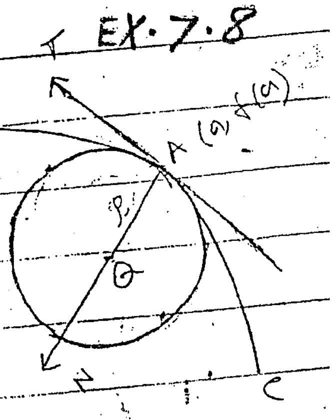
EX·7·8

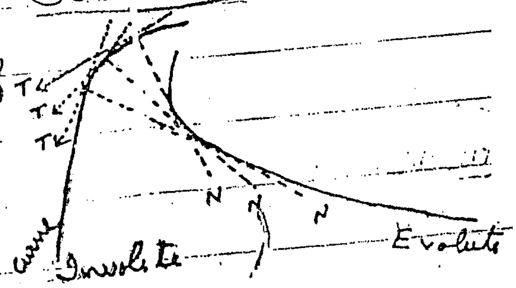


The centre of curvature to the curve C defined by $y=f(x)$ at $A=(a,f(a))$ is the pt Q on the normal to the curve at $A(a,f(a))$ on concave side of curve s.t $|AQ|=\rho$

The circle of curvature of C at $(a,f(a))$ is the circle with centre Q and radius ρ

Circle of Curvature is also called OSCULATING CIRCLE

EVOLUTE is the locus of centres of curvature of a curve and the curve is called Involute of this locus of centres of curvature.



The Normals to a curve (Involute) are Tangents to its Evolute.

Coordinates (α,β) of the centre of Curvature are

$$\alpha = x - \frac{y'\left[1+y'^2\right]}{y''}$$

$$\rho = \frac{\left[1+(y')^2\right]^{3/2}}{|y''|}$$

$$\beta = y + \frac{\left[1+y'^2\right]}{y''}$$

Eq of Osculating Circle $\quad (x-\alpha)^2+(y-\beta)^2=\rho^2$

Ex 7.8

Eq of Osculating circle
$(x-\alpha)^2+(y-\beta)^2=\rho^2$

① $y=\ln x$ at $(1,0)$

$y'=\frac{1}{x}$ $\quad y''=-\frac{1}{x^2}$

$y'(1,0)=\frac{1}{1}=1$ $\quad y''(1,0)=\frac{-1}{1}=-1$

$\rho=\frac{[1+(y')^2]^{3/2}}{|y''|}=\frac{\{1+(\frac{1}{x})^2\}^{3/2}}{\left|-\frac{1}{x^2}\right|}=\frac{\left[\frac{x^2+1}{x^3}\right]}{\frac{1}{x^2}}$

$=\frac{(x^2+1)^{3/2}}{x^3}\cdot\frac{x^2}{1}=\frac{(x^2+1)^{3/2}}{x}$

ρ at $(1,0)=\frac{(1+1)^{3/2}}{1}=2^{3/2}=2\sqrt{2}$

ρ at $(1,0)=\frac{[1+\{y'(1,0)\}^2]^{3/2}}{|y''(1,0)|}=\frac{(1+1)^{3/2}}{|-1|}=\frac{2^{3/2}}{1}=2\sqrt{2}$

Centre of curvature (α,β) is

$\alpha=x-\frac{y'(1+y'^2)}{y''}$

$\alpha_{(1,0)}=1-\frac{1(1+1)}{-1}=1-\frac{2}{-1}=1+2=3$

$\beta=y+\frac{(1+y'^2)}{y''}$

$\beta_{(1,0)}=0+\frac{(1+1)}{-1}=-2$

Eq of Osculating Circle (circle of curvature) is $(x-\alpha)^2+(y-\beta)^2=\rho^2$

$\therefore (x-3)^2+[y-(-2)]^2=(2\sqrt{2})^2 \Rightarrow (x-3)^2+(y+2)^2=8$

(2) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at $(-2,0)$

Check it
find y' at $(-2,0)$
y'' at $(-2,0)$
ρ at $(-2,0)$ using
α at $(-2,0)$ using
b at $(-2,0)$ using

$$\frac{2x}{4} - \frac{2yy'}{9} = 0$$

$$\frac{x}{2} - \frac{2yy'}{9} = 0$$

$$\frac{9x - 4yy'}{18} = 0 \Rightarrow 9x - 4yy' = 0 \Rightarrow \frac{9x}{4y} = y'$$

$$y'' = \frac{9}{4}\left[\frac{y(1) - xy'}{y^2}\right]$$

$$= \frac{9}{4}\left[\frac{y - x\frac{9x}{4y}}{y^2}\right] = \frac{9}{4}\frac{(4y^2 - 9x^2)}{4y^3} = \frac{9(4y^2 - 9x^2)}{16y^3}$$

$$\rho = \frac{\left[1 + y'^2\right]^{3/2}}{|y''|} = \frac{\left[1 + \left(\frac{9x}{4y}\right)^2\right]^{3/2}}{\left|\frac{9}{16}\frac{(4y^2 - 9x^2)}{y^3}\right|} = \frac{\left[(4y)^2 + (9x)^2\right]^{3/2} 16y^3}{\left|\left[(4y)^2\right]^{3/2} 9(4y^2 - 9x^2)\right|}$$

$$\rho = \frac{(16y^2 + 81x^2)^{3/2} \cancel{16y^3}}{\left|4\,64y^3 \; 9(4y^2 - 9x^2)\right|} = \frac{(16y^2 + 81x^2)^{3/2}}{\left|36(4y^2 - 9x^2)\right|}$$

$$\rho \text{ at } (-2,0) = \frac{\left[16(0)^2 + 81(-2)^2\right]^{3/2}}{\left|36(4(0) - 9(-2)^2)\right|} = \frac{\left[81(4)\right]^{3/2}}{\left|36(-36)\right|} = \frac{\left[9^2 2^2\right]^{3/2}}{\left|36(-36)\right|}$$

$$= \frac{9^3 2^3}{|36(-36)|} = \frac{9(8)}{|4(-4)|} = \frac{9(2)}{|(-1)|} = \frac{9}{|-2|} = \frac{9}{2}$$

Centre of Curvature (α, β) is

$$\alpha = x - \frac{y'(1+y'^2)}{y''} = x - \frac{\frac{9x}{4y}\left(1+\frac{81x^2}{16y^2}\right)}{\frac{9}{16y^3}(4y^2-9x^2)}$$

$$\alpha = x - \frac{\frac{9x}{4y}\left(\frac{16y^2+81x^2}{16y^2}\right)}{\frac{9}{16y^3}(4y^2-9x^2)} ; = x - \frac{\frac{9x(16y^2+81x^2)}{4\cdot 64 y^3}}{\frac{9(4y^2-9x^2)}{16y^3}}$$

$$\alpha = x - \frac{9x(16y^2+81x^2)}{36(4y^2-9x^2)}$$

$$\alpha_{(-2,0)} = (-2) - \frac{9(-2)\left[0+81(-2)^2\right]}{36\left(0-9(-2)^2\right)} = (-2) + \frac{18\left[81(4)\right]}{(36)(-36)}$$

$$(-2) + \frac{81(4)}{2(-36)} \quad -2 + \frac{81(2)}{-36} = -2 + \frac{81}{-18} = -2 + \frac{9}{-2} = \frac{+4+9}{-2} = \boxed{\frac{13}{-2}}$$

$$\beta = y + \frac{(1+y'^2)}{y''} = y + \frac{\left(1+\left(\frac{9x}{4y}\right)^2\right)}{\frac{9(4y^2-9x^2)}{16y^3}} = y + \frac{\frac{(16y^2+81x^2)}{16y^2}}{\frac{9(4y^2-9x^2)}{16y^3}}$$

$$= y + \frac{(16y^2+81x^2)}{\cancel{y^2}} \cdot \frac{\cancel{y^3}}{9(4y^2-9x^2)}$$

$$\beta_{(-2,0)} = 0 - 0 \quad = 0$$

Eq of the Osculating circle is

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

$$\left[x - \left(\frac{-13}{2}\right)\right]^2 + [y-0]^2 = \left(\frac{9}{2}\right)^2$$

$$\left(x + \frac{13}{2}\right)^2 + y^2 = \frac{81}{4}$$

③ $x = a\cos\theta$ $\qquad y = b\sin\theta$ (i)

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{b}{a}\left[-\operatorname{cosec}^2\theta \frac{d\theta}{dx}\right] = -\frac{b}{a}\left(\cancel{-}\operatorname{cosec}^2\theta \cdot \frac{1}{\cancel{-}a\sin\theta}\right)$$

$$y'' = \frac{-b}{a^2}\frac{\operatorname{cosec}^2\theta}{\sin\theta} = \frac{-b}{a^2\sin^3\theta} \qquad \text{(ii)}$$

$$\alpha = x - \frac{y'(1+(y')^2)}{y''} \qquad Q(\alpha,\beta) = \text{Centre of Curvature}$$

$$\alpha = a\cos\theta - \frac{\left(-\frac{b}{a}\cot\theta\right)\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{\frac{\cancel{-}b}{a^2\sin^3\theta}}$$

$$\alpha = a\cos\theta - \frac{\cot\theta}{a}\left(\frac{a^2+b^2\cot^2\theta}{\cancel{a^2}}\right)\cancel{a^2}\sin^3\theta$$

$$= \frac{a^2\cos\theta - \cot\theta\,(a^2+b^2\cot^2\theta)\sin^3\theta}{a}$$

$$= \frac{1}{a}\left[a^2\cos\theta - (\cot\theta\, a^2 + b^2\cot^3\theta)\sin^3\theta\right]$$

$$= \frac{1}{a}\left[a^2\cos\theta - \left(\frac{\cos\theta}{\sin\theta}a^2\sin^3\theta + b^2\frac{\cos^3\theta}{\sin^3\theta}\sin^3\theta\right)\right]$$

$$= \frac{1}{a}\left[a^2\cos\theta - a^2\cos\theta\sin^2\theta - b^2\cos^3\theta\right]$$

$$= \frac{1}{a}\left[a^2\cos\theta(1-\sin^2\theta) - b^2\cos^3\theta\right]$$

$$\alpha = \frac{1}{a}\left[a^2\cos^3\theta - b^2\cos^3\theta\right] \qquad = \frac{1}{a}\cos^3\theta(a^2-b^2)$$

$$a\alpha = \cos^3\theta(a^2-b^2) \qquad \text{--- } 1$$

$$\beta = y + \frac{1+(y')^2}{y''}$$

$$= b\sin\theta + \frac{1+\left(-\frac{b}{a}\cot\theta\right)^2}{\frac{-b}{a^2\sin^3\theta}}$$

$$= b\sin\theta + \left(\frac{a^2+b^2\cot^2\theta}{\cancel{a^2}}\right)\left(\frac{\cancel{a^2}\sin^3\theta}{-b}\right)$$

$$= b\sin\theta - \frac{1}{b}\left[a^2\sin^3\theta + b^2\sin^3\theta\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= b\sin\theta - \frac{1}{b}\left(a^2\sin^3\theta + b^2\sin\theta\cos^2\theta\right)$$

$$= \frac{b^2\sin\theta - a^2\sin^3\theta - b^2\sin\theta\cos^2\theta}{b}$$

$$b\beta = b^2\sin\theta\left(1-\cos^2\theta\right) - a^2\sin^3\theta$$

$$b\beta = b^2\sin^3\theta - a^2\sin^3\theta = (b^2-a^2)\sin^3\theta$$

$$b\beta = (a^2-b^2)(-\sin^3\theta) \quad \text{———} \quad (ii)$$

For Evolute we eliminate θ from (i) & (ii)

Taking power $\frac{2}{3}$ on both sides of (i) & (ii) then adding

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}(\cos^3\theta)^{\frac{2}{3}} + (a^2-b^2)^{\frac{2}{3}}(-\sin^3\theta)^{2/3}$$

$$= (a^2-b^2)^{\frac{2}{3}}\left[(\cos\theta)^{3\times\frac{2}{3}} + (-\sin\theta)^{3\times\frac{2}{3}}\right]$$

$$= (a^2-b^2)^{\frac{2}{3}}\left[(\cos\theta)^2 + (-\sin\theta)^2\right]$$

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}\left(\cos^2\theta + \sin^2\theta\right)$$

Put $\alpha = x$, $\beta = y$

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$$ proved.

Q4 $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$

$x = a\sec\theta \qquad y = b\tan\theta$

$\dfrac{dx}{d\theta} = a\sec\theta\tan\theta \qquad \dfrac{dy}{d\theta} = b\sec^2\theta$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\,\frac{\sec\theta}{\tan\theta}$$

$$y' = \frac{b}{a}\,\frac{1}{\cos\theta\,\frac{\sin\theta}{\cos\theta}} = \frac{b}{a\sin\theta} = \frac{b}{a}\operatorname{cosec}\theta.$$

$$y'' = \frac{d^2y}{dx^2} = \frac{b}{a}\left[-\operatorname{cosec}\theta\cot\theta\,\frac{d\theta}{dx}\right] = \frac{b}{a}\left[\frac{-\operatorname{cosec}\theta\cot\theta}{a\sec\theta\tan\theta}\right]$$

$$y'' = \frac{-b}{a^2}\left[\frac{1}{\sin\theta}\,\frac{\cos\theta}{\sin\theta}\,\cos\theta\,\frac{\cos\theta}{\sin\theta}\right] = \frac{-b}{a^2}\,\frac{\cos^3\theta}{\sin^3\theta}$$

$Q(\alpha,\beta)$ be Centre of Curvature

$$\alpha = x - \frac{y'(1+y'^2)}{y''} = a\sec\theta - \frac{\frac{b}{a}\operatorname{cosec}\theta\left(1+\frac{b^2\operatorname{cosec}^2\theta}{a^2}\right)}{\frac{-b}{a^2}\,\frac{\cos^3\theta}{\sin^3\theta}}$$

$$= a\sec\theta - \frac{b}{a}\operatorname{cosec}\theta\left(\frac{a^2+b^2\operatorname{cosec}^2\theta}{\cancel{a^2}}\right)\frac{\cancel{a^2}\sin^3\theta}{(-\cancel{b})\cos^3\theta}.$$

$$= a\sec\theta + \frac{1}{a\sin\theta\cos^3}\cdot\sin^3\theta\left(a^2+\frac{b^2}{\sin^2\theta}\right)$$

$$\alpha = \frac{a}{\cos\theta} + \frac{\sin^{\cancel{2}}\theta}{a\cos^3\theta}\left(\frac{a^2\sin^2\theta+b^2}{\cancel{\sin^2\theta}}\right) = \frac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$$

$$\alpha = \frac{(a^2+b^2)}{a\cos^3\theta} \qquad \because \sin^2\theta + \cos^2\theta = 1$$

$$\alpha = \frac{(a^2+b^2)\sec^3\theta}{a} \Rightarrow a\alpha = (a^2+b^2)\sec^3\theta \quad \text{--- (I)}$$

$$\beta = y + \frac{(1+y'^2)}{y''} = b\tan\theta + \frac{\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)}{\frac{-b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}}$$

$$= b\tan\theta - \frac{\cancel{a^2}\sin^3\theta\left(a^2+b^2\operatorname{cosec}^2\theta\right)}{b\cos^3\theta\ \cancel{a^2}} \quad \text{--- (II)}$$

$$= \frac{b\sin\theta}{\cos\theta} - \frac{a^2\sin^3\theta - b^2\sin^3\theta\frac{1}{\sin^2\theta}}{b\cos^3\theta}$$

$$= \frac{b^2\cos^2\theta\sin\theta - a^2\sin^3\theta - b^2\sin\theta}{b\cos^3\theta} = \frac{b^2\sin\theta(\cos^2\theta-1) - a^2 S}{b\cos^3\theta}$$

$$= \frac{b^2\sin\theta(-\sin^2\theta) - a^2\sin^3\theta}{b\cos^3\theta} = \frac{-\sin^3\theta\,(b^2+a^2)}{b\cos^3\theta}.$$

$$\beta = \frac{-\tan^3\theta\,(b^2+a^2)}{b} \Rightarrow b\beta = -\tan^3\theta\,(b^2+a^2) \quad \text{--- (II)}$$

Taking power $\frac{2}{3}$ of (I) & (II) then subtracting (To eliminate θ)

$$(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}(\sec^3\theta)^{\frac{2}{3}} - (a^2+b^2)^{\frac{2}{3}}(-\tan^3\theta)^{\frac{2}{3}}$$

$$= (a^2+b^2)^{\frac{2}{3}}\left[\sec^2\theta - \tan^2\theta\right] \qquad \because \sec^2\theta - \tan^2\theta = 1$$

Replace α by 'x' & β by 'y' $\quad (ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$ proved

Q5 $2xy = a^2$

$$y = \frac{a^2}{2x} \quad \text{--- (I)}$$

$$y' = \frac{dy}{dx} = -\frac{a^2}{2x^2}$$

$$y'' = \frac{-a^2}{\cancel{2}}\left(\frac{-\cancel{2}}{x^3}\right) = \frac{a^2}{x^3}$$

Now $$\alpha = x - \frac{y'(1+y'^2)}{y''}$$

$$\frac{1+(y')^2}{y''} = \frac{1+\left(\frac{-a^2}{2x^2}\right)^2}{\frac{a^2}{x^3}}$$

$$= \left(1+\frac{a^4}{4x^4}\right)\left(\frac{x^3}{a^2}\right)$$

$$\frac{1+y'^2}{y''} = \left(\frac{4x^4+a^4}{4x^4}\right)\left(\frac{x^3}{a^2}\right) = \left(\frac{4x^4+a^4}{4a^2x}\right)$$

using (II)

$$\alpha = x - \left(\frac{-a^2}{2x^2}\right)\left(\frac{4x^4+a^4}{4a^2x}\right)$$

$$\frac{1+y'^2}{y''} = \frac{4x^4+a^4}{4a^2x} \quad \text{--- (II)}$$

$$\alpha = x + \frac{4x^4+a^4}{8x^3}$$

$$= \frac{8x^4+4x^4+a^4}{8x^3} = \frac{12x^4+a^4}{8x^3} \quad \text{--- (III)}$$

$$\beta = y + \frac{(1+y'^2)}{y''}$$

using (I) & (II)

$$= \frac{a^2}{2x} + \frac{4x^4+a^4}{4a^2x} = \frac{2a^2(a^2)+4x^4+a^4}{4a^2x}$$

$$\beta = \frac{3a^4+4x^4}{4a^2x} \quad \text{--- (IV)}$$

$a(1-\cos\theta)^2$

For the evolute we eliminate x from (III) & (IV)

$$\alpha+\beta = \frac{12x^4+a^4}{8x^3} + \frac{3a^4+4x^4}{4a^2x}$$

$$= \frac{12x^4a^2+a^6+6x^2a^4+8x^6}{8a^2x^3}$$

$$\alpha+\beta = \frac{(2x^2+a^2)^3}{8a^2x^3}$$

$(a+b)^3 = a^3+b^3+3ab(a+b)$

$(2x^2+a^2)^3 = (2x^2)^3+(a^2)^3+3(2x^2a^2)(2x^2+a^2)$

$= 8x^6+a^6+12x^4a^2+6x^2a^4.$

$$(\alpha+\beta)^{\frac{2}{3}} = \frac{\left[(2x^2+a^2)^3\right]^{\frac{2}{3}}}{(2^3a^2x^3)^{\frac{2}{3}}} = \frac{(2x^2+a^2)^2}{2^2a^{\frac{4}{3}}x^2}$$

$$(\alpha+\beta)^{2/3} = \frac{(2x^2+a^2)^2}{4x^2a^{4/3}} \quad \text{(V)}$$

$$\alpha-\beta = \frac{12x^4+a^4}{8x^3} - \frac{3a^4+4x^4}{4a^2x} = \frac{12x^4a^2+a^6-6x^2a^4-8x^6}{8a^2x^3}$$

$$\alpha-\beta = \frac{(a^2-2x^2)^3}{8a^2x^3} \quad \text{(VI)}$$

$$(\alpha-\beta)^{\frac{2}{3}} = \frac{\left[(a^2-2x^2)^3\right]^{\frac{2}{3}}}{(2^3a^2x^3)^{2/3}} = \frac{(a^2-2x^2)^2}{2^2a^{\frac{4}{3}}x^2} = \frac{(a^2-2x^2)^2}{4x^2a^{4/3}} \quad \text{(VII)}$$

$$(\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{2/3} = \frac{(2x^2+a^2)^2}{4x^2a^{4/3}} - \frac{(a^2-2x^2)^2}{4x^2a^{4/3}}$$

$$= \frac{4x^4+a^4+4x^2a^2-(a^4+4x^4-4a^2x^2)}{4x^2a^{4/3}}$$

$$(\alpha+\beta)^{\frac{2}{3}} (\alpha-\beta)^{2/3} = \frac{8x^2a^2}{4x^2a^{4/3}} = 2a^{2-\frac{4}{3}}$$

$$(\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{2/3} = 2a^{2/3}$$

Put $x=\alpha$ $y=\beta$

$$(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}} = 2a^{2/3}$$

(6) $x = a(\theta - \sin\theta)$ $\qquad y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta) \qquad \frac{dy}{d\theta} = a(\sin\theta)$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{\cancel{a}(\sin\theta)}{\cancel{a}(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$y'' = \frac{d^2y}{dx^2} = \left(\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}\right)\frac{d\theta}{dx}$$

$$= \left(\frac{\cos\theta - \cos^2\theta - \sin^2\theta}{(1-\cos\theta)^2}\right)\frac{1}{a(1-\cos\theta)} = \frac{\cos\theta - (\cos^2\theta + \sin^2\theta)}{a(1-\cos\theta)^3}$$

$$y'' = \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-(1-\cos\theta)}{a(1-\cos\theta)^3} = \frac{-1}{a(1-\cos\theta)^2}$$

$$\alpha = x - y'\left(\frac{1+y'^2}{y''}\right) = a(\theta - \sin\theta) - \left(\frac{\sin\theta}{1-\cos\theta}\right)\frac{\left(1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\right)}{\frac{-1}{a(1-\cos\theta)^2}}$$

$$= a(\theta - \sin\theta) + \frac{\sin\theta}{(1-\cos\theta)}\left(\frac{(1-\cos\theta)^2 + \sin^2\theta}{\cancel{(1-\cos\theta)^2}}\right) a\cancel{(1-\cos\theta)^2}$$

$$= a\theta - a\sin\theta + \frac{a\sin\theta}{(1-\cos\theta)}\left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta\right]$$

$$= a\theta - a\sin\theta + \frac{a\sin\theta}{(1-\cos\theta)}(2-2\cos\theta)$$

$$= a\theta - a\sin\theta + \frac{2a\sin\theta\cancel{(1-\cos\theta)}}{\cancel{(1-\cos\theta)}}$$

$$\alpha = a\theta + a\sin\theta \qquad = a(\theta + \sin\theta) \text{ ——— (I)}$$

$$\beta = y + \frac{(1+y'^2)}{y''} \qquad = a(1-\cos\theta) + \frac{\left[1 + \frac{\sin^2\theta}{(1-\cos\theta)^2}\right]}{\frac{-1}{a(1-\cos\theta)}}$$

$$= a(1-\cos\theta) - \left[\frac{(1-\cos\theta)^2 + \sin^2\theta}{\cancel{(1-\cos\theta)^2}}\right] a\cancel{(1-\cos\theta)^2}$$

$$= a(1-\cos\theta) - a\left[1+\cos^2\theta - 2\cos\theta + \sin^2\theta\right]$$

$$= a(1-\cos\theta) - a(1+1-2\cos\theta) \quad = a(1-\cos\theta) - 2a(1-\cos\theta)$$

$$\beta = (1-\cos\theta)[a-2a] \quad = -a(1-\cos\theta) \text{ ——— (II)}$$

Elimination of θ is not possible. (I) & (II) are parametric Eq^s evolute, which is cycloid.

7.8-14

Q.No 7 $\quad x = a\cos^3\theta \quad y = a\sin^3\theta$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = -\tan\theta$$

$$y'' = -\sec^2\theta\frac{d\theta}{dx} = +\sec^2\theta\frac{1}{+3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}.$$

$\therefore$

$$\alpha = x - y'\left[\frac{1+y'^2}{y''}\right]$$

$$= a\cos^3\theta - (-\tan\theta)\left(\frac{1+\tan^2\theta}{\frac{1}{3a\cos^4\theta\sin\theta}}\right) = a\cos^3\theta + \frac{\sin\theta}{\cos\theta}(\sec^2\theta)3a\cos^4\theta\sin\theta$$

$$= a\cos^3\theta + \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos^2\theta}\right)3a\cos^4\theta\sin\theta$$

$$\alpha = a\cos^3\theta + 3a\cos\theta\sin^2\theta \qquad (1)$$

$$\beta = y + \frac{1+y'^2}{y''}$$

$$= a\sin^3\theta + \left[\frac{1+\tan^2\theta}{\frac{1}{3a\cos^4\theta\sin\theta}}\right] = a\sin^3\theta + (\sec^2\theta)[3a\cos^4\theta\sin\theta]$$

$$= a\sin^3\theta + \frac{1}{\cos^2\theta}3a\cos^4\theta\sin\theta$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta \qquad (2)$$

$$\alpha+\beta = a\cos^3\theta + 3a\sin^2\theta\cos\theta + a\sin^3\theta + 3a\cos^2\theta\sin\theta$$

$$= a(\cos^3\theta + \sin^3\theta + 3\cos^2\theta\sin\theta + 3\sin^2\theta\cos\theta)$$

$$= a\left[\cos^3\theta + \sin^3\theta + 3\cos\theta\sin\theta(\cos\theta+\sin\theta)\right]$$

$$\alpha+\beta = a(\cos\theta+\sin\theta)^3$$

$$(\alpha+\beta)^{2/3} = a^{\frac{2}{3}}\left[(\cos\theta+\sin\theta)^3\right]^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos\theta+\sin\theta)^2 \quad \text{(III)}$$

Now consider

$$\alpha-\beta = a\cos^3\theta + 3a\sin^2\theta\cos\theta - a\sin^3\theta - 3a\cos^2\theta\sin\theta$$

$$= a\left[\cos^3\theta - \sin^3\theta - 3\cos^2\theta\sin\theta + 3\sin^2\theta\cos\theta\right]$$

$$= a\left[\cos^3\theta - \sin^3\theta - 3\cos\theta\sin\theta(\cos\theta - \sin\theta)\right]$$

$$\alpha-\beta = a(\cos\theta - \sin\theta)^3$$

$$(\alpha-\beta)^{2/3} = a^{2/3}\left[(\cos\theta - \sin\theta)^3\right]^{2/3} = a^{2/3}(\cos\theta - \sin\theta)^2 \quad \text{(IV)}$$

Adding (III) & (IV) $(\alpha+\beta)^{2/3} + (\alpha-\beta)^{2/3} = a^{2/3}(\cos\theta+\sin\theta)^2 + a^{2/3}(\cos\theta-\sin\theta)^2$

$$= a^{2/3}\left[\cos^2\theta + \sin^2\theta + \cancel{2\sin\theta\cos\theta} + \cos^2\theta + \sin^2\theta - \cancel{2\sin\theta\cos\theta}\right]$$

$$= a^{2/3}[1+1]$$

$$(\alpha+\beta)^{2/3} + (\alpha-\beta)^{2/3} = 2a^{2/3}$$

Replace α by x & β by y

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

$$= a\left[\cos t + \ln \tan \frac{t}{2}\right] - \frac{1}{\cos t}\,\frac{1}{\cos^2 t}\,a\cos^4 t$$

$$= \cancel{a\cos t} + a\ln\tan\frac{t}{2}) - \cancel{a\cos t}$$

$$\alpha = a\ln\tan\frac{t}{2} \quad \text{———} \quad (1)$$

$$\beta = y + \left[\frac{1+y'^2}{y''}\right]$$

$$= a\sin t + \frac{(1+\tan^2 t)}{\frac{\sin t}{a\cos^4 t}} = a\sin t + (\sec^2 t)\frac{(a\cos^4 t)}{\sin t}$$

$$= a\sin t + \frac{1}{\cos^2 t}\,\frac{a\cos^4 t}{\sin t} = a\sin t + \frac{a\cos^2 t}{\sin t}$$

$$\beta = \frac{a\sin^2 t + a\cos^2 t}{\sin t} = \frac{a(\sin^2 t + \cos^2 t)}{\sin t} = \frac{a}{\sin t}$$

$$\beta = \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}} = \frac{\frac{a}{\cos^2 t/2}}{\frac{2\sin\frac{t}{2}\cos t/2}{\cos^2 t/2}} = \frac{a\sec^2 t/2}{2\tan t/2}$$

$$\beta = a\frac{(1+\tan^2 t/2)}{2\tan t/2} \quad \text{———} \quad (11)$$

For evolute eliminate t from (1) & (11).

As $\alpha = a\ln\tan t/2$ $\qquad \frac{\alpha}{a} = \ln\tan\frac{t}{2}$

$$e^{\frac{\alpha}{a}} = \tan\frac{t}{2}$$

$\cancel{(x+y)} + (x-y) = \alpha a$

Q8 $x = a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right]$; $y = a\sin t$.

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}\sec^2\left(\frac{t}{2}\right)\frac{1}{2}\right] \qquad \frac{dy}{dt} = a\cos t.$$

$$= a\left[-\sin t + \frac{1}{\cos^2\left(\frac{t}{2}\right)}\frac{\cos(t/2)}{\sin\left(\frac{t}{2}\right)}\frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\cos\frac{t}{2}\sin\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] = a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$\frac{dx}{dt} = a\frac{\cos^2 t}{\sin t}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t$$

$$y'' = \sec^2 t\,\frac{dt}{dx} = \sec^2 t\,\frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$

(i) Centre of curvature (α, β)

$$\alpha = x - y'\left(\frac{1+y'^2}{y''}\right)$$

$$= a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right] - \tan t\left(\frac{1+\tan^2 t}{\frac{\sin t}{a\cos^4 t}}\right)$$

$$= a\left[\cos t + \ln\tan\frac{t}{2}\right] - \frac{\sin t}{\cos t}\left(\sec^2 t\right)\frac{a\cos^4 t}{\sin t}$$

$\therefore$ (III) becomes $\beta = a\left(\dfrac{1+e^{\frac{2\alpha}{a}}}{2e^{\frac{\alpha}{a}}}\right)$

$$\beta = \frac{a}{2}\left[\frac{1}{e^{\alpha/a}} + \frac{e^{\frac{2\alpha}{a}}}{e^{\alpha/a}}\right]$$

$$= \frac{a}{2}\left[e^{-\frac{\alpha}{a}} + e^{\alpha/a}\right] = a\left(\frac{e^{-\frac{\alpha}{a}} + e^{\frac{\alpha}{a}}}{2}\right)$$

$$\beta = a \cosh \frac{\alpha}{a}$$

Replacing α by x & β by 'y'

$$y = a \cosh \frac{x}{a}$$

x————x

Q.) $x^3+y^3 = 3axy$.

Centre of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is $\left(\frac{21a}{16}, \frac{21a}{16}\right)$

$$3x^2+3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right)$$

$$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay-3x^2$$

$$3\frac{dy}{dx}(y^2-ax) = 3(ay-x^2)$$

$$\frac{dy}{dx} = \frac{\cancel{3}(ay-x^2)}{\cancel{3}(y^2-ax)} \quad \text{——— (1)}$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ $\quad \dfrac{dy}{dx} = \dfrac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = \dfrac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}}$

$$\frac{dy}{dx} = \frac{-\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}{\left(\frac{9a^2}{4} - \left(\frac{3a^2}{2}\right)\right)} = -1$$

~~from~~ (1)

$$\frac{d^2y}{dx^2} = \frac{1}{(y^2-ax)^2}\left[(y^2-ax)(ay'-2x) - (ay-x^2)(2yy'-a)\right]$$

~~At~~ $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\left(a(-1) - 2\left(\frac{3a}{2}\right)\right) - \left(\frac{3a^2}{2} - \frac{9a^2}{4}\right)\left(2\left(\frac{3a}{2}\right)(-1) - a\right)}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^2}$$

$$= \frac{\cancel{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}\left[(-a-3a) + (-3a-a)\right]}{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)^{\cancel{2}}}$$

$$= \frac{-8a}{\frac{9a^2}{4} - \frac{3a^2}{2}} = \frac{-8a}{\frac{9a^2-6a^2}{4}} = \frac{-32a}{3a^2}$$

$$y'' = \frac{-32}{3a}$$

$$\alpha = x - y'\left[\frac{1+y'^2}{y''}\right] = \frac{3a}{2} - (-1)\left[\frac{1+(-1)^2}{\frac{-32}{3a}}\right]$$

$$= \frac{3a}{2} + 1\frac{(1+1)}{-32}3a = \frac{3a}{2} - \frac{6a}{32} = \frac{3a}{2} - \frac{3a}{16} = \frac{24a-3a}{16} = \boxed{\frac{21a}{16}}$$

Similarly $\beta = y + \left[\frac{1+y'^2}{y''}\right] = \frac{3a}{2} + \frac{(1+1)}{\frac{-32}{3a}} = \frac{3a}{2} - \frac{6a}{32}$

$$\beta = \frac{3a}{2} - \frac{3a}{16} = \frac{24a-3a}{16} = \boxed{\frac{21a}{16}}$$

7.8-20

Q10 Let $P(x,y)$ be any pt on a curve $y=f(x)$
We know by Theorem that coord of the centre of curvature $C(\alpha,\beta)$ corresponding to the pt P are

$$\alpha = x - \rho\sin\psi \quad \text{——— (1)}$$

$$\beta = y + \rho\cos\psi \quad \text{——— (2)}$$

~~Diff (1) & (2) w.r.t 'x'~~

$$\frac{d\alpha}{dx} = 1 - \left[\rho\cos\psi\frac{d\psi}{dx} + \sin\psi\frac{d\rho}{dx}\right]$$

$\rho = \frac{ds}{d\psi}$, $\tan\psi = \frac{dy}{dx}$, $\cos\psi = \frac{dx}{ds}$, $\sin\psi = \frac{dy}{ds}$

$$= 1 - \left[\frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx} + \sin\psi\frac{d\rho}{dx}\right]$$

$$= 1 - \cancel{\frac{dx}{ds}}\cdot\frac{ds}{d\psi}\cdot\cancel{\frac{d\psi}{dx}} - \sin\psi\frac{d\rho}{dx}$$

$$= 1 - 1 - \sin\psi\frac{d\rho}{dx}$$

$$\frac{d\alpha}{dx} = -\sin\psi\frac{d\rho}{dx} \quad \text{——— (1)}$$

(4)

$$\frac{d\beta}{dx} = \frac{dy}{dx} - \rho\sin\psi\frac{d\psi}{dx} + \frac{d\rho}{dx}\cos\psi$$

$$= \frac{dy}{dx} - \frac{ds}{d\psi}\cdot\frac{dy}{ds}\cdot\frac{d\psi}{dx} + \frac{d\rho}{dx}\cos\psi$$

$$\frac{d\beta}{dx} = \frac{dy}{dx} - \frac{dy}{dx} + \frac{d\rho}{dx}\cos\psi$$

$$= \frac{d\rho}{dx}\cos\psi \quad \text{——— (2)}$$

$$\frac{d\beta}{d\alpha} = \frac{d\beta/dx}{d\alpha/dx} = \frac{\frac{d\rho}{dx}\cos\psi}{\frac{d\rho}{dx}(-\sin\psi)}$$

$$\frac{d\beta}{d\alpha} = -\cot\psi = \frac{-1}{\tan\psi} = \frac{-1}{\frac{dy}{dx}} \text{(slope of Normal)}$$

(like $\frac{dy}{dx}$ = sl

Now $\frac{d\beta}{d\alpha}$ is slope of the tangent to the evolute at $C(\alpha, \beta)$ and it is equal to the slope of the normal PC to the curve $y = f(x)$ at $P(x, y)$.

×———————×