CHAPTER # 08

ANALYTIC GEOMETRY OF THREE DIMENSONS

Exercise #8.1

Show that the three given points are the vertices of a right triangle, or the vertices of an isosceles triangle, or both.

Q#4: $A(1,5,0)$, $B(6,6,4)$, $C(0,9,5)$

Solution:

Consider a ΔABC with vertices $A(1,5,0), B(6,6,4), C(0,9,5)$

For required proof, we have to find the following magnitudes

$$|AB| = \sqrt{(6-1)^2 + (6-5)^2 + (4-0)^2} = \sqrt{5^2+1^2+4^2} = \sqrt{25+1+6} \implies |AB| = \sqrt{42}$$

$$|BC| = \sqrt{(0-6)^2 + (9-6)^2 + (5-4)^2} = \sqrt{6^2+3^2+1^2} = \sqrt{36+9+1} \implies |BC| = \sqrt{46}$$

$$|AC| = \sqrt{(1-0)^2 + (5-9)^2 + (0-5)^2} = \sqrt{1^2+4^2+5^2} = \sqrt{1+16+25} \implies |AC| = \sqrt{42}$$

Since $|AB| = |AC|$ Hence given triangle is isosceles.

Q#5: $A(4,9,4)$, $B(0,11,2)$, $C(1,0,1)$

Solution:

Consider a ΔABC with vertices $A(4,9,4), B(0,11,2), C(1,0,1)$

For required proof, we have to find the following magnitudes

$$|AB| = \sqrt{(0-4)^2 + (11-9)^2 + (2-4)^2} = \sqrt{4^2+2^2+2^2} = \sqrt{16+4+4} \implies |AB| = \sqrt{24}$$

$$|BC| = \sqrt{(1-0)^2 + (0-11)^2 + (1-2)^2} = \sqrt{1^2+11^2+1^2} = \sqrt{1+121+1} \implies |BC| = \sqrt{123}$$

$$|AC| = \sqrt{(1-4)^2 + (0-9)^2 + (1-4)^2} = \sqrt{1^2+11^2+1^2} = \sqrt{9+81+9} \implies |AC| = \sqrt{99}$$

Since $|AB|^2 + |AC|^2 = 24 + 99 = 123 = \sqrt{(123)^2} = |BC|^2$

$$\implies |AB|^2 + |AC|^2 = |BC|^2$$

Hence ΔABC is a right triangle with right angle at vertex A.

Q#6: $A(1,0,2)$, $B(4,3,2)$, $C(0,7,6)$

Do yourself as above

Q#7: $A(2,3,4)$, $B(8,-1,2)$, $C(4,1,0)$

Do yourself as above

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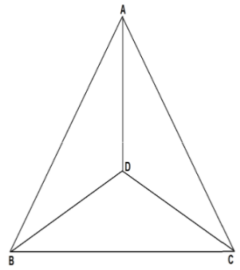
Q#8: Show that the points $(1,6,1),(1,3,4),(4,3,1)$ and $(0,2,0)$ are the vertices of regular tetrahedron.

Solution: Let given points are $A=(1,6,1), B=(1,3,4), C=(4,3,1)$ & $D=(0,2,0)$



to show that given points are the vertices of regular tetrahedron, for this we have to show that

$|AB|=|AC|=|AD|=|BC|=|CD|=|BD|$

Now

$|AB|=\sqrt{(1-1)^2+(3-6)^2+(4-1)^2}=\sqrt{0^2+3^2+3^2}=\sqrt{0+9+9}=\sqrt{18} \Rightarrow |AB|=3\sqrt{2}$

$|AC|=\sqrt{(4-1)^2+(3-6)^2+(1-1)^2}=\sqrt{3^2+3^2+0^2}=\sqrt{9+9+0}=\sqrt{18} \Rightarrow |AC|=3\sqrt{2}$

$|AD|=\sqrt{(0-1)^2+(2-6)^2+(0-1)^2}=\sqrt{1^2+4^2+1^2}=\sqrt{1+16+1}=\sqrt{18} \Rightarrow |AD|=3\sqrt{2}$

$|BC|=\sqrt{(4-1)^2+(3-3)^2+(1-4)^2}=\sqrt{3^2+0^2+3^2}=\sqrt{9+0+9}=\sqrt{18} \Rightarrow |BC|=3\sqrt{2}$

$|CD|=\sqrt{(0-4)^2+(2-3)^2+(0-1)^2}=\sqrt{4^2+1^2+1^2}=\sqrt{16+1+1}=\sqrt{18} \Rightarrow |CD|=3\sqrt{2}$

$|BD|=\sqrt{(0-1)^2+(2-3)^2+(0-4)^2}=\sqrt{1^2+1^2+4^2}=\sqrt{1+1+16}=\sqrt{18} \Rightarrow |BD|=3\sqrt{2}$

Since $|AB|=|AC|=|AD|=|BC|=|CD|=|BD|$

Hence proved the given points are the vertices of a regular tetrahedron.

Q#9: Show that the points $(3,-1,3),(1,-1,2),(2,1,0)$ and $(4,1,1)$ are the vertices of rectangle.

Solution: Suppose given points are $A=(3,-1,3), B=(1,-1,2), C=(2,1,0), D=(4,1,1)$

For a rectangle, we have to show that $|AB|=|CD|$ & $|BC|=|AD|$ & $\angle A=90^o$

Now

$|AB|=\sqrt{(1-3)^2+(-1+1)^2+(2-3)^2}$
$=\sqrt{4+0+1}$
$\Rightarrow |AB|=\sqrt{5}$

$|CD|=\sqrt{(4-2)^2+(1-1)^2+(1-0)^2}$
$=\sqrt{4+0+1}$
$\Rightarrow |CD|=\sqrt{5}$

Now

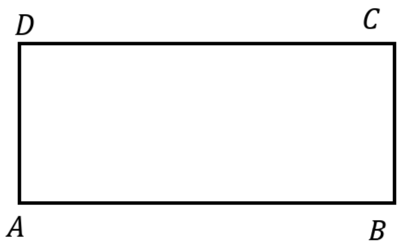
$|BC|=\sqrt{(2-1)^2+(1+1)^2+(0-2)^2}$
$=\sqrt{1+4+4}=\sqrt{9}$
$\Rightarrow |BC|=3$

& $|AD|=\sqrt{(4-3)^2+(1+1)^2+(1-3)^2}$
$=\sqrt{1+4+4}=\sqrt{9}$
$\Rightarrow |AD|=3$

Hence $|AB|=|CD|$ & $|BC|=|AD|$

Now we have to prove $\angle A=90^o$

Consider $|AB|^2+|AD|^2=5+9=14$ --------------- (1)



Since $|BD|=\sqrt{(4-1)^2+(1+1)^2+(1-2)^2}$

$|BD|=\sqrt{9+4+1}=\sqrt{14}$

$|BD|^2=14$

Putting in equation (1)

$|AB|^2+|AD|^2=|BD|^2$

So $\angle A=90^o$ Hence given points are the vertices of rectangle.

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Q#10: Under what conditions on x, y and z is the point $P(x, y, z)$ equidistant from the points $(3, -1, 4)$ and $(-1, 5, 0)$?

Solution:

Suppose the given points are $A(3,-1,4)$ and $B(-1,5,0)$ & Let $P(x, y, z)$ be any point which is equidistance from A and B.

According to given condition $|PA| = |PB|$

$$\Rightarrow \quad \sqrt{(x-3)^2 + (y+1)^2 + (z-4)^2} = \sqrt{(x+1)^2 + (y-5)^2 + (z-0)^2}$$

Square on both sides

$$\Rightarrow \quad (x-3)^2 + (y+1)^2 + (z-4)^2 = (x+1)^2 + (y-5)^2 + (z-0)^2$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 + z^2 - 8z + 16 = x^2 + 2x + 1 + y^2 - 10y + 25 + z^2$$

$$-6x + 2y - 8z + 26 = 2x - 10y + 26$$

$$-8x + 12y - 8z = 0$$

$$-4(2x - 3y + 2z) = 0$$

$$\Rightarrow \quad 2x - 3y + 2z = 0$$ is the required condition

Q#11: Find the coordinates of the point dividing the join of $A(-3, 1, 4)$ and $B(5, -1, 6)$ in the ratio $3: 5$.

Solution: Given points are $A(-3,1,4)$ & $B(5,-1,6)$.

Let $P(x, y, z)$ be the required point dividing the line AB in ratio 3: 5

As we know that the $P(x, y, z)$ divide the join of $A(x_1, y_1, z_1)$ & $B\ (x_2, y_2, z_2)$ in the ratio $m_1: m_2$ is

$$\left(\frac{m_2x_1 + m_1x_2}{m_1 + m_2}, \frac{m_2y_1 + m_1y_2}{m_1 + m_2}, \frac{m_2z_1 + m_1z_2}{m_1 + m_2}\right)$$

Hence coordinates of point P are

$$P\left(\frac{5(-3) + 3(5)}{3+5}, \frac{5(1) + 3(-1)}{3+5}, \frac{5(4) + 3(6)}{3+5}\right)$$

$$P\left(\frac{-15+15}{8}, \frac{5-3}{8}, \frac{20+18}{8}\right) \Rightarrow P = \left(0,1,\frac{19}{4}\right)$$ *is required point.*

Q#12: Find the ratio in which the yz-plane divides the segment joining the points $A(-2, 4, 7)$ and $B(3, -5, 8)$.

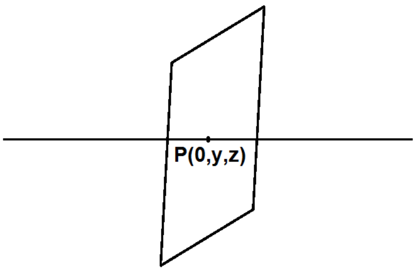
Solution: Given points are $A(-2,4,7)$ & $B(3,-5,8)$

Let the yz -plane divides the join of the given points in the ratio $m_1: m_2$

Now the x-coordinate of the point P dividing the join of given points in the ratio $m_1: m_2$ is

$$x = \frac{3m_1 + (-2)m_2}{m_1 + m_2} = \frac{3m_1 - 2m_2}{m_1 + m_2}$$

Since this point lies on yz -plane so $x = 0$



$$\Rightarrow \quad \frac{3m_1 - 2m_2}{m_1 + m_2} = 0 \Rightarrow 3m_1 - 2m_2 = 0$$

$$\Rightarrow \quad 3m_1 = 2m_2$$

$$\Rightarrow \quad \frac{m_1}{m_2} = \frac{2}{3}$$

$$\Rightarrow \quad m_1: m_2 = 2: 3$$ is required ratio

Q#13: Show that the centroid of the triangle whose vertices are $(x_i, y_i, z_i), i = 1, 2, 3$; is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

Solution: Let the given vertices of the triangle are $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ & $C(x_3, y_3, z_3)$.

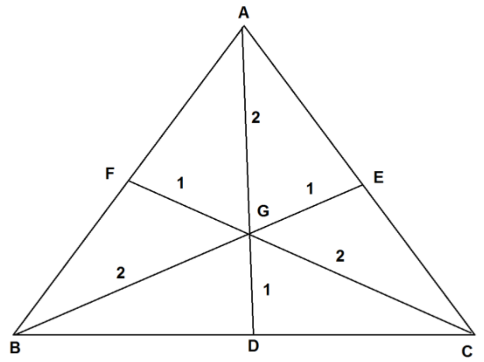
Suppose D, E, F are the mid points of the sides BC, AC & AB respectively.

Now coordinates of D are $D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$

Suppose G is the centroid of ΔABC, Then coordinates of point G dividing AD in the ratio $2:1$ are

$$G\left(\frac{1.x_1 + 2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1.y_1 + 2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1.z_1 + 2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$



Now the coordinates of the points E and F are

$$E\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right) \ \& \ F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

Then the coordinates of the centroid G dividing BE in the ratio $2:1$ are

$$G\left(\frac{1.x_2 + 2\left(\frac{x_1+x_3}{2}\right)}{1+2}, \frac{1.y_2 + 2\left(\frac{y_1+y_3}{2}\right)}{1+2}, \frac{1.z_2 + 2\left(\frac{z_1+z_3}{2}\right)}{1+2}\right)$$

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Similarly the coordinates of cancroids G dividing CF in the ratio $2:1$ are

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

Hence coordinates of centroid G are

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

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Q#14: Find the centroid of the tetrahedron whose vertices are $(x_i, y_i, z_i), i = 1, 2, 3, 4$.

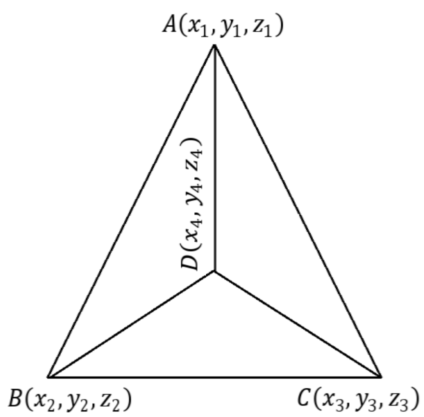
Solution: Let the vertices of the tetrahedron are

$A = (x_1, y_1, z_1)$

$B = (x_2, y_2, z_2)$

$C = (x_3, y_3, z_3)$

$D = (x_4, y_4, z_4)$



Let E, F, G, H are the centroids of

the triangle BCD, ACD, ABD & ABC respectively, Then their coordinates are

$$E=\left(\frac{x_2+x_3+x_4}{3},\frac{y_2+y_3+y_4}{3},\frac{z_2+z_3+z_4}{3}\right)$$

$$F=\left(\frac{x_1+x_3+x_4}{3},\frac{y_1+y_3+y_4}{3},\frac{z_1+z_3+z_4}{3}\right)$$

$$G=\left(\frac{x_1+x_2+x_4}{3},\frac{y_1+y_2+y_4}{3},\frac{z_1+z_2+z_4}{3}\right)$$

$$H=\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3},\frac{z_1+z_2+z_3}{3}\right)$$

Now coordinates of centroid dividing the line AE in ratio $3:1$ are

$$\left(\frac{1.x_1+3\left(\frac{x_2+x_3+x_4}{3}\right)}{1+3},\frac{1.y_1+3\left(\frac{y_2+y_3+y_4}{3}\right)}{1+3},\frac{1.z_1+3\left(\frac{z_2+z_3+z_4}{3}\right)}{1+3}\right)$$

$$\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

Now coordinates of centroid dividing the line BF in the ratio $3:1$ are

$$\left(\frac{1.x_2+3\left(\frac{x_1+x_3+x_4}{3}\right)}{1+3},\frac{1.y_2+3\left(\frac{y_1+y_3+y_4}{3}\right)}{1+3},\frac{1.z_2+3\left(\frac{z_1+z_3+z_4}{3}\right)}{1+3}\right)$$

$$\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

Similarly we can prove that co-ordinates of centroid in case of CG and DG are

$$\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

So co-ordinates of centroid are

$$\left(\frac{x_1+x_2+x_3+x_4}{4},\frac{y_1+y_2+y_3+y_4}{4},\frac{z_1+z_2+z_3+z_4}{4}\right)$$

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