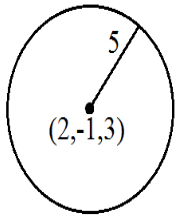
The Sphere

Definition: The set of all points in space that are equidistant from a fixed point is called a **sphere**. The constant distance is called the **radius** of the sphere and the fixed point is called the **centre** of the sphere.

Example#36: Find the centre and radius of the sphere $x^2+y^2+z^2-4x+2y-6z-11=0$.

Solution: Given equation of sphere is



$$x^2+y^2+z^2-4x+2y-6z-11=0$$

$$x^2-4x+y^2+2y+z^2-6z-11=0$$

$$x^2-2(2)(x)+(2)^2+y^2+2(1)(y)+(1)^2+z^2-2(3)(z)+(3)^2-11=(2)^2+(1)^2+(3)^2$$

$$(x-2)^2+(y+1)^2+(z-3)^2=4+1+9+11$$

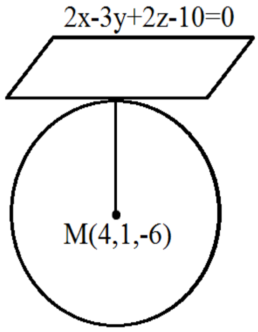
$$(x-2)^2+(y+1)^2+(z-3)^2=25$$

Which represents the equation of a sphere with centre $(2,-1,3)$ and radius $=\sqrt{25}=5$

Example#37: Find an equation of the sphere with centre at $M(4,1,-6)$ and tangent to the plane $2x-3y+2z-10=0$.

Solution: Given equation of the plane $2x-3y+2z-10=0$ and centre of sphere Is at $M(4,1,-6)$.

According to given condition, Given plane is tangent to the sphere, So



radius of the sphere = Distance from plane to the Centre

$$r=\frac{|2(4)-3(1)+2(-6)-10|}{\sqrt{2^2+(-3)^2+2^2}} \Rightarrow r=\frac{|8-3-12-10|}{\sqrt{4+9+4}} \Rightarrow r=\frac{|-17|}{\sqrt{17}} \Rightarrow r=\frac{17}{\sqrt{17}} \Rightarrow r=\sqrt{17}$$

Hence, equation of the sphere with centre $(4,1,-6)$ and radius $\sqrt{17}$ will become

$$(x-4)^2+(y-1)^2+(z+6)^2=\left(\sqrt{17}\right)^2$$

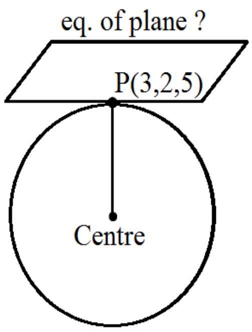
$$x^2+16-8x+y^2+1-2y+z^2+36+12z=17$$

$$x^2+y^2+z^2-8x-2y+12z+36+17=17$$

$$x^2+y^2+z^2-8x-2y+12z+36=0 \quad \text{required eq. of sphere}$$

Example#38: Find an equation of the tangent plane to the sphere $x^2+y^2+z^2-4x+2y-6z=0$ at the point $(3,2,5)$.

Solution: Given equation of the sphere



$x^2+y^2+z^2-4x+2y-6z=0$ at the point $(3,2,5)$

Hence Centre of given sphere is $(-f,-g,-h)=(2,-1,3)$

Given point on the tangent plane is $P(3,2,5)$

Hence $\overrightarrow{CP}$ is the normal vector of required plane

$$\overrightarrow{CP}=\overrightarrow{OP}-\overrightarrow{OC} \Rightarrow \overrightarrow{CP}=(3,2,5)-(2,-1,3) \Rightarrow \overrightarrow{CP}=\hat{i}+3\hat{j}+2\hat{k}$$

$\overrightarrow{CP}=\hat{i}+3\hat{j}+2\hat{k}$ here $a=1, b=3, c=2$ are direction ratios of normal vector of the plane

Now equation of the tangent at the point $(3,2,5)$ with direction ratios a, b & c of the normal vector of the plane.

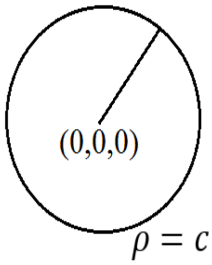
$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \Rightarrow 1(x-3)+3(y-2)+2(z-5)=0$$

$$\Rightarrow x-3+3y-6+2z-10=0 \Rightarrow x+3y+2z-19=0 \quad \text{required eq. of the plane}$$

Exercise #8.11

Question#1: Show that $\rho = c$ is an equation of a sphere of radius c and Centre at (0,0,0).

Solution: Given $\rho = c$



As we know that $\rho = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \sqrt{x^2 + y^2 + z^2} = c$

Now squaring on both sides

$\Rightarrow x^2 + y^2 + z^2 = c^2$

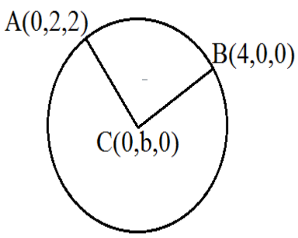
$\Rightarrow (x-0)^2 + (y-0)^2 + (z-0)^2 = c^2$

This equation shows that radius r = c & Centre is (0,0,0)

Question#2: Find an equation of the sphere whose Centre is on the y-axis and which passes through the points (0,2,2) & (4,0,0).

Solutoin:

Consider a sphere whose Centre C (0,b,0) which passes through the points A(0,2,2) & B (4,0,0) as shown in the figure.



from figure , we can see that

$$|CA| = |CB| \qquad \{\because \text{radius} = |CA| = |CB|\}$$

$$\sqrt{(0-0)^2 + (b-2)^2 + (0-2)^2} = \sqrt{(0-4)^2 + (b-0)^2 + (0-0)^2}$$

Squaring on both sides

$\Rightarrow 0 + b^2 - 4b + 4 + 4 = 16 + b^2 + 0 \Rightarrow -4b + 8 = 16 \quad \Rightarrow 4b = -8$ $\quad \boxed{\Rightarrow b = -2}$

Thus the co-ordinates of the Centere of sphere are C(0,-2,0)

Now radius $= |CA| = \sqrt{(0-0)^2 + (2-(-2))^2 + (2-0)^2}$

$= \sqrt{(0-0)^2 + (2+2)^2 + (2-0)^2}$

$= \sqrt{0 + 16 + 4}$

$\Rightarrow \text{radius} = \sqrt{20}$

Now the equation of sphere having Centere $(0,-2,0)$ and radius $\sqrt{20}$

$\Rightarrow (x-0)^2 + (y+2)^2 + (z-0)^2 = (\sqrt{20})^2$

$\Rightarrow x^2 + y^2 + 4 + 4y + z^2 = 20$

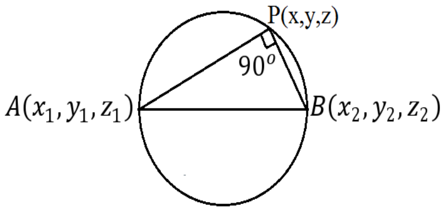
$\Rightarrow x^2 + y^2 + z^2 + 4y - 16 = 0$

This is required equation of the sphere

Question#3: Show that an equation of the sphere having the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

Solution:Consider a sphere having the straight line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as diameter.

Consider a point P on the sphere and meet A and B.



From figure AP is perpendicular to BP

Now direction ratios of line AP are $x - x_1, y - y_1, z - z_1$

& *direction ratios of line BP are* $x - x_2, y - y_2, z - z_2$

Therefore $\overrightarrow{AP} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$ & $\overrightarrow{BP} = (x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k}$

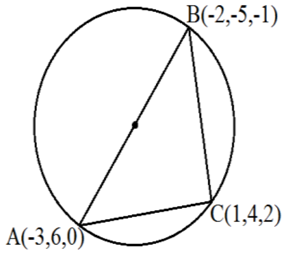
From figure $\overrightarrow{AP} \perp \overrightarrow{BP}$ then $\overrightarrow{AP}.\overrightarrow{BP} = 0$

$\Rightarrow [(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}].[(x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k}] = 0$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$ is required equation of sphere.

Question#4: Find an equation of the sphere which passes through the points $A(-3, 6, 0)$, $B(-2, -5, -1)$ and $C(1, 4, 2)$ and whose centre lies on the hypotenuse of the right triangle ABC.

Solution: Given points are $A(-3,6,0), B(-2,-5,-1) \& C(1,4,2)$



For required proof wehve to find

$|\overrightarrow{AB}| = \sqrt{(-2+3)^2 + (-5-6)^2 + (-1-0)^2} = \sqrt{(1)^2 + (-11)^2 + (-1)^2} \Rightarrow |\overrightarrow{AB}| = \sqrt{123}$

$|\overrightarrow{BC}| = \sqrt{(1+2)^2 + (4+5)^2 + (2+1)^2} = \sqrt{(3)^2 + (9)^2 + (3)^2} \Rightarrow |\overrightarrow{BC}| = \sqrt{99}$

$|\overrightarrow{CA}| = \sqrt{(-3-1)^2 + (6-4)^2 + (0-2)^2} = \sqrt{(-4)^2 + (2)^2 + (-2)^2} \Rightarrow |\overrightarrow{CA}| = \sqrt{24}$

From above equations its clear that $|\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

So by Pethagoras theorem , AB is the hypotenuse of ΔABC.

So $|\overrightarrow{AB}|$ act as a diameter of sphere.The mid point of AB is M called Centre of sphere.

$M = \left(\frac{-2-3}{2}, \frac{-5+6}{2}, \frac{0-1}{2}\right) = \left(\frac{-5}{2}, \frac{1}{2}, \frac{-1}{2}\right)$

Now radius of sphere = $|\overrightarrow{AM}| = \sqrt{\left(\frac{-5}{2}+3\right)^2 + \left(\frac{1}{2}-6\right)^2 + \left(\frac{-1}{2}-0\right)^2} = \sqrt{\frac{1}{4} + \frac{121}{4} + \frac{1}{4}} = \sqrt{\frac{123}{4}}$

Hence equation of sphere with Centre $M = \left(\frac{-5}{2}, \frac{1}{2}, \frac{-1}{2}\right)$ radius=$|\overrightarrow{AM}| = \sqrt{\frac{123}{4}}$.

$$\left[\left(x - \left(\frac{-5}{2}\right)\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \left(\frac{-1}{2}\right)\right)^2\right] = \left(\sqrt{\frac{123}{4}}\right)^2 \Rightarrow \left(x + \frac{5}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z + \frac{1}{2}\right)^2 = \frac{123}{4}$$

$$x^2 + \frac{25}{4} + 5x + y^2 + \frac{1}{4} - y + z^2 + \frac{1}{4} + z = \frac{123}{4}$$

$$x^2 + y^2 + z^2 + 5x - y + z = \frac{123}{4} - \frac{25}{4} - \frac{1}{4} - \frac{1}{4}$$

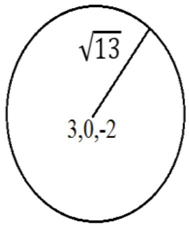
$$x^2 + y^2 + z^2 + 5x - y + z = \frac{94}{4} \qquad \Rightarrow x^2 + y^2 + z^2 + 5x - y + z = 24$$

$\Rightarrow x^2 + y^2 + z^2 + 5x - y + z - 24 = 0$ **required equation of sphere.**

Question#5: Prove that each of the following equation represents a sphere. Find the centre and radius of each:

(i) $\mathbf{x^2+y^2+z^2-6x+4z=0}$

Solution: Given equation of sphere is



$$x^2+y^2+z^2-6x+4z=0$$

$$x^2-6x+y^2+z^2+4z=0$$

$$x^2-2(x)(3)+(3)^2+(y-0)^2+z^2+2(z)(2)+(2)^2=(3)^2+(2)^2$$

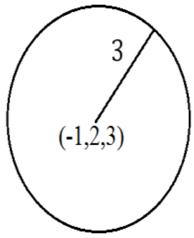
$$(x-3)^2+(y-0)^2+(z+2)^2=13$$

$$(x-3)^2+(y-0)^2+(z-(-2))^2=\left(\sqrt{13}\right)^2$$

Which represents the equation of a sphere with centre $(3,0,-2)$ and radius $\sqrt{13}$.

(ii) $\mathbf{x^2+y^2+z^2+2x-4y-6z+5=0}$

Solution: Given equation of sphere is



$$x^2+y^2+z^2+2x-4y-6z+5=0$$

$$x^2+2x+y^2-4y+z^2-6z+5=0$$

$$x^2+2(x)(1)+(1)^2+y^2-2(y)(2)+(2)^2+z^2-2(z)(3)+(3)^2+5=(1)^2+(2)^2+(3)^2$$

$$(x+1)^2+(y-2)^2+(z-3)^2=1+4+9-5$$

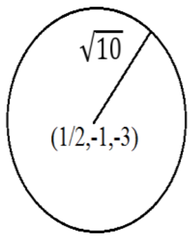
$$(x-(-1))^2+(y-2)^2+(z-3)^2=9$$

$$(x-(-1))^2+(y-2)^2+(z-3)^2=(3)^2$$

Which represents the equation of a sphere with centre $(-1,2,3)$ and radius (3).

(iii) $\mathbf{4x^2+4y^2+4z^2-4x+8y+24z+1=0}$

Solution: Given equation of sphere is



$$4x^2+4y^2+4z^2-4x+8y+24z+1=0$$

Now dividing both sides by 4

$$x^2+y^2+z^2-x+2y+6z+\tfrac{1}{4}=0$$

$$x^2-x+y^2+2y+z^2+6z+\tfrac{1}{4}=0$$

$$x^2-2(x)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+y^2+2(y)(1)+(1)^2+z^2+2(z)(3)+(3)^2+\frac{1}{4}=\left(\frac{1}{2}\right)^2+(1)^2+(3)^2$$

$$\left(x-\frac{1}{2}\right)^2+(y+1)^2+(z+3)^2=\frac{1}{4}+1+9-\frac{1}{4}$$

$$\left(x-\tfrac{1}{2}\right)^2+\left(y-(-1)\right)^2+(z-(-3)^2=10$$

$$\left(x-\tfrac{1}{2}\right)^2+\left(y-(-1)\right)^2+(z-(-3)^2=\left(\sqrt{10}\right)^2$$

Which represents the equation of a sphere with centre $\left(\frac{1}{2},-1,-3\right)$ and radius $\sqrt{10}$.

Question#6: find an equation of the sphere through the points $(0,0,0),(0,1,-1),(-1,2,0)$ and $(1,2,3)$. Also find its centre and radius.

Solution: Let equation of the sphere is $x^2+y^2+z^2+2fx+2gy+2hz+c=0$

It passes through the points

$(0,0,0)$; $0+0+0+0+0+0+c=0 \quad \Rightarrow \mathbf{c=0}$

$(0,1,-1)$; $0+1+1+0+2g+2h(-1)+0=0 \Rightarrow 2g-2h=-2 \quad \Rightarrow g-h=-1 \quad ---(1)$

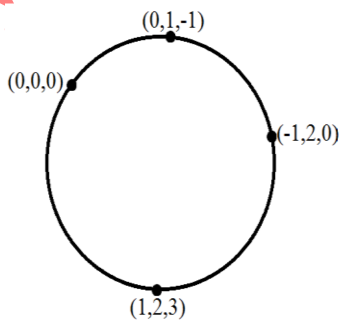
$(-1,2,0)$; $1+4+0-2f+4g+0+0=0 \quad \Rightarrow \quad -2f+4g=-5 \quad ---(2)$

$(1,2,3)$; $1+4+9+2f+4g+6h+0=0 \Rightarrow 2f+4g+6h=-14 \quad \Rightarrow f+2g+3h=-7 \quad ---(3)$

From equation (1) $\Rightarrow \quad g=h-1 \quad ---(4)$

Now putting the value of g in equation (2)

$$-2f+4(h-1)=-5 \quad \Rightarrow -2f+4h-4=-5 \quad \Rightarrow 4h+1=2f \quad \Rightarrow f=\tfrac{1}{2}(4h+1) \quad ---(5)$$



Now using equations (4) and (5) in equation (3)

$$\frac{1}{2}(4h+1)+2(h-1)+3h=-7 \quad \Rightarrow 2h+\frac{1}{2}+2h-2+3h=-7$$

$$7h=-7-\frac{1}{2}+2 \quad \Rightarrow 7h=-7+\frac{3}{2} \quad \Rightarrow 7h=\frac{-11}{2} \quad \Rightarrow \mathbf{h}=\frac{\mathbf{-11}}{\mathbf{14}}$$

Putting the value of **h** in eq (5)

$$f=\tfrac{1}{2}\left(4\left(\tfrac{-11}{14}\right)+1\right) \quad \Rightarrow f=\tfrac{1}{2}\left(\tfrac{-44}{14}+1\right) \quad \Rightarrow f=\tfrac{1}{2}\left(\tfrac{-30}{14}\right) \Rightarrow \mathbf{f}=-\tfrac{\mathbf{15}}{\mathbf{14}}$$

Putting value of **h** in eq . (1)

$$\Rightarrow g--\frac{11}{14}=-1 \quad \Rightarrow g+\frac{11}{14}=-1 \quad \Rightarrow g=-1-\frac{11}{14} \quad \Rightarrow \mathbf{g}=-\frac{\mathbf{25}}{\mathbf{14}}$$

Put values of f , g ,h in eq, (A)

$$x^2+y^2+z^2+2\left(-\frac{15}{14}\right)x+2\left(-\frac{25}{14}\right)y+2\left(-\frac{11}{14}\right)z+0=0$$

$$14x^2+14y^2+14z^2-30x-50y-22z=0$$

$7x^2+7y^2+7z^2-15x-25y-11z=0$ is required equation

Dividing both sides by 7

$$x^2+y^2+z^2-\frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z=0$$

Its centre is $\left(-\frac{15}{7(-2)},-\frac{25}{7(-2)},-\frac{11}{7(-2)}\right)$

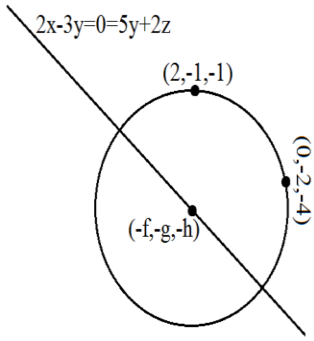
$$\text{centre}=\left(\frac{15}{14},\frac{25}{14},\frac{11}{14}\right)$$

Now radius is $=\sqrt{\left(\frac{15}{14}\right)^2+\left(\frac{25}{14}\right)^2+\left(\frac{11}{14}\right)^2-0}=\sqrt{\frac{225}{196}+\frac{625}{196}+\frac{121}{196}}=\sqrt{\frac{225+625+121}{196}}$

$$\text{radius}=\frac{\sqrt{971}}{14}$$

Question#7: Find an equation of the sphere passing through the points $(0,-2,-4),(2,-1,-1)$ and having its centre on the straight line $\mathbf{2x-3y=0=5y+2z}$.

Solution: Let the general equation of sphere is



$$x^2+y^2+z^2+2fx+2gy+2hz+c=0 \qquad ---(A)$$

Here $(-f,-g,-h)$ is a Centre of equation (A)

The Centre of the sphere passing through this line

$2x-3y=0\,,5y+2z=0$

$2x-3y=0 \quad \Rightarrow 2(-f)-3(-g)=0 \quad -2f+3g=0 \qquad ---(1)$

$5y+2z=0 \quad \Rightarrow 5(-g)+2(-h)=0 \quad \Rightarrow -5g-2h=0 \Rightarrow 5g+2h=0 \qquad ---(2)$

Now equation of the sphere passing through the points $(0,-2,-4),(2,-1,-1)$.

$(0,-2,-4); \qquad 0+4+16+0-4g-8h+c=0 \quad \Rightarrow -4g-8h+c=-20 \qquad ---(3)$

$(2,-1,-1); \qquad 4+1+1+4f-2g-2h+c=0 \quad \Rightarrow 4f-2g-2h+c=-6 \qquad ---(4)$

Now subtracting eq(3) & eq(4) $\qquad -4f-2g-6h=-14$

Dividing by "-2" above $\qquad 2f+g+3h=7 \qquad ----(5)$

Now using eq(1) & eq(2)

from (1) $\quad 3g=2f \qquad \Rightarrow f=\frac{3}{2}g \qquad ---(6)$

from (2) $\quad 2h=-5g \qquad \Rightarrow h=\frac{-5}{2}g \qquad ---(7)$

Using equ(6) & equ(7) in equation (5)

$$2\left(\frac{3}{2}g\right)+g+3\left(\frac{-5}{2}g\right)=7 \quad \Rightarrow 3g+g-\frac{15}{2}g=7 \quad \Rightarrow 4g-\frac{15}{2}=7 \quad \Rightarrow g\left(\frac{8-15}{2}\right)=7$$

$g\left(\frac{-7}{2}\right)=7 \qquad \Rightarrow \mathbf{g=-2}$

eq(6) becomes $\quad f=\frac{3}{2}(-2) \qquad \Rightarrow \mathbf{f=3}$

eq(7) becomes $\quad h=\frac{-5}{2}(-2)=\frac{10}{2} \qquad \Rightarrow \mathbf{h=5}$

Putting these values in eq(3)

$-4(-2)-8(5)+c=-20 \quad \Rightarrow c=-20+40-8 \qquad \Rightarrow \mathbf{c=12}$

Now equation (A) becomes

$$x^2+y^2+z^2+2(-3)x+2(-2)y+2(5)z++12=0$$

$$x^2+y^2+z^2-6x-4y+10z++12=0$$

Question#8: Find an equation of the sphere which passes through the circle $x^2+y^2+z^2=9,\quad 2x+3y+4z=5$, and the point $(1,2,3)$
$[$Hint: $(x^2+y^2+z^2-9)+k(2x+3y+4z-5)=0$ defines a sphere through the circle for each $k\epsilon R$.$]$

Solution: Let equation of the sphere passes through the circle is

$$(x^2+y^2+z^2-9)+k(2x+3y+4z-5)=0 \qquad ---(1)$$

This equation of the sphere passes through the point (1,2,3).

$$(1+4+9-9)+k(2+6+12-5)=0 \quad \Rightarrow 5+15k=0 \quad \Rightarrow 5=-15k \Rightarrow \mathbf{k=-\frac{1}{3}}$$

Using the value of k in equation (1)

$$(x^2+y^2+z^2-9)-\frac{1}{3}(2x+3y+4z-5)=0$$

$$3x^2+3y^2+3z^2-27-2x-3y-4z+5=0$$

$$3x^2+3y^2+3z^2-2x-3y-4z-22=0$$

Question#9: Find an equation of the sphere through the circle $x^2+y^2+z^2=1, 2x+4y+5z-6=0$ and touching the plane $z=0$.

Solution: Let equation of the sphere passes through the circle is

$$(x^2+y^2+z^2-1)+k(2x+4y+5z-6)=0 \qquad ---(1)$$

$$x^2+y^2+z^2-1+2kx+4ky+5kz-6k=0$$

$$x^2+y^2+z^2+2kx+4ky+5kz+\{-(1+6k)\}=0$$

Here the Centre of this equation is $\left(-k,-2k,-\frac{5}{2}k\right)$ & center of the g. eq is $\left(-f,-g,-h\right)$

And radius is $r=\sqrt{f^2+g^2+h^2-c}$

$$r=\sqrt{(-k)^2+(-2k)^2+\left(-\frac{5}{2}k\right)^2-(1+6k)}=\sqrt{k^2+4k^2+\frac{25k^2}{4}+1+6k}=\sqrt{\frac{4k^2+16k^2+25k^2+4+24k}{4}}$$

$$r=\sqrt{\frac{45k^2+24k+4}{4}}$$

As the sphere touches the plane $z=0$, So Radius of the sphere = Distance between plane & Centre

$$\sqrt{\frac{45k^2+24k+4}{4}}=\left|\frac{0+0-\frac{5k}{2}}{\sqrt{0^2+0^2+1^2}}\right| \quad \Rightarrow \sqrt{\frac{45k^2+24k+4}{4}}=\left|\frac{-\frac{5k}{2}}{1}\right| \quad \Rightarrow \sqrt{\frac{45k^2+24k+4}{4}}=\frac{5k}{2}$$

Now squaring on both sides

$$\frac{45k^2+24k+4}{4}=\frac{25k^2}{4} \Rightarrow 45k^2+24k+4=25k^2 \Rightarrow 45k^2-25k^2+24k+4=0 \Rightarrow 20k^2+24k+4=0$$

dividing by 4 $\Rightarrow 5k^2+6k+1=0 \quad \Rightarrow 5k^2+6k+1=0 \quad \Rightarrow 5k^2+5k+k+1=0$

$$5k(k+1)+(k+1)=0 \quad \Rightarrow (5k+1)(k+1)=0 \quad \Rightarrow 5k+1=0, k+1=0 \Rightarrow \mathbf{k=-1,-\frac{1}{5}}$$

Put $k=-1,-\frac{1}{5}$ in Equation (1)

$$x^2+y^2+z^2-1-1(2x+4y+5z-6)=0$$

$$x^2+y^2+z^2-1-2x-4y-5z+6=0$$

$$x^2+y^2+z^2-2x-4y-5z+5=0$$

$$x^2+y^2+z^2-1-\frac{1}{5}(2x+4y+5z-6)=0$$

$$5x^2+5y^2+5z^2-5-2x-4y-5z+6=0$$

$$5x^2+5y^2+5z^2-2x-4y-5z+1=0$$

Question#10: Show that the two circles $x^2 + y^2 + z^2 = 9, x - 2y + 4z - 13 = 0$ and $x^2 + y^2 + z^2 + 6y - 6z + 21 = 0, x + y + z + 2 = 0$ lie on the same sphere. Also find its equation.

Solution: Let equation of the sphere passes through the first circle is

$$(x^2 + y^2 + z^2 - 9) + k(x - 2y + 4z - 13) = 0 \qquad ---(A)$$

$$x^2 + y^2 + z^2 + kx - 2ky + 4kz - (9 + 13k) = 0 \qquad ---(1)$$

$$(x^2 + y^2 + z^2 + 6y - 6z + 21) + h(x + y + z + 2) = 0 \qquad ---(B)$$

$$x^2 + y^2 + z^2 + hx + (h + 6)y + (h - 6)z + (2h + 21) = 0 \qquad --(2)$$

The given circles lie on the same sphere then eq.(1) & eq.(2) must be identical.

Comparing co-efficients of x, y, z & constants in (1) & (2)

$$k = h \qquad ---(i)$$

$$-2k = h + 6 \qquad ---(ii)$$

$$4k = h - 6 \qquad ---(iii)$$

$$-(9 + 13k) = 2h + 21 \qquad ---(iv)$$

From (i) $k = h$ put in (ii)

$-2h = h + 6 \quad \Rightarrow -3h = 6 \quad \Rightarrow \mathbf{h = -2}$ & also $\mathbf{k = -2}$

Putting these values in (iii) & (iv)

We see that eqs. (iii) & (iv) are satisfied.

Values of k & h shows that the both circles lie on the same sphere and equation of the sphere is obtained by putting $k = -2$ in (A) or in (B).

$$(x^2 + y^2 + z^2 - 9) - 2(x - 2y + 4z - 13) = 0$$

$$x^2 + y^2 + z^2 - 9 - 2x + 4y - 8z + 26 = 0$$

$$x^2 + y^2 + z^2 - 2x + 4y - 8z + 17 = 0$$

Question#11: Find an equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y - 4z - 8 = 0$ is a great circle.
[Hint : A great circle of sphere is such that its plane passes through the centre of the sphere]

Solution: Let equation of the sphere passes through a given circle is

$$(x^2 + y^2 + z^2 + 7y - 2z + 2) + k(2x + 3y + 4z - 8) = 0 \qquad ---(1)$$

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + 2kx + 3ky + 4kz - 8k = 0$$

$$x^2 + y^2 + z^2 + 2kx + (3k + 7)y + (4k - 2)z + 2 - 8k = 0$$

Now Centre of this sphere is $(-f, -g, -h) = \left(-k, -\frac{3k+7}{2}, 1 - 2k\right)$

According to given condition that the given circle is a great circle.

Then we know that plane of the circle passes through Centre of the sphere.

Then $\quad 2(-k) + 3\left(-\frac{3k+7}{2}\right) + 4(1 - 2k) - 8 = 0$

$$\Rightarrow -2k - \frac{3}{2}(3k + 7) + 4(1 - 2k) - 8 = 0$$

$\Rightarrow -4k - 3(3k + 7) + 8 - 16k - 16 = 0$

$\Rightarrow -4k - 9k - 21 + 8 - 16k - 16 = 0$

$\Rightarrow -29k - 29 = 0 \quad \Rightarrow k + 1 = 0 \quad \Rightarrow \mathbf{k = -1}$

Using value of k in eq.(1)

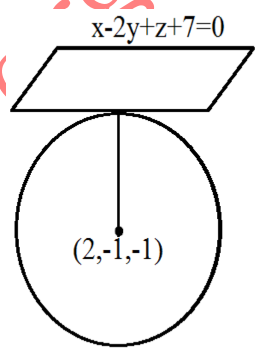
$(x^2 + y^2 + z^2 + 7y - 2z + 2) - 1(2x + 3y + 4z - 8) = 0$

$x^2 + y^2 + z^2 + 7y - 2z + 2 - 2x - 3y - 4z + 8 = 0$

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$ required equation pf sphere

Question#12: Find an equation of the sphere with centre $(2, -1, -1)$ and tangent to the plane $x - 2y + z + 7 = 0$.

Solution:



Given equation of the plane $x - 2y + z + 7 = 0$ and centre of sphere is $(2, -1, -1)$.

According to given condition , Given plane is tangent to the sphere

So radius of the sphere = Distance from plane to the Centre

$$r = \frac{|2-2(-1)+(-1)+7|}{\sqrt{1^2+(-2)^2+1^2}}$$

$$r = \frac{|2+2-1+7|}{\sqrt{1+4+1}}$$

$$r = \frac{|10|}{\sqrt{6}}$$

$$r = \frac{10}{\sqrt{6}}$$

Hence equation of the sphere with centre $(2, -1, -1)$ and radius $\frac{10}{\sqrt{6}}$ will become

$$(x - 2)^2 + (y + 1)^2 + (z + 1)^2 = \left(\frac{10}{\sqrt{6}}\right)^2$$

$$x^2 + 4 - 4x + y^2 + 1 + 2y + z^2 + 1 + 2z = \frac{100}{6}$$

$$x^2 + y^2 + z^2 - 4x + 2y + 2z + 6 = \frac{50}{3}$$

$$3(x^2 + y^2 + z^2 - 4x + 2y + 2z + 6) = 50$$

$$3x^2 + 3y^2 + 3z^2 - 12x + 6y + 6z + 18 = 50$$

$3x^2 + 3y^2 + 3z^2 - 12x + 6y + 6z - 32 = 0$ is required eq. of sphere

Question#13: Find an equation of the plane tangent to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$.

Solution: Given equation of the sphere

$3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$

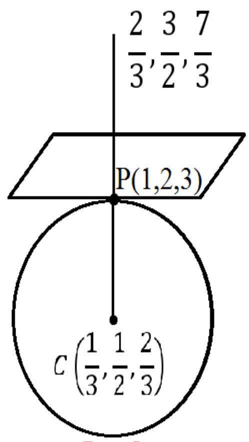
Dividing by 3 on both sides

$$(x^2+y^2+z^2)-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0$$

Hence Centre of given sphere is $(-f,-g,-h)=\left(\frac{1}{3},\frac{1}{2},\frac{2}{3}\right)$

Given points on the tangent plane are $P(1,2,3)$

Hence $\overrightarrow{CP}$ is the normal vector of required plane



$\overrightarrow{CP}=\overrightarrow{OP}-\overrightarrow{OC} \quad \Rightarrow \overrightarrow{CP}=(1,2,3)-\left(\frac{1}{3},\frac{1}{2},\frac{2}{3}\right)$

$\overrightarrow{CP}=\left(\frac{2}{3}\hat{\imath}+\frac{3}{2}\hat{\jmath}+\frac{7}{3}\hat{k}\right)$ Here $a=\frac{2}{3}, b=\frac{3}{2}, c=\frac{7}{3}$ are direction ratios of normal vector of the plane

Now equation of the tangent at the point $(1,2,3)$ with direction ratios a, b& c of the normal vector of the plane.

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

$$\frac{2}{3}(x-1)+\frac{3}{2}(y-2)+\frac{7}{3}(z-3)=0$$

Multiplying both sides by 6

$$4(x-1)+9(y-2)+14(z-3)=0$$

$$4x-4+9y-18+14z-42=0$$

$$4x+9y+14z-64=0$$ required eq. of the plane

Question#14: Find an equation of the sphere with centre at the point $(-2,4,-6)$ and tangent to the
(a) $xy-$plane (b) $yz-$plane (c) $zx-$plane

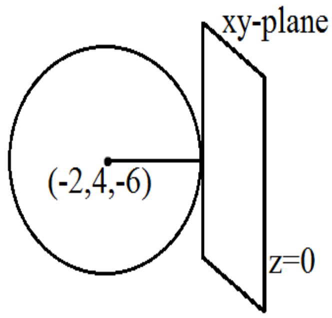
Solution (a):

Equation of $xy-$plane is $z=0$

As $xy-$plane is tangent to the required sphere with centre at the point $P(-2,4,-6)$

So radius of the sphere = Distance between centre of sphere and plane

$$\text{radius}=\frac{|0x+0y+(-6)z+0|}{\sqrt{0+0+1}}$$



$\text{radius}=\frac{|-6|}{\sqrt{1}} \quad \Rightarrow \text{radius}=6$

Now equation of the sphere with centre $(-2,4,-6)$ and radius 6 will become

$$(x+2)^2+(y-4)^2+(z+6)^2=(6)^2$$

$$x^2+4+4x+y^2+16-8y+z^2+36+12z=36$$

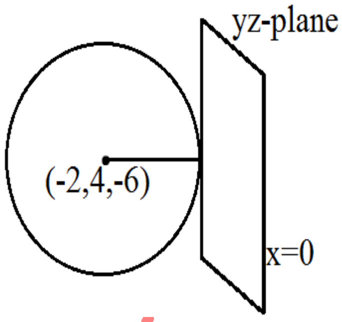
$$x^2+y^2+z^2+4x-8y+12z+20=0$$

Solution(b): Equation of yz − plane is $x = 0$

As yz − plane is tangent to the required sphere with centre at the point $P(-2,4,-6)$

So radius of the sphere = Distance between centre of sphere and plane

$$\text{radius} = \frac{|(-2)x+0y+0z+0|}{\sqrt{1+0+0}} \Rightarrow \quad \text{radius} = \frac{|-2|}{1} \quad \Rightarrow \text{radius} = 2$$



Now equation of the sphere with centre $(-2,4,-6)$ and radius 2 will become

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = (2)^2$$

$$x^2 + 4 + 4x + y^2 + 16 - 8y + z^2 + 36 + 12z = 4$$

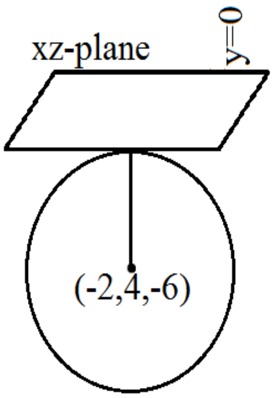
$$x^2 + y^2 + z^2 + 4x - 8y + 12z + 52 = 0$$

Solution(c): Equation of zx − plane is $y = 0$

As zx − plane is tangent to the required sphere with centre at the point $P(-2,4,-6)$

So radius of the sphere = Distance between centre of sphere and plane

$$\text{radius} = \frac{|0x+(4)y+0z+0|}{\sqrt{0+1+0}} \Rightarrow \text{radius} = \frac{|4|}{1} \quad \Rightarrow \text{radius} = 4$$



Now equation of the sphere with centre $(-2,4,-6)$ and radius 4 will become

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = (4)^2$$

$$x^2 + 4 + 4x + y^2 + 16 - 8y + z^2 + 36 + 12z = 16$$

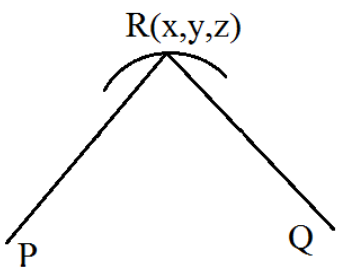
$$x^2 + y^2 + z^2 + 4x - 8y + 12z + 40 = 0$$

Question#15: Find an equation of the surface whose points are equidistant from $P(7,8,2)$ and $Q(5,2,-6)$.

Solution: Given points are $P(7,8,2)$ and $Q(5,2,-6)$

Let $R(x,y,z)$ be a point on the required surface then by given condition

$$|\overrightarrow{RP}| = |\overrightarrow{PQ}|$$



$$\sqrt{(7-x)^2 + (8-y)^2 + (2-z)^2} = \sqrt{(5-x)^2 + (2-y)^2 + (-6-z)^2}$$

Taking square on both sides

$$(7-x)^2 + (8-y)^2 + (2-z)^2 = (5-x)^2 + (2-y)^2 + (-6-z)^2$$

$$(7-x)^2 + (8-y)^2 + (2-z)^2 = (5-x)^2 + (2-y)^2 + \left(-(6+z)\right)^2$$

$$(7-x)^2 + (8-y)^2 + (2-z)^2 = (5-x)^2 + (2-y)^2 + (6+z)^2$$

$$49 - 14x + x^2 + 64 - 16y + y^2 + 4 - 4z + z^2 = 25 + x^2 - 10x + 4 - 4y + y^2 + 36 + z^2 + 12z$$

$$117 - 14x - 16y - 4z - 65 + 10x + 4y - 12z = 0$$

$$\Rightarrow -4x - 12y - 16z + 52 = 0$$

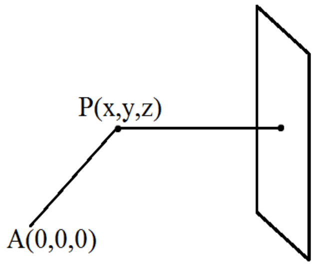
$$\Rightarrow -4(x + 3y + 4z + 13) = 0$$

$$\Rightarrow x + 3y + 4z + 13 = 0$$

Question#16: A point P moves such that the square of its distance from the origin is proportional to its distance from a fixed plane. Show that P always lie on a sphere.

Solution: Let equation of the fixed plane is $lx+my+nz=p \quad ---(1)$

Where l, m & n are the direction cosines of the normal vector of the plane (1)



Let $P(x,y,z)$ be any point on the locus then according to given condition

$$\left|\overrightarrow{OP}\right|^2 \propto \left|\overrightarrow{PR}\right|$$

$$\left|\overrightarrow{OP}\right|^2 = k\left|\overrightarrow{PR}\right| \quad \text{where k is constant.}$$

$$(x-0)^2+(y-0)^2+(z-0)^2 = k\left|\frac{lx+my+nz-p}{\sqrt{l^2+m^2+n^2}}\right|$$

$$x^2+y^2+z^2 = k|lx+my+nz-p| \qquad \because l^2+m^2+n^2=1$$

$$x^2+y^2+z^2-k(lx+my+nz-p)=0$$

$$x^2+y^2+z^2-klx-kmy-knz+kp=0$$

Which is clearly the equation of a sphere .

Hence proved that the point P always lies on a sphere.

Question#17: A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

Solution: Let equation of the sphere passes through the origin is

$$x^2+y^2+z^2+2fx+2gy+2hz=0$$

Centre of sphere is $(-f,-g,-h)$. & radius is $r=\sqrt{f^2+g^2+h^2}$

Given radius $r=k$ *then above expression will becomes*

$$k=\sqrt{f^2+g^2+h^2} \quad \Rightarrow \quad k^2=f^2+g^2+h^2 \quad ---(1)$$

As the sphere cuts axes in point's A, B, C.So the co-ordinates of the points A, B& C are

$A(-2f,0,0)$, $B(0,-2g,0)$ & $C(0,0,-2h)$.

As we know that centroid of ΔABC is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z+z_3}{3}\right)$.

Now Centroid of ΔABC is $\left(\frac{-2f}{3}, \frac{-2g}{3}, \frac{-2h}{3}\right)$.

Here $x=\frac{-2f}{3}, y=\frac{-2g}{3}, z=\frac{-2h}{3}$

Putting in given equation $9(x^2+y^2+z^2)=4k^2$

$$9\left(\frac{4f^2}{9}+\frac{4g^2}{9}+\frac{4h^2}{9}\right)=4k^2$$

$$\frac{9}{9}(4f^2+4g^2+4h^2)=4k^2$$

$$4(f^2+g^2+h^2)=4k^2$$

$$\Rightarrow 4k^2=4k^2 \qquad \textit{from eq.}(1)$$

As this equation is satisfied

So the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.

Question#18: The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B, C. Find an equation of the circumcircle of the triangle ABC. Also find the co-ordinates of the centre of the circle.

Solution: The required circle is the intersection of the given plane by a sphere through origin O and points A, B & C Now given equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad ---(1)$$

As this plane cuts co-ordinate axes in points A.B & C.

So co-ordinates of these points are $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ & $O(0,0,0)$

Now equation of the sphere OABC is

$$x^2+y^2+z^2-ax-by-cz=0 \quad ---(2)$$

The eq.(1) and eq.(2) are the equations of required circumcircle

Now centre of the circle is the foot of perpendicular from the centre of sphere (1) to plane (2).

Here centre of sphere (2) is $G\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

Now direction ratios of line perpendicular to plane (2) are $\frac{1}{a},\frac{1}{b},\frac{1}{c}$

So equation of line PG is

$$\frac{x-\frac{a}{2}}{1/a}=\frac{y-\frac{b}{2}}{1/b}=\frac{z-\frac{c}{2}}{1/c}=t$$

$$x=\frac{a}{2}+\frac{t}{a}$$

$$y=\frac{b}{2}+\frac{t}{b}$$

$$z=\frac{c}{2}+\frac{t}{c}$$

So co-ordinates of point P are $P\left(\frac{a}{2}+\frac{t}{a},\frac{b}{2}+\frac{t}{b},\frac{c}{2}+\frac{t}{c}\right)$

As the point P lies on the plane (2)

So $$\frac{1}{a}\left(\frac{a}{2}+\frac{t}{a}\right)+\frac{1}{b}\left(\frac{b}{2}+\frac{t}{b}\right)+\frac{1}{c}\left(\frac{c}{2}+\frac{t}{c}\right)=1$$

$$\frac{1}{2}+\frac{t}{a^2}+\frac{1}{2}+\frac{t}{b^2}+\frac{1}{2}+\frac{t}{c^2}=1$$

$$\frac{3}{2}+t\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1$$

$$t\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=-\frac{1}{2}$$

$$t=\frac{-1}{2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}$$

By putting this value of t in eq.(A) we get co-ordinates of centre of circle.

Question#19: A plane passes through a fixed point (a, b, c) and cuts the axes of coordinates in A, B , C.Find the locus of the centre of the sphere $OABC$ for different positons of the planes, O being the origin.

Solution: Let equation of the fixed plane is $lx + my + nz = p \quad ---(1)$

As this plane passes through a point (a, b, c) , so $la + mb + nc = p \quad ---(2)$

Equation(1) cuts at A, B & C Then $A\left(-\frac{p}{l}, 0,0\right), B\left(0, -\frac{p}{m}, 0\right)$ & $C\left(0,0, -\frac{p}{n}\right)$

Thus equation of sphere OABC will be $x^2 + y^2 + z^2 - \frac{p}{l}x - \frac{p}{m}y - \frac{p}{n}z = 0$

Hence centre of sphere will be $(-f, -g, -h) = \left(\frac{p}{2l}, \frac{p}{2m}, \frac{p}{2n}\right)$

Let $x_1 = \frac{p}{2l}$, $y_1 = \frac{p}{2m}$, $z_1 = \frac{p}{2n}$

$\Rightarrow l = \dfrac{p}{2x_1}$, $m = \dfrac{p}{2y_1}$, $n = \dfrac{p}{2z_1}$

Using these values in $eq.(2)$

$\Rightarrow \frac{p}{2x_1}a + \frac{p}{2y_1}b + \frac{p}{2z_1}c = p \Rightarrow \frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1 \Rightarrow \frac{1}{2}\left(\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1}\right) = 1 \Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$ This is the required equation.

Question#20: Find an equation of the sphere circumscribing the tetrahedron whose faces are $x = 0, y = 0, z = 0$ and $lx + my + nz + p = 0$.

Solution: Let equation of the sphere is

$x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + c = 0 \quad ---(1)$

Given equation of the plane is $lx + my + nz + p = 0$

Let equation of the sphere passes through the vertices of tetrahedron OABC.

$O(0,0,0)$; $c = 0$

$A\left(\frac{-p}{l}, 0,0\right)$: $\frac{p^2}{l^2} - \frac{2fp}{l} = 0 \Rightarrow \frac{p}{l}\left(\frac{p}{l} - 2f\right) \Rightarrow \frac{p}{l} = 2f \Rightarrow f = \frac{p}{2l}$

$B\left(0, \frac{-p}{m}, 0\right)$: $\frac{p^2}{m^2} - \frac{2gp}{m} = 0 \Rightarrow \frac{p}{m}\left(\frac{p}{m} - 2g\right) \Rightarrow \frac{p}{m} = 2g \Rightarrow g = \frac{p}{2m}$

$C\left(0,0, \frac{-p}{n}\right)$: $\frac{p^2}{n^2} - \frac{2hp}{n} = 0 \Rightarrow \frac{p}{n}\left(\frac{p}{n} - 2h\right) \Rightarrow \frac{p}{n} = 2h \Rightarrow h = \frac{p}{2n}$

Using In equation (1), we get $x^2 + y^2 + z^2 + \frac{p}{l}x + \frac{p}{m}y + \frac{p}{n}z = 0$

Checked by: Sir Hameed ullah *(hameedmath2017 @ gmail.com)*

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