A(2,1,3)$ & $B(-1,2,-4)$

is $\dfrac{x-2}{2+1} = \dfrac{y-1}{1-2} = \dfrac{z-3}{3+4}$

or $\dfrac{x-2}{3} = \dfrac{y-1}{-1} = \dfrac{z-3}{7}$ ——— (L)

Now eq. of line M through $P(5,1,-2)$ & $Q(0,4,3)$ is

$\dfrac{x-5}{5-0} = \dfrac{y-1}{1-4} = \dfrac{z+2}{-2-3}$

$\dfrac{x-5}{5} = \dfrac{y-1}{-3} = \dfrac{z+2}{-5}$ ——— (M)

which are req. eqs. in symmetric form of L & M.

Now we write eqs. of L & M in parametric form.

let $\dfrac{x-2}{3} = \dfrac{y-1}{-1} = \dfrac{z-3}{7} = p$

$\Rightarrow \left.\begin{aligned} x &= 2+3p \\ y &= 1-p \\ z &= 3+7p \end{aligned}\right\}$ ——— (L)

Also let $\dfrac{x-5}{5} = \dfrac{y-1}{-3} = \dfrac{z+2}{-5} = q$

$\Rightarrow \left.\begin{aligned} x &= 5+5q \\ y &= 1-3q \\ z &= -2-5q \end{aligned}\right\}$ ——— (M)

Let the lines L & M intersect at (x_0, y_0, z_0)

So this pt. will lie on both lines.

So $\left.\begin{aligned} x_0 &= 2+3p \\ y_0 &= 1-p \\ z_0 &= 3+7p \end{aligned}\right]$ & $\left.\begin{aligned} x_0 &= 5+5q \\ y_0 &= 1-3q \\ z_0 &= -2-5q \end{aligned}\right]$

$$\Rightarrow \quad \begin{aligned} 2+3p &= 5+5q \\ 1-p &= 1-3q \\ 3+7p &= -2-5q \end{aligned}$$

or
$$3p-5q = 3 \quad \text{———} \quad (1)$$
$$p-3q = 0 \quad \text{———} \quad (2)$$
$$7p+5q = -5 \quad \text{———} \quad (3)$$

Multiplying (2) by 3

$$\cancel{3p}-5q = 3 \quad \text{———} \quad (1)$$
$$\underline{\underset{-}{\cancel{3p}} \underset{+}{-} 9q \underset{-}{=} 0} \quad \text{———} \quad (2)$$
$$4q = 3$$
$$\boxed{q = \frac{3}{4}}$$

Put in (2)

$$p - 3\left(\frac{3}{4}\right) = 0$$
$$\boxed{p = \frac{9}{4}}$$

We see these values of p & q do not satisfy eq (3)

Hence given lines do not intersect.

So pt. of intersection of lines does not exist.

Q11 $L : \underline{r} = (3\hat{i}+2\hat{j}-\hat{k}) + t(6\hat{i}-4\hat{j}-3\hat{k})$

$M : \underline{r} = (5\hat{i}+4\hat{j}+7\hat{k}) + s(14\hat{i}-6\hat{j}+2\hat{k})$

Sol. Given lines are

$L : \underline{r} = (3\hat{i}+2\hat{j}-\hat{k}) + t(6\hat{i}-4\hat{j}-3\hat{k})$

$M : \underline{r} = (5\hat{i}+4\hat{j}+7\hat{k}) + s(14\hat{i}-6\hat{j}+2\hat{k})$

$L : \underline{r} = (3+6t)\hat{i} + (2-4t)\hat{j} + (-1-3t)\hat{k}$

$M : \underline{r} = (5+14s)\hat{i} + (4-6s)\hat{j} + (7+2s)\hat{k}$

Now parametric eqs. for L are

$$\left.\begin{aligned} x &= 3+6t \\ y &= 2-4t \\ z &= -1-3t \end{aligned}\right\} \text{———— (L)}$$

& parametric eqs. for line M are

$$\left.\begin{aligned} x &= 5+14s \\ y &= 4-6s \\ z &= 7+2s \end{aligned}\right\} \text{———— (M)}$$

Now eqs. of L & M in symmetric forms are

$$\frac{x-3}{6} = \frac{y-2}{-4} = \frac{z+1}{-3} \text{ ———— (L)}$$

$$\& \quad \frac{x-5}{14} = \frac{y-4}{-6} = \frac{z-7}{2} \text{ ———— (M)}$$

Let the lines L & M intersect at pt. (x_0, y_0, z_0)

so (x_0, y_0, z_0) will lie on both lines L & M

Hence it satisfies both eqs.

So $\left.\begin{aligned} x_0 &= 3+6t \\ y_0 &= 2-4t \\ z_0 &= -1-3t \end{aligned}\right\}$ & $\left.\begin{aligned} x_0 &= 5+14s \\ y_0 &= 4-6s \\ z_0 &= 7+2s \end{aligned}\right\}$

Hence

$$\left.\begin{aligned} 3+6t &= 5+14s \\ 2-4t &= 4-6s \\ -1-3t &= 7+2s \end{aligned}\right\}$$

or $\quad 6t-14s = 2$

$$3t-7s = 1 \quad \text{——— (1)}$$
$$4t-6s = -2 \quad \text{——— (2)}$$
$$3t+2s = -8 \quad \text{——— (3)}$$

Subt. (1) & (3)

$$-9s = 9 \Rightarrow \boxed{s=-1}$$

Put in (1)

$$3t-7(-1) = 1$$
$$\cancel{3t = 1-7}$$
$$3t = -6$$
$$\Rightarrow \boxed{t=-2}$$

These values of t & s satisfy eq. (2)
Hence lines L & M intersect.

Put $t=-2$ in above eq.

$$(x_0, y_0, z_0) = (3+6(-2),\ 2-4(-2),\ -1-3(-2))$$
$$= (-9, 10, 5)$$

Find the distance of the given pt. P from the given line L : (Problems 13 – 14). 32

Q13 $P = (3,-2,1)$ $\quad L : \begin{cases} x = 1+t \\ y = 3-2t \\ z = -2+2t \end{cases}$

Sol:- Given pt. & given line are

$$P = (3,-2,1) \qquad L : \begin{cases} x = 1+t \\ y = 3-2t \\ z = -2+2t \end{cases}$$

A point on the line L is $A = (1,3,-2)$

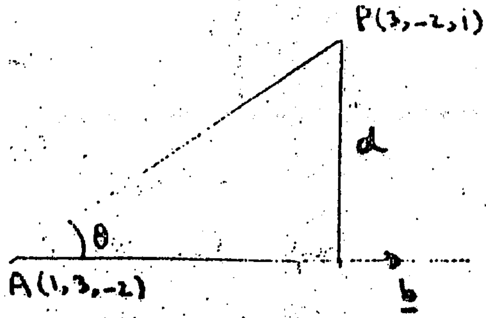
Now $\overrightarrow{AP} = [3-1, -2-3, 1+2]$

$$= [2,-5,3]$$

Now direction vector of the line L is $\underline{b} = [1,-2,2]$

Let d be the req. distance of pt. from line then

$$d = \frac{|\overrightarrow{AP} \times \underline{b}|}{|\underline{b}|} \quad \text{——— (1)}$$



Now $\overrightarrow{AP} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -5 & 3 \\ 1 & -2 & 2 \end{vmatrix}$

$$= \hat{i}(-10+6) - \hat{j}(4-3) + \hat{k}(-4+5)$$

$$= -4\hat{i} - \hat{j} + \hat{k}$$

So $|\overrightarrow{AP} \times \underline{b}| = \sqrt{16+1+1}$

$= \sqrt{18}$

$= 3\sqrt{2}$

Now $|\underline{b}| = \sqrt{1+4+4}$

$= \sqrt{9}$

$= 3$

Put in ①

$d = \dfrac{3\sqrt{2}}{3}$

$\boxed{d = \sqrt{2}}$

Q14 $P = (0,-2,1)$, $L : \dfrac{x-1}{4} = \dfrac{y+3}{-2} = \dfrac{z+1}{5}$

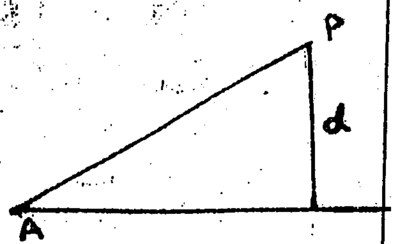
Sol. Given pt. & line are

$P = (0,-2,1)$; $L: \dfrac{x-1}{4} = \dfrac{y+3}{-2} = \dfrac{z+1}{5}$

A pt. on the line L is $A = (1,-3,-1)$

Now $\overrightarrow{AP} = [0-1, -2+3, 1+1]$

$= [-1, 1, 2]$



Now direction vector of line L is $\underline{b} = [4,-2,5]$

Let d be the req. distance then

$d = \dfrac{|\overrightarrow{AP} \times \underline{b}|}{|\underline{b}|}$ ——— ①

Now $\vec{AP}\times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix}$

$= \hat{i}(5+4) - \hat{j}(-5-8) + \hat{k}(2-4)$

$= 9\hat{i} + 13\hat{j} - 2\hat{k}$

& $|\vec{AP}\times \underline{b}| = \sqrt{81+169+4}$

$= \sqrt{254}$

Now $|\underline{b}| = \sqrt{16+4+25}$

$= \sqrt{45}$

Put in ①

$d = \dfrac{\sqrt{254}}{\sqrt{45}}$ —— Ans.

Q15 If the edges of a rectangular parallelopipped are a, b, c, show that the angles b/w the four diagonals are given by

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

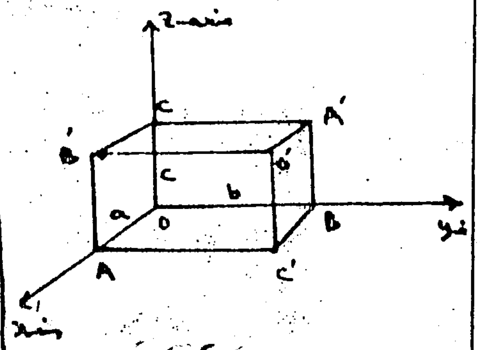
Sol. Here the lengths of the edges OA, OB & OC are a, b, c resp.

The Co-ords. of the vertices of parallelopipped are $O=(0,0,0)$, $O'=(a,b,c)$, $A=(a,0,0)$, $A'=(0,b,c)$

$B = (0, b, 0)$, $B' = (a, 0, c)$, $C = (0, 0, c)$ & $C' = (a, b, 0)$ 35

The four diagonals of parallelopipped are

AA' , BB' , CC' & OO'



D.rs. of AA' are $-a, b, c$

D.rs. of BB' are $a, -b, c$

D.rs. of CC' are $a, b, -c$

D.rs. of OO' are a, b, c

So d.cs. of AA' are $\dfrac{-a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

d.cs. of BB' are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{-b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

d.cs. of CC' are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{-c}{\sqrt{a^2+b^2+c^2}}$

d.cs. of OO' are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$

Angle b/w diagonal AA' & BB' is

$$\cos\alpha = \frac{-a^2}{a^2+b^2+c^2} + \frac{-b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2}$$

$$\cos\alpha = \frac{-a^2-b^2+c^2}{a^2+b^2+c^2}$$

The angle b/w diagonal BB' & CC' is

$$\cos\beta = \frac{a^2}{a^2+b^2+c^2} + \frac{-b^2}{a^2+b^2+c^2} + \frac{-c^2}{a^2+b^2+c^2}$$

$$\cos\beta = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \beta = \cos^{-1}\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right) \quad \text{——— (2)}$$

The angle b/w diagonals CC' & OO' is

$$\cos\gamma = \frac{a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{-c^2}{a^2+b^2+c^2}$$

$$\cos\gamma = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \gamma = \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right) \quad \text{——— (3)}$$

The angle b/w the diagonal AA' & OO' is

$$\cos\delta = \frac{-a^2}{a^2+b^2+c^2} + \frac{b^2}{a^2+b^2+c^2} + \frac{c^2}{a^2+b^2+c^2}$$

$$\cos\delta = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \delta = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right) \quad \text{——— (4)}$$

from (1), (2), (3) & (4), we see that the angles b/w four diagonals of a Cube are

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

Q16 A st. line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that 37

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Sol. Consider a cube & let a be the length of one edge of cube.

Then the co-ords. of different vertices of cube

$$O = (0,0,0)\,,\quad O' = (a,a,a)$$
$$A = (a,0,0)\,,\quad A' = (0,a,a)$$
$$B = (0,a,0)\,,\quad B' = (a,0,a)$$
$$C = (0,0,a)\,,\quad C' = (a,a,0)$$

Then AA', BB', CC' & OO' are four diagonals of the cube.

d.rs. of diagonal AA' are $-a, a, a$

d.rs. of diagonal BB' are $a, -a, a$

d.rs. of ~~diagonal~~ CC' are $a, a, -a$

d.rs. of diagonal OO' are a, a, a

Now $\sqrt{a^2+a^2+a^2} = \sqrt{3a^2} = \sqrt{3}\,a$

So

d.cs. of diagonal AA' are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

d.cs. of diagonal BB' are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

d.cs. of diagonal CC' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

& d.cs. of diagonal OO' are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Since $L \parallel AB$

So d.rs. of line L are $-1, 3, -2$

Hence eq. of line L passing through $P(0,-3,2)$ & having d.rs. $-1,3,-2$ is

$$\frac{x-0}{-1} = \frac{y+3}{3} = \frac{z+1}{-2}$$

or $$\frac{x}{1} = \frac{y+3}{-2} = \frac{z+1}{2}$$

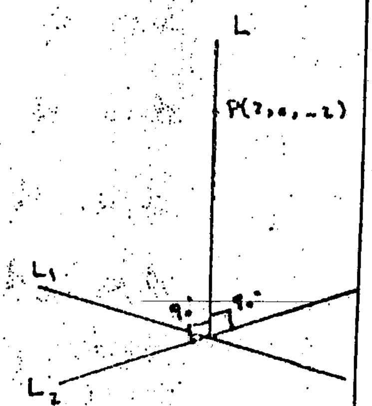
Q18 Find eq. of the st. line passing through the pt. $P(2,0,-2)$ & perpendicular to each of the st. lines.

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \text{ \& } \frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}$$

Sol. Given lines are

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \text{ ——— } (L_1)$$

$$\frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2} \text{ ——— } (L_2)$$



D.rs. of L_1 are $2,2,2$

D.rs. of L_2 are $3,-1,2$

Suppose L be the req. line with d.rs. c_1, c_2, c_3

Since $L \perp L_1$ & $L \perp L_2$

So by condition of perpendicularity,

$$\left.\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0 \\ \text{\&} \quad 3c_1 - c_2 + 2c_3 &= 0 \end{aligned}\right]$$

Let l, m, n be the d.cs. of the line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals AA', BB', CC' & OO' then

$$\cos\alpha = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{-l+m+n}{\sqrt{3}} \quad \text{---} (1)$$

$$\cos\beta = \frac{l}{\sqrt{3}} + \frac{-m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l-m+n}{\sqrt{3}} \quad \text{---} (2)$$

$$\cos\gamma = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}} = \frac{l+m-n}{\sqrt{3}} \quad \text{---} (3)$$

$$\cos\delta = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}} \quad \text{---} (4)$$

Sq. (1), (2), (3) & (4) & adding.

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4l^2+4m^2+4n^2}{3}$$

$$= \frac{4(l^2+m^2+n^2)}{3}$$

$$= \frac{4}{3}$$

Q17 Find the eq. of the st. line passing through the pt. $P(0,-3,2)$ & parallel to the st. line joining the pts. $A(3,4,7)$ & $B(2,7,5)$.

Sol.

Let L be the req. line passing through $P(0,-3,2)$

Now d.rs. of line AB are $2-3, 7-4, 5-7$

or $-1, 3, -2$

$$\frac{c_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{-c_2}{\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{c_3}{\begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix}}$$

$$\frac{c_1}{4+2} = \frac{-c_2}{4-6} = \frac{c_3}{-2-6}$$

$$\frac{c_1}{6} = \frac{c_2}{2} = \frac{c_3}{-8}$$

$$\frac{c_1}{3} = \frac{c_2}{1} = \frac{c_3}{-4}$$

So d.rs. of line L are $3, 1, -4$

Hence eq. of line L is

$$\frac{x-2}{3} = \frac{y-0}{1} = \frac{z+2}{-4}$$

Find eqs. of the st. line through the given pt A & perpendicular to the given st. line (Problems 19 – 20):

Q19 $A = (11, 4, -6)$ & $x = 4-t,\ y = 7+2t,\ z = -1+t$

Sol:- Given pt. & line are

$A = (11, 4, -6)$ & $x = 4-t,\ y = 7+2t,\ z = -1+t$

Let L be the req. line passing through $A(11, 4, -6)$ & perp. to given line. Suppose it meets the given line at pt. B.

Now any pt. on given line is $(4-t, 7+2t, -1+t)$

So coords. of B are $(4-t, 7+2t, -1+t)$

D.rs. of AB are $4-t-11,\ 7+2t-4,\ -1+t+6$

or $-t-7,\ 2t+3,\ t+5$

Here d.rs. of given line are $-1, 2, 1$

Since $AB \perp$ given line

So $(-1)(-t-7)+2(2t+3)+1(t+5)=0$

$$t+7+4t+6+t+5=0$$

$$6t+18=0$$

$$t+3=0$$

$$\boxed{t=-3}$$

Hence d.rs. of AB are $3-7,\ 2(-3)+3,\ -3+5$

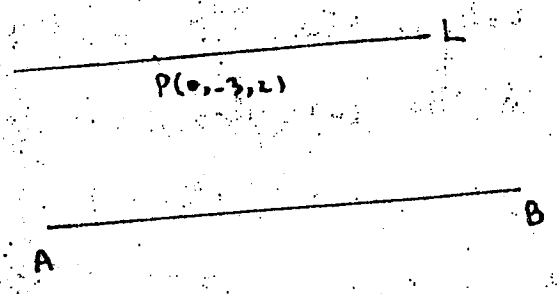
$$= -4, -3, 2$$

So eq. of line AB is

$$\frac{x-11}{-4}=\frac{y-4}{-3}=\frac{z+1}{2}$$

or $$\frac{x-11}{4}=\frac{y-4}{3}=\frac{z+1}{-2}$$

Fig



Q20 $A = (5,-4,4)$ & $\frac{x}{-1} = \frac{y-1}{1} = \frac{z}{-2} = t$

Sol:- Given pt. & line is

$A = (5,-4,4)$ & $\frac{x}{-1} = \frac{y-1}{1} = \frac{z}{-2} = t$

The parametric eqs. of given line are

$$\left.\begin{aligned} x &= -t \\ y &= 1+t \\ z &= -2t \end{aligned}\right\}$$

Let L be the req. line passing through $A(5,-4,4)$ & perp. to given line. Suppose it meets the given line at pt. B

Any pt. on given line are $(-t, 1+t, -2t)$

So Co-ords. of pt. B are $B(-t, 1+t, -2t)$

D.rs. of line AB are $5+t, -4-1-t, 4+2t$

$$= 5+t, -5-t, 4+2t$$

Here d.rs. of given line are $-1, 1, -2$

Since $AB \perp$ given line

So $-1(5+t)+1(-5-t)+(-2)(4+2t) = 0$

$$-5-t-5-t-8-4t = 0$$

$$-6t-18 = 0$$

$$t+3 = 0 \Rightarrow \boxed{t = -3}$$

So d.rs. of AB are $5-3,\ -5+3,\ 4-6$ 43

or $2, -2, -2$

or $1, -1, -1$

Hence req. eq. of line AB is

$$\frac{x-5}{1} = \frac{y+4}{-1} = \frac{z-4}{-1}$$

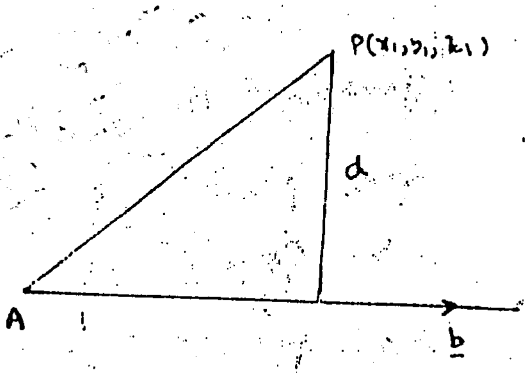
Q21 Find the length of the perpendicular from the pt. (x_1, y_1, z_1) to the st. line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ where $l^2+m^2+n^2 = 1$

Sol.

Given pt. & given line are

$P(x_1, y_1, z_1)$ &

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$



Here $A = (\alpha, \beta, \gamma)$ ~~is~~ a pt. of given line

Since l, m, n are d.cs. of given line

So direction vector of given line is $\underline{b} = [l, m, n]$

Now $\overrightarrow{AP} = [x_1-\alpha, y_1-\beta, z_1-\gamma]$

Let d be the req. distance then

$$d = \frac{|\vec{AP} \times \underline{b}|}{|\underline{b}|} \quad \text{———} \quad (1)$$

Now $\vec{AP} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1-\alpha & y_1-\beta & z_1-\gamma \\ l & m & n \end{vmatrix}$

$$= \hat{i}\left[n(y_1-\beta) - m(z_1-\gamma)\right] - \hat{j}\left[n(x_1-\alpha) - l(z_1-\gamma)\right] + \hat{k}\left[m(x_1-\alpha) - l(y_1-\beta)\right]$$

$$\vec{AP} \times \underline{b} = \left[n(y_1-\beta) - m(z_1-\gamma)\right]\hat{i} + \left[l(z_1-\gamma) - n(x_1-\alpha)\right] + \left[m(x_1-\alpha) - l(y_1-\beta)\right]\hat{k}$$

$$\Rightarrow |\vec{AP} \times \underline{b}| = \sqrt{\left[n(y_1-\beta) - m(z_1-\gamma)\right]^2 + \left[l(z_1-\gamma) - n(x_1-\alpha)\right]^2 + \left[m(x_1-\alpha) - l(y_1-\beta)\right]^2}$$

$$= \sqrt{\Sigma\left[n(y_1-\beta) - m(z_1-\gamma)\right]^2}$$

Put in (1)

$$d = \frac{\sqrt{\Sigma\left[n(y_1-\beta) - m(z_1-\gamma)\right]^2}}{\sqrt{l^2+m^2+n^2}}$$

$$d = \sqrt{\Sigma\left[n(y_1-\beta) - m(z_1-\gamma)\right]^2}$$

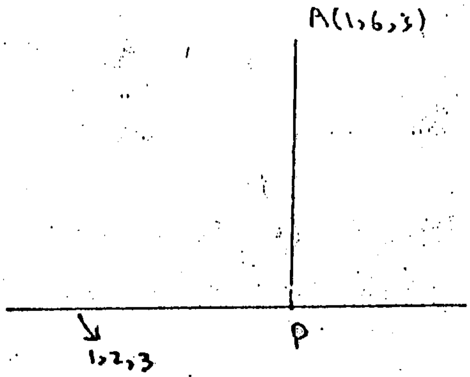
is the req. distance.

Q22 Find eq. of the perpendicular from the pt. $(1,6,3)$ to the st. line 45

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

Also obtain its length & Co-ords. of the foot of the perpendicular.

Sol:-



Given pt. & given line are $A = (1,6,3)$ &

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

Let AP be the length of perpendicular from pt. A to given line.

$\Rightarrow \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = t$

So parametric eqs. of given line are

$$\left.\begin{aligned} x &= t \\ y &= 1+2t \\ z &= 2+3t \end{aligned}\right|$$

Any pt. on this line is $(t, 1+2t, 2+3t)$

So Co-ords. of pt. P are $P(t, 1+2t, 2+3t)$

D.rs. of line AP are $t-1,\ 1+2t-6,\ 2+3t-3$

$$= t-1,\ 2t-5,\ 3t-1$$

& drs. of given line are $1,2,3$.

Since AP ⊥ given line

So $1(t-1)+2(2t-5)+3(3t-1)=0$

$$t-1+4t-10+9t-3=0$$

$$14t-14=0$$

$$t-1=0$$

$$\boxed{t=1}$$

So Co-ords. of pt. P are $P(1,3,5)$.

Length of perp. $= |AP|$

$$= \sqrt{(1-1)^2+(3-1)^2+(5-3)^2}$$

$$= \sqrt{0+9+4}$$

$$= \sqrt{13}$$

Now eq. of perpendicular AP is

$$\frac{x-1}{1-1} = \frac{y-6}{3-6} = \frac{z-3}{5-3}$$

or $$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

Q23 Find a necessary & sufficient Condition that the pts. $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ & $R(x_3,y_3,z_3)$ are Collinear.

Sol:- Given pts. are

$P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ & $R(x_3,y_3,z_3)$

Suppose that the pts. P, Q & R are Collinear
Now eq. of line through $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Since pts. P, Q & R are Collinear
So $R(x_3, y_3, z_3)$ lies on above line

i.e. $$\frac{x_3-x_1}{x_2-x_1} = \frac{y_3-y_1}{y_2-y_1} = \frac{z_3-z_1}{z_2-z_1} = t \text{ (say)}$$

$$\Rightarrow \left.\begin{aligned} x_3-x_1 &= t(x_2-x_1) \\ y_3-y_1 &= t(y_2-y_1) \\ z_3-z_1 &= t(z_2-z_1) \end{aligned}\right]$$

or $$\left.\begin{aligned} x_3-x_1-tx_2+tx_1 &= 0 \\ y_3-y_1-ty_2+ty_1 &= 0 \\ z_3-z_1-tz_2+tz_1 &= 0 \end{aligned}\right]$$

or $$\left.\begin{aligned} (t-1)x_1-tx_2+x_3 &= 0 \\ (t-1)y_1-ty_2+y_3 &= 0 \\ (t-1)z_1-tz_2+z_3 &= 0 \end{aligned}\right]$$

Eliminating t from above eqs.

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0$$

or $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$

which is a necessary condition for three pts. P, Q & R to be Collinear.

Conversely suppose that for three pts. $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ & $R(x_3, y_3, z_3)$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Q24 If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are d.cs. of three mutually perpendicular lines, prove that the line whose d.cs. are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$ & $n_1+n_2+n_3$ makes congruent angles with them.

Sol:-

Suppose that L_1, L_2 & L_3 are given lines s.t.

d.cs. of L_1 are l_1, m_1, n_1

d.cs. of L_2 are l_2, m_2, n_2

d.cs. of L_3 are l_3, m_3, n_3

Since lines L_1, L_2, L_3 are mutually perpendicular

So
$$\left.\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= 0 \\ l_2l_3 + m_2m_3 + n_2n_3 &= 0 \\ l_1l_3 + m_1m_3 + n_1n_3 &= 0 \end{aligned}\right]$$

Let L be the line having d.cs. $l_1+l_2+l_3$, $m_1+m_2+m_3$ & $n_1+n_2+n_3$

Let θ_1 be the angle b/w L & L_1 then

$$\cos\theta_1 = \frac{l_1(l_1+l_2+l_3) + m_1(m_1+m_2+m_3) + n_1(n_1+n_2+n_3)}{\sqrt{l_1^2+m_1^2+n_1^2}\cdot\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{(l_1^2+m_1^2+n_1^2)+(l_1l_2+m_1m_2+n_1n_2)+(l_1l_3+m_1m_3+n_1n_3)}{\sqrt{(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+(l_3^2+m_3^2+n_3^2)+2(l_1l_2+m_1m_2+n_1n_2)+2(l_2l_3+m_2m_3+n_2n_3)+2(l_1l_3+m_1m_3+n_1n_3)}}$$

$$= \frac{1+0+0}{\sqrt{1+1+1+0+0+0}}$$

$$\cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ ——— } (1)$$

let θ_2 be the angle b/w L & L_2 then

$$\cos\theta_2 = \frac{l_2(l_1+l_2+l_3)+m_2(m_1+m_2+m_3)+n_2(n_1+n_2+n_3)}{\sqrt{l_2^2+m_2^2+n_2^2}\cdot\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{(l_1l_2+m_1m_2+n_1n_2)+(l_2^2+m_2^2+n_2^2)+(l_2l_3+m_2m_3+n_2n_3)}{\sqrt{(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+(l_3^2+m_3^2+n_3^2)+2(l_1l_2+m_1m_2+n_1n_2)+2(l_2l_3+m_2m_3+n_2n_3)+2(l_1l_3+m_1m_3+n_1n_3)}}$$

$$= \frac{0+1+0}{\sqrt{1+1+1+0+0+0}}$$

$$\cos\theta_2 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)} \text{ ——— } (2)$$

Now let θ_3 be the angle b/w L & L_3 then

$$\cos\theta_3 = \frac{l_3(l_1+l_2+l_3)+m_3(m_1+m_2+m_3)+n_3(n_1+n_2+n_3)}{\sqrt{l_3^2+m_3^2+n_3^2}\cdot\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{(l_1l_3+m_1m_3+n_1n_3)+(l_2l_3+m_2m_3+n_2n_3)+(l_3^2+m_3^2+n_3^2)}{1.\sqrt{3}}$$

$$= \frac{0+0+1}{\sqrt{3}}$$

$$\cos\theta_3 = \frac{1}{\sqrt{3}}$$

$$\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ ——— } (3)$$

from (1), (2) & (3) it is proved that

$$\theta_1 = \theta_2 = \theta_3 \text{ —— Ans.}$$

Q25 A variable line in two adjacent positions has direction cosines l, m, n & $l+\delta l$, $m+\delta m$ & $n+\delta n$. Show that the measure of the small angle $\delta\theta$ b/w the two positions is given by

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

Sol.
Let OA & OB be the two adjacent positions of the line. Let PQ be the arc of the circle with centre at O & radius 1.

Then the co-ords. of pts. P & Q are $P(l, m, n)$ & $Q(l+\delta l, m+\delta m, n+\delta n)$

Let $\delta\theta$ be the angle b/w two positions of line.

Now $\delta\theta = \widehat{PQ} =$ chord PQ   Since $\delta\theta \to 0$

So $\delta\theta = |PQ|$

$$= \sqrt{(l+\delta l - l)^2 + (m+\delta m - m)^2 + (n+\delta n - n)^2}$$

$$\delta\theta = \sqrt{\delta l^2 + \delta m^2 + \delta n^2}$$

$$\Rightarrow \boxed{\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2}$$