Ex 7.2. 35

Locate the pts of relative extrema

(i) Put $f'(x)=0$ to find the values of x.
(ii) Put the values of x from step (i) into $f''(x)$.
(iii) $f''(x)>0$ Rel Min
$f''(x)<0$ Rel Max

$f(x) = 2x^3 - 15x^2 + 36x + 10$

$f'(x) = 6x^2 - 30x + 36$

$f''(x) = 12x - 30$

Put $f'(x) = 0$. $\quad 6x^2 - 30x + 36 = 0$

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$x(x-3) - 2(x-3) = 0$

$(x-3)(x-2) = 0 \qquad x = 2, 3$

At $x=2$. $\quad f''(x) = 12(2) - 30 = -6 < 0$

$\therefore x = 2$ is pt of relative Max.

At $x = 3$. $\quad f''(x) = 12(3) - 30 = 6 > 0$

$\therefore x = 3$ is pt of relative Min.

(2) $f(x) = 3x^4 - 4x^3 + 5$

$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

Put $f'(x) = 0$. $\quad 12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$12x^2 = 0 \Rightarrow \boxed{x=0} \qquad x-1 = 0 \Rightarrow \boxed{x=1}$

$x=0 \quad f''(x) = 36(0) - 24(0) = 0$.

$x=0$ is neither point of relative Max nor pt of relative Min

Ex 7.2.

Locate the pts of relative extrema

$f(x) = 2x^3 - 15x^2 + 36x + 10$

$f'(x) = 6x^2 - 30x + 36$

$f''(x) = 12x - 30$

$f'(x) = 0$. $6x^2 - 30x + 36 = 0$ | $f''(x) < 0$ Rel Max

$x^2 - 5x + 6 = 0$

$x^2 - 3x - 2x + 6 = 0$

$x(x-3) - 2(x-3) = 0$

$(x-3)(x-2) = 0$ $x = 2, 3$.

At $x = 2$. $f''(x) = 12(2) - 30 = -6 < 0$

$\therefore$ $x = 2$ is pt of relative Max.

At $x = 3$ $f''(x) = 12(3) - 30 = 6 > 0$

$\therefore$ $x = 3$ is pt of relative Min.

(2) $f(x) = 3x^4 - 4x^3 + 5$

$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

Put $f'(x) = 0$. $12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$12x^2 = 0 \Rightarrow \boxed{x = 0}$ $x - 1 = 0 \Rightarrow \boxed{x = 1}$

$f''(x) = 36(0) - 24(0) = 0$.

$x = 0$ is neither point of relative Max nor pt of relative Min.

$f(x) = (x-1)(x-2)(x-3)$

$= (x^2 - 2x - x + 2)(x-3)$

$= x^3 - 2x^2 - x^2 + 2x - 3x^2 + 6x + 3x - 6$

$= x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f''(x) = 6x - 12$

Put $f'(x) = 0$

$3x^2 - 12x + 11 = 0$

$x = \dfrac{+12 \pm \sqrt{144 - 4\cdot 3\cdot 11}}{6} = \dfrac{12 \pm \sqrt{144 - 132}}{6}$

$= \dfrac{12 \pm \sqrt{12}}{6} = \dfrac{12 \pm 2\sqrt{3}}{6} = \dfrac{6 \pm \sqrt{3}}{3}$

$x = \dfrac{6}{3} \pm \dfrac{\sqrt{3}}{3} = \boxed{2 \pm \dfrac{\sqrt{3}}{3}}$

At $x = 2 + \dfrac{\sqrt{3}}{3}$

$f''(x) = 6\left(2 + \dfrac{\sqrt{3}}{3}\right) - 12$

$= \cancel{12} + \dfrac{6\sqrt{3}}{3} - \cancel{12} = 2\sqrt{3} > 0$

So $x = 2 + \dfrac{\sqrt{3}}{3}$ is the pt of Rel Min

At $x = 2 - \dfrac{\sqrt{3}}{3}$

$f''(x) = 6\left(2 - \dfrac{\sqrt{3}}{3}\right) - 12$

$= \cancel{12} - \dfrac{6\sqrt{3}}{3} - \cancel{12} = -2\sqrt{3} < 0$

So $x = 2 - \dfrac{\sqrt{3}}{3}$ is the pt of Relative Max.

Q25. $f(x) = \sin x \cos 2x$

$f'(x) = \cos x \cos 2x + \sin x(-\sin 2x)2.$

$= \cos x \cos 2x - 2\sin x \sin 2x.$

$f''(x) = \{-\sin x \cos 2x + \cos x(-\sin 2x)2\} - 2\{\cos x \sin 2x + \sin x \cos 2x(2)\}$

$= -\sin x \cos 2x - 2\cos x \sin 2x - 2\cos x \sin 2x - 4\sin x \cos 2x.$

$f''(x) = -5\sin x \cos 2x - 4\cos x \sin 2x.$

$f'(x) = 0$. $\cos x \cos 2x - 2\sin x \sin 2x = 0$

$\cos x \cos 2x - 2\sin x(2\sin x \cos x) = 0$

$\cos x(\cos 2x - 4\sin^2 x) = 0.$

$\cos x(1 - 2\sin^2 x - 4\sin^2 x) = 0$

$\cos x(1 - 6\sin^2 x) = 0$

either $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \cos(2\pi - \frac{\pi}{2}) = \cos(\frac{3\pi}{2})$

or $1 - 6\sin^2 x = 0 \Rightarrow \sin^2 x = \frac{1}{6}$

$\sin x = \pm\frac{1}{\sqrt{6}}$

$\therefore x = \sin^{-1}(\frac{1}{\sqrt{6}}), \sin^{-1}(\frac{-1}{\sqrt{6}})$

$= 24°5', -24°5'$

At $x = \frac{\pi}{2}$ $\quad f''(x) = -5\sin(\frac{\pi}{2})\cos 2(\frac{\pi}{2}) - 4\cos(\frac{\pi}{2})\sin 2(\frac{\pi}{2})$

$= -5(1)(-1) - 4(0)$

$= 5 > 0$

So $x = \frac{\pi}{2}$ is the pt of relative Min

At $x = \frac{3\pi}{2}$ $\quad f''(x) = -5\sin\left(\frac{3\pi}{2}\right)\cos 2\left(\frac{3\pi}{2}\right) - 4\cos\left(\frac{3\pi}{2}\right)\sin 2\left(\frac{3\pi}{2}\right)$

$$= -5(-1)(-1) - 4(0)(0)$$

$$= -5 < 0$$

So $x = \frac{\pi}{2}$ is the pt of relative Max.

At $x = 24°5'$ $\quad f''(x) = -5\sin(24°5')\cos 2(24°5') - 4\cos(24°5')\sin^2(24°5')$

$$= -5(.40806)(.66696) - 4(.9129)(.7450)$$

$$= -1.36079 - 2.7207 = -4.08 < 0$$

So $x = \sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$ or $x = 24°5'$ is the pt of rel Max.

At $x = -24°5'$ $\quad f''(x) = -5(-.40806)(.66696) - 4(.9129)(-.7450)$

$$= 1.360790 + 2.7207 = 4.08151 > 0$$

So $x = \sin\left(\frac{-1}{\sqrt{6}}\right)$ or $x = 24°5'$ is the pt of rel Minimum

(Second Method see in R.)

$\times$——$\times$

Q6 $f(x) = a\sec x + b\operatorname{cosec} x$

$f'(x) = a\sec x\tan x - b\operatorname{cosec} x\cot x$

$f''(x) = a\sec x\tan x\tan x + a\sec x\sec^2 x - b\{-\operatorname{cosec} x\cot x\cot x - \operatorname{cosec} x\operatorname{cosec}^2 x\}$

$f''(x) = a\sec x\tan^2 x + a\sec^3 x + b\operatorname{cosec} x\cot^2 x + b\operatorname{cosec}^3 x.$

Put $f'(x) = 0$.

$$a \sec x \tan x - b \operatorname{cosec} x \cot x = 0$$

$$a \frac{1}{\cos x} \frac{\sin x}{\cos x} - b \frac{1}{\sin x} \frac{\cos x}{\sin x} = 0$$

$$\frac{a \sin x}{\cos^2 x} - \frac{b \cos x}{\sin^2 x} = 0$$

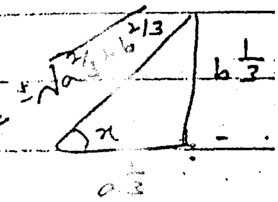
$$\frac{a \sin^3 x - b \cos^3 x}{\cos^2 x \sin^2 x} = 0$$

$$a \sin^3 x = b \cos^3 x.$$

$$\frac{\sin^3 x}{\cos^3 x} = \frac{b}{a}.$$

$$\tan^3 x = \frac{b}{a}. \Rightarrow \tan x = \left(\frac{b}{a}\right)^{1/3}$$

$$\tan x = \frac{b^{1/3}}{a^{1/3}}$$



$$\cos = \frac{a^{1/3}}{\pm\sqrt{a^{2/3}+b^{2/3}}}, \quad \sin x = \frac{b^{1/3}}{\pm\sqrt{a^{2/3}+b^{2/3}}}$$

At +ve values of $\cos x$ & $\sin x$

$$f''(x) = a \sec x \tan^2 x + a \sec^3 x + b \operatorname{cosec} x \cot^2 x + b \operatorname{cosec}^3 x.$$

$$= a \frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} \left(\frac{b^{2/3}}{a^{2/3}}\right) + a\left(\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}\right)^3 + b \frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}} \frac{a^{2/3}}{b^{2/3}} + b\left(\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}\right)^3$$

$$= \frac{\cancel{a}}{\cancel{a}} b^{2/3}\sqrt{a^{2/3}+b^{2/3}} + \frac{\cancel{a}}{\cancel{a}}\left(a^{2/3}+b^{2/3}\right)^{\frac{3}{2}} + a^{2/3} \frac{\cancel{b}}{\cancel{b}} \sqrt{a^{2/3}+b^{2/3}} + \frac{\cancel{b}}{\cancel{b}} \left(a^{2/3}+b^{2/3}\right)^{3/2}$$

$$= \sqrt{a^{2/3}+b^{2/3}}\left(a^{2/3}+b^{2/3}\right) + 2\left(a^{2/3}+b^{2/3}\right)^{3/2} = +ve > 0$$

$$\sec x = \frac{\pm\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$$

$$\operatorname{cosec} x = \frac{\pm\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$$

$$\tan x = \frac{b^{1/3}}{a^{1/3}}$$

$$\tan^2 x = \frac{b^{2/3}}{a^{2/3}}$$

$$\cot^2 x = \frac{a^{2/3}}{b^{2/3}}$$

pt of rel Min is at +ve values of $\cos x$ & $\sin x$.

At −ve values of $\cos x$ & $\sin x$:

$$f''(x) = a\sec x\tan^2 x + a\sec^3 x + b\operatorname{cosec} x\cot^2 x + b\operatorname{cosec}^3 x.$$

$$= a\left(-\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}\right)\left(\frac{b^{2/3}}{a^{2/3}}\right) + a\left(-\frac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}\right)^3 + b\left(-\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}\right)\frac{a^{2/3}}{b^{2/3}} + b\left(-\frac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}\right)^3$$

$$= -\sqrt{a^{2/3}+b^{2/3}}\, b^{2/3} - (a^{2/3}+b^{2/3})^{3/2} - \sqrt{a^{2/3}+b^{2/3}}\, a^{2/3} - (a^{2/3}+b^{2/3})^{3/2}$$

$$= -\sqrt{a^{2/3}+b^{2/3}}\,(a^{2/3}+b^{2/3}) - 2(a^{2/3}+b^{2/3})^{3/2} = -ve < 0$$

∴ pt of rel Max is at −ve values of $\sin x$ & $\cos x$.

Q7 $f(x) = \sin x\cos^2 x.$

$$f'(x) = \cos x\cos^2 x + \sin x\, 2\cos x(-\sin x)$$

$$= \cos x(1-\sin^2 x) - 2\sin^2 x\cos x.$$

$$= \cos x - \cos x\sin^2 x - 2\sin^2 x\cos x.$$

$$= \cos x - 3\cos x\sin^2 x.$$

$$f'(x) = \cos x(1 - 3\sin^2 x)$$

$$f''(x) = -\sin x(1-3\sin^2 x) + \cos x(-6\sin x\cos x)$$

$$= -\sin x + 3\sin^3 x - 6\cos^2 x\sin x$$

$$= -\sin x + 3\sin^3 x - 6(1-\sin^2 x)\sin x$$

$$= -\sin x + 3\sin^3 x - 6\sin x + 6\sin^3 x$$

$$f''(x) = -7\sin x + 9\sin^3 x.$$

Put $f'(x)=0$ $\quad \cos x(1-3\sin^2 x)=0.$

either $\cos x = 0 \Rightarrow x = \cos^{-1}(0) = \boxed{\frac{\pi}{2}, \frac{3\pi}{2}}$

or $(1-3\sin^2 x) = 0.$

$3\sin^2 x = 1$

$\sin^2 x = \frac{1}{3}$ $\qquad \sin x = \pm\frac{1}{\sqrt{3}}$ $\qquad x = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right), \sin^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

At $x=\frac{\pi}{2}$ $\quad f''(x) = -7\sin\left(\frac{\pi}{2}\right) + 9\sin^3\left(\frac{\pi}{2}\right)$

$= -7+9 = 2 > 0.$

So $x = \frac{\pi}{2}$ is the pt of Rel Minimum.

At $x = \frac{3\pi}{2}$ $\quad f''(x) = -7\sin\left(\frac{3\pi}{2}\right) + 9\sin^3\left(\frac{3\pi}{2}\right)$

$= -7(-1) + 9(-1) = -2 < 0.$

So $x = \frac{3\pi}{2}$ is the pt of Rel Max.

At $x = \sin^{-1}\frac{1}{\sqrt{3}}$ $\quad f''(x) = -7\left(\frac{1}{\sqrt{3}}\right) + 9\left(\frac{1}{\sqrt{3}}\right)^3 = \frac{-7}{\sqrt{3}} + \frac{9}{3\sqrt{3}} = \frac{-21+9}{3\sqrt{3}}$

$= \frac{-4}{\sqrt{3}} < 0$

So $x = \sin^{-1}\frac{1}{\sqrt{3}}$ is the p of Rel Max.

At $x = \sin^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ $\quad f''(x) = -7\left(\frac{-1}{\sqrt{3}}\right) + 9\left(\frac{-1}{\sqrt{3}}\right)^3$

$= \frac{7}{\sqrt{3}} - \frac{9}{3\sqrt{3}} = \frac{21-9}{3\sqrt{3}} = \frac{12}{3\sqrt{3}} = \frac{4}{\sqrt{3}} > 0$

So at $x = \sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is the pt of Rel Min.

Q8 $f(x) = e^x \cos(x-a)$

$$f'(x) = e^x \cos(x-a) + e^x(-\sin(x-a))$$

$$= e^x[\cos(x-a) - \sin(x-a)]$$

either $e^x = 0$ ~~but it can never be possible~~

So $\cos(x-a) - \sin(x-a) = 0$

$$\Rightarrow \cos(x-a) = \sin(x-a)$$

$$1 = \frac{\sin(x-a)}{\cos(x-a)} \Rightarrow \tan(x-a) = 1$$

$$x-a = \tan^{-1}(1) = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$x = \frac{\pi}{4}+a, \quad \frac{5\pi}{4}+a$$

$$f''(x) = e^x[\cos(x-a) - \sin(x-a)] + e^x[-\sin(x-a) - \cos(x-a)]$$

$$= e^x \cancel{\cos(x-a)} - e^x \sin(x-a) - e^x \sin(x-a) - e^x \cancel{\cos(x-a)}$$

$$f''(x) = -2e^x \sin(x-a)$$

At $x = \frac{\pi}{4}+a$ $\quad f''(x) = -2e^{\frac{\pi}{4}+a} \sin\left(\frac{\pi}{4}+a-a\right) = -2e^{\frac{\pi}{4}+a} \sin\left(\frac{\pi}{4}\right)$

$$= -2e^{\frac{\pi}{4}+a} \frac{1}{\sqrt{2}} = < 0$$

So at $x = \frac{\pi}{4}+a$ is the pt of Rel Max.

At $x = \frac{5\pi}{4}+a$ $\quad f''(x) = -2e^{\frac{5\pi}{4}+a} \sin\left(\frac{5\pi}{4}+a-a\right) = -2e^{\frac{5\pi}{4}+a} \sin\left(\frac{5\pi}{4}\right)$

$$= -2e^{\frac{5\pi}{4}+a}\left(-\frac{1}{\sqrt{2}}\right) = > 0$$

So $x = \frac{5\pi}{4}+a$ is the pt of Rel Min.

Q9 $f(x) = x^{x}$.

$\ln f(x) = x \ln x$

$\frac{1}{f(x)} f'(x) = x \cdot \frac{1}{x} + \ln x$

$f'(x) = f(x)\left[1 + \ln x\right]$

$f'(x) = x^{x}\left[1 + \ln x\right]$

Put $f'(x) = 0$ $\quad x^{x}\left[1 + \ln x\right] = 0$.

$1 + \ln x = 0$

$\ln x = -1$

$\log_e x = -1 \Rightarrow e^{-1} = x$.

$\Rightarrow \boxed{\frac{1}{e} = x}$

$f''(x) = x^{x}\left[1+\ln x\right]\left[1+\ln x\right] + x^{x}\left[\frac{1}{x}\right]$

$f''(x) = x^{x}\left[1+\ln x\right]^{2} + x^{x-1}$

$= \left(\frac{1}{e}\right)^{\frac{1}{e}}\left[1 + \ln e^{-1}\right]^{2} + \left(\frac{1}{e}\right)^{\frac{1}{e}-1} > 0.$

$\therefore x = e^{-1}$ is the pt of Relative Minimum

Q12 $\quad \dfrac{c^4}{r^2} = \dfrac{a^2}{\sin^2\theta} + \dfrac{b^2}{\cos^2\theta}$

Diff w.r.t. 'θ': $\quad -2r^{-3}c^4 \dfrac{dr}{d\theta} = a^2\left(-2\sin^{-3}\theta\cos\theta\right) + b^2\left(-2\cos^{-3}\theta(-\sin\theta)\right)$

$$\frac{-2c^4}{r^3}\frac{dr}{d\theta} = -\frac{2a^2\cos\theta}{\sin^3\theta} + \frac{2b^2\sin\theta}{\cos^3\theta}$$

$$\frac{dr}{d\theta} = \left(\frac{r^3}{-2c^4}\right)(-2)\left(\frac{a^2\cos\theta}{\sin^3\theta} - \frac{b^2\sin\theta}{\cos^3\theta}\right)$$

$$\frac{dr}{d\theta} = \frac{r^3}{c^4}\left(\frac{a^2\cos^4\theta - b^2\sin^4\theta}{\sin^3\theta\cos^3\theta}\right)$$

For extreme values Put $\dfrac{dr}{d\theta} = 0$.

$$\frac{r^3}{c^4}\left(\frac{a^2\cos^4\theta - b^2\sin^4\theta}{\sin^3\theta\cos^3\theta}\right) = 0$$

$$\frac{a^2\cos^4\theta - b^2\sin^4\theta}{\sin^3\theta\cos^3\theta} = 0$$

$$a^2\cos^4\theta - b^2\sin^4\theta = 0$$

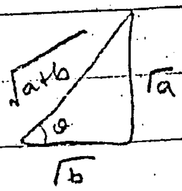
$$a^2\cos^4\theta = b^2\sin^4\theta$$

$$\frac{a^2}{b^2} = \frac{\sin^4\theta}{\cos^4\theta} \Rightarrow \tan^4\theta = \frac{a^2}{b^2}$$

$$\Rightarrow \tan^2\theta = \pm\frac{a}{b}$$

$$\Rightarrow \tan\theta = \pm\sqrt{\frac{a}{b}}$$

$\sin\theta = \frac{\sqrt{a}}{\sqrt{a+b}}$ $\cos\theta = \frac{\sqrt{b}}{\sqrt{a+b}}$.



$$\frac{dr}{d\theta} = \frac{r^3}{c^4}\left(\frac{a^2\cos^4\theta - b^2\sin^4\theta}{\sin^3\theta\cos^3\theta}\right)$$

$$\frac{d^2r}{d\theta^2} = \frac{d}{d\theta}\left[\frac{r^3}{c^4}\left(\frac{a^2\cos^4\theta - b^2\sin^4\theta}{\sin^3\theta\cos^3\theta}\right)\right]$$

$$= \frac{d}{d\theta}\left[\left(\frac{r^3}{c^4\sin^3\theta\cos^3\theta}\right)\left(a^2\cos^4\theta - b^2\sin^4\theta\right)\right]$$

$\frac{d}{d\theta}(u \cdot v) = uv' + vu'$

$$= \left(\frac{r^3}{c^4\sin^3\theta\cos^3\theta}\right)\left(a^2 4\cos^3\theta(-\sin\theta) - b^2 4\sin^3\theta(\cos\theta)\right)$$

$$+ \left(a^2\cos^4\theta - b^2\sin^4\theta\right)\frac{d}{d\theta}\left(\frac{r^3}{c^4\sin^3\theta\cos^3\theta}\right)$$

$$\therefore = \left(\frac{r^3}{c^4\sin^3\theta\cos^3\theta}\right)\left[-\left(4a^2\cos^3\theta\sin\theta + b^2 4\sin^3\theta\cos\theta\right)\right] + \cdots$$

$= -ve$

$\therefore$ when we put $\sin\theta = \frac{\sqrt{a}}{\sqrt{a+b}}$, $\cos\theta = \frac{\sqrt{b}}{\sqrt{a+b}}$ in $\frac{d^2r}{d\theta^2}$

it becomes $-ve$ as Ist part is $-ve$ clearly

and 2nd part involving $(a^2\cos^4\theta - b^2\sin^4\theta)$ becomes

zero as $\sin\theta = \frac{\sqrt{a}}{\sqrt{a+b}}$ & $\cos\theta = \frac{\sqrt{b}}{\sqrt{a+b}}$ have been derived

from $a^2\cos^4\theta - b^2\sin^4\theta = 0$

$\therefore$ r is max for all values of θ & maximum value of r is given by

$$\frac{c^4}{r^2} = \frac{a^2}{\frac{a}{a+b}} + \frac{b^2}{\frac{b}{a+b}}$$

$\therefore \sin^2\theta = \frac{a}{a+b}$

$\cos^2\theta = \frac{b}{a+b}$

$$\frac{c^4}{r^2} = \frac{a^2(a+b)}{a} + \frac{b^2(a+b)}{b}$$

$$\frac{c^4}{r^2} = a(a+b) + b(a+b) = (a+b)(a+b)$$

$$\frac{c^4}{(a+b)^2} = r^2 \Rightarrow r = \frac{c^2}{(a+b)}$$

Q11 Let $Q(a,b)$ be a pt on the line $2x - 7y + 5 = 0$ — ①

The distance of $Q(a,b)$ from origin is

$p = \sqrt{(a-0)^2+(b-0)^2} = \sqrt{a^2+b^2}$ — ②

Since $Q(a,b)$ lies on ① ∴ $2a - 7b + 5 = 0$ — ③

The pt $Q(a,b)$ is closest to origin if the distance p is minimum

from ③ $b = \frac{2a+5}{7}$ put in ②

$$p = \sqrt{a^2+\left(\frac{2a+5}{7}\right)^2} = \sqrt{\frac{49a^2+4a^2+25+20a}{49}} = \sqrt{\frac{53a^2+20a+25}{49}}$$

Squaring

$$49p^2 = 53a^2 + 20a + 25$$

Diff w.r.t 'a'

$$98p\frac{dp}{da} = 106a + 20$$ — ④

Put $\frac{dp}{da} = 0 \Rightarrow 106a + 20 = 0 \Rightarrow a = \frac{-20}{106} = \boxed{\frac{-10}{53}}$

Diff w.r.t 'a'

$$98\left(\frac{dp}{da}\right)\left(\frac{dp}{da}\right) + 98p\left(\frac{d^2p}{da^2}\right) = 106$$

$$98p\left(\frac{d^2p}{da^2}\right) = 106 - 98\left(\frac{dp}{da}\right)^2$$ {It is an eq free of 'a' & also tve. So where to put value of a.

∴ $\frac{d^2p}{da^2}$ is +ve at 'a' ∴ p is minimum at $a = \frac{-10}{53}$

To find 'b' $2a - 7b + 5 = 0 \Rightarrow 2\left(\frac{-10}{53}\right) + 5 = 7b \Rightarrow b = \frac{35}{53}$

∴ $Q\left(\frac{-10}{53}, \frac{35}{53}\right)$

Concavity

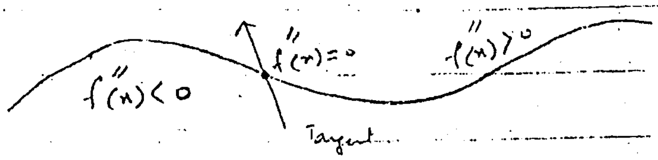
$y = f(x)$ is concave up if $f''(x) > 0$ — face up → graph of fn above tangents

$Y = f(x)$ is concave down if $f''(x) < 0$ — face down → graph of fn $Y = f(x)$ below Tangents

Point of Inflection

For pts of Inflection Put $y'' = 0$ and $y''' \neq 0$ at pts given by $y'' = 0$.

Put that pt given by $y'' = 0$ in $Y = f(x)$ we get Pt of Inflection



Q13. $y = \dfrac{x^3 - x}{3x^2 + 1}$ ——(1)

$$\frac{dy}{dx} = \frac{(3x^2+1)(3x^2-1) - (x^3-x)(6x)}{(3x^2+1)^2} = \frac{9x^4 - 1 - 6x^4 + 6x^2}{(3x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{3x^4 + 6x^2 - 1}{(3x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(3x^2+1)^2\left[12x^3 + 12x\right] - \left[3x^4 + 6x^2 - 1\right]\left[2(3x^2+1)6x\right]}{(3x^2+1)^4}$$

$$= \frac{(3x^2+1)}{(3x^2+1)^4}\left[(3x^2+1)(12x^3+12x) - (3x^4+6x^2-1)12x\right]$$

$$= \frac{1}{(3x^2+1)^3}(12x)\left[(3x^2+1)(x^2+1) - 3x^4 - 6x^2+1\right]$$

$$= \frac{12x}{(3x^2+1)^3}\left[\cancel{3x^4} + 3x^2 + x^2 + 1 - \cancel{3x^4} - 6x^2+1\right]$$

$$\frac{d^2y}{dx^2} = \frac{12x(-2x^2+2)}{(3x^2+1)^3} = \frac{24x(1-x^2)}{(3x^2+1)^3}$$

For pt of Inflection Put $\frac{d^2y}{dx^2} = 0$.

$$\frac{24x(1-x^2)}{(3x^2+1)^3} = 0$$

$$24x(1-x^2) = 0.$$

either $24x = 0 \Rightarrow x = 0$

or $1 - x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$$\frac{d}{d} = \frac{24x(1-x^2)}{(3x^2+1)^3} = \frac{24(x-x^3)}{(3x^2+1)^3}$$

$$\frac{d^3y}{dx^3} = \frac{(3x^2+1)^3\left[24(1-3x^2)\right] - 24(x-x^3)\,3(3x^2+1)^2(6x)}{(3x^2+1)^6}$$

$$= \frac{24(3x^2+1)^2}{(3x^2+1)^6}\left[(3x^2+1)(1-3x^2) - 18x(x-x^3)\right]$$

$\therefore Q\left(\frac{-10}{\sqrt{3}}, \frac{20}{3\sqrt{3}}\right)$

$$\frac{d^3y}{dx^3} = \frac{24\left[(1-9x^4) - 18x(x-x^3)\right]}{(3x^2+1)^4}$$

At $x=0$ $\quad \frac{d^3y}{dx^3} = \frac{24(1-0)-0}{(1)^4} = 24 \neq 0$

At $x = 1$ $\quad \frac{d^3y}{dx^3} = \frac{24\left[(1-9)-18(1)(1-1)\right]}{(3(1)^2+1)^4} \neq 0$

At $x = -1$ $\quad \frac{d^3y}{dx^3} = \frac{24\left[(1-9)-18(-1)(-1+1)\right]}{\left[3(1)+1\right]^4} \neq 0$

$\therefore$ the pts of Inflection are at $x = 0, \pm 1$

$$y = \frac{x^3 - x}{3x^2+1} \quad \text{--------} (1)$$

Put $x = 0$ in (1) $\quad y = \frac{0}{1} = 0$.

Put $x = 1$ in (1) $\quad y = \frac{1-1}{3+1} = \frac{0}{4} = 0$.

Put $x = -1$ in (1) $\quad y = \frac{-1-(-1)}{3+1} = \frac{0}{4} = 0$

$\therefore$ $(0,0)$ $(1,0)$ $(-1,0)$ are the pts of Inflection

curve $x = f(y)$
$\frac{d^2x}{dy^2} = 0$
& check $\frac{d^3x}{dy^2} \neq 0$

14) $x = (y-1)(y-2)(y-3)$

$x = (y^2-3y+2)(y-3)$

$= y^3 - 3y^2 - 3y^2 + 9y + 2y - 6$

$x = y^3 - 6y^2 + 11y - 6$ —— 1

$\frac{dx}{dy} = 3y^2 - 12y + 11$

$\frac{d^2x}{dy^2} = 6y - 12.$ $\frac{d^3x}{dy^3} = 6.$

Put $\frac{d^2x}{dy^2} = 0$ for point of Inflection

$6y - 12 = 0 \Rightarrow y = \frac{12}{6} = 2.$

$\frac{d^3x}{dy^3} = 6 \neq 0$ at $y = 2.$

Now in 1 : $y^3 - 6y^2 + 11y - 6.$ Put $y = 2$

$= (2)^3 - 6(2)^2 + 11(2) - 6$

$= 8 - 24 + 22 - 6$

$x = 0$

Pt of Inflection $(0, 2)$

$\therefore Q\left(\frac{-10}{\sqrt{3}}, \frac{20}{\sqrt{3}}\right)$

15 $y^2 = x(x+1)^2$ ... (1)

$y = \sqrt{x}(x+1) = x^{3/2} + x^{1/2}$

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2\sqrt{x}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}\cdot\frac{1}{2\sqrt{x}} + \frac{1}{2}\left(\frac{-1}{2}\right)\frac{1}{x^{3/2}}$$

$$= \frac{3}{4\sqrt{x}} - \frac{1}{4.x^{3/2}} \qquad = \frac{1}{4\sqrt{x}}\left[3 - \frac{1}{x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{4\sqrt{x}}\left[\frac{3x-1}{x}\right] = \frac{3x-1}{4x^{3/2}}$$

Put $\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{3x-1}{4x^{3/2}} = 0$

$3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

$$\frac{d^3y}{dx^3} = \frac{(4x^{3/2})(3) - (3x-1)\left(4\left(\frac{3}{2}\right)x^{\frac{1}{2}}\right)}{(4x^{3/2})^2} = \frac{12x^{3/2} - (3x-1)6x^{\frac{1}{2}}}{16x^3}$$

at $x = \frac{1}{3}$ $\quad \frac{d^3y}{dx^3} = \frac{12\left(\frac{1}{3}\right)^{3/2} - \left(3\left(\frac{1}{3}\right)-1\right)6\left(\frac{1}{3}\right)^{\frac{1}{2}}}{16\left(\frac{1}{3}\right)^3}$

$$= \frac{\frac{12}{3\sqrt{3}} - 0}{\frac{16}{27}} \neq 0$$

So pt of Inflection is at $x = \frac{1}{3}$

$$y^2 = x(x+1)^2 \quad \text{--- (1)}$$

$$y^2 = \frac{1}{3}\left(\frac{1}{3}+1\right)^2 \qquad \text{at } x = \frac{1}{3}$$

$$y^2 = \frac{1}{3}\left(\frac{4}{3}\right)^2$$

$$y^2 = \frac{1}{3}\,\frac{16}{9} = \frac{16}{27}$$

$$y = \pm\frac{4}{3\sqrt{3}}$$

$\therefore$ Pt of Inflection are $\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right)\left(\frac{1}{3}, -\frac{4}{3\sqrt{3}}\right)$

16. $a^2y^2 = x^2(a^2-x^2)$

$$y^2 = \frac{x^2}{a^2}(a^2-x^2) \implies y = \frac{x}{a}\sqrt{a^2-x^2} \quad \text{--- (1)}$$

$$\frac{dy}{dx} = \frac{1}{a}\sqrt{a^2-x^2} + \frac{x}{a}\left(\frac{1}{2}\right)\frac{(-2x)}{\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{a} - \frac{x^2}{a\sqrt{a^2-x^2}} = \frac{a^2-x^2-x^2}{a\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{a^2-2x^2}{a\sqrt{a^2-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{a\sqrt{a^2-x^2}(0-4x) - (a^2-2x^2)\frac{a\,(-2x)}{2\sqrt{a^2-x^2}}}{a^2(a^2-x^2)}$$

$\therefore$ (2) $\left(\frac{-10}{\sqrt{3}}, \frac{30}{\sqrt{3}}\right)$

$$= \frac{2a(a^2-x^2)(-4x) + 2ax(a^2-2x^2)}{2a^2(a^2-x^2)^{3/2}}$$

$$= \frac{-8a^3x + 8ax^3 + 2a^3x - 4ax^3}{2a^2(a^2-x^2)^{3/2}} = \frac{4ax^3 - 6a^3x}{2a^2(a^2-x^2)^{3/2}}$$

$$= \frac{\cancel{2a}x(2x^2-3a^2)}{\cancel{2}a^{\cancel{2}}(a^2-x^2)^{3/2}} = \frac{x(2x^2-3a^2)}{a(a^2-x^2)^{3/2}}$$

Put $\frac{d^2y}{dx^2} = 0$.

$$\frac{x(2x^2-3a^2)}{a(a^2-x^2)^{3/2}} = 0$$

$$x(2x^2-3a^2) = 0.$$

either $\boxed{x=0}$ or $2x^2-3a^2=0$

$$x^2 = \frac{3a^2}{2}$$

$$\boxed{x = \pm\sqrt{\frac{3}{2}}\,a}$$

$$\frac{d^2y}{dx^2} = \frac{(2x^3-3xa^2)}{a(a^2-x^2)^{3/2}}$$

$$\frac{d^3y}{dx^3} = \frac{a(a^2-x^2)^{3/2}[6x^2-3a^2] - [2x^3-3xa^2]a\frac{3}{\cancel{2}}(a^2-x^2)^{\frac{1}{2}}(-\cancel{2}x)}{a^2(a^2-x^2)^3}$$

$$= \frac{(a^2-x^2)^{\frac{1}{2}}\, a\, \left[(a^2-x^2)(6x^2-3a^2) + 3x(2x^3-3xa^2)\right]}{a^2(a^2-x^2)^3}$$

$$\frac{d^3y}{dx^3} = \frac{\left[(a^2-x^2)(6x^2-3a^2) + 3x(2x^3-3xa^2)\right]}{a(a^2-x^2)^{\frac{5}{2}}}$$

At $x = 0$ $\quad \frac{(a^2)(-3a^2) + 0}{a^6} = \frac{-3}{a} \neq 0.$

At $x = \pm\sqrt{\frac{3}{2}}\,a$ $\quad \frac{d^3y}{dx^3} = \frac{\left(a^2-\left(\frac{3}{2}\right)a^2\right)\left(6\left(\frac{3}{2}\right)a^2 - 3a^2\right) + 3\sqrt{\frac{3}{2}}\,a\left(2\left(\frac{3}{2}\right)^{\frac{3}{2}}a^3 - 3\sqrt{\frac{3}{2}}\,a^3\right)}{a\left(a^2-\left(\frac{3}{2}\right)a^2\right)^{\frac{5}{2}}}$

$\neq 0.$

So pts of inflection are at $0, \pm\sqrt{\frac{3}{2}}\,a.$

From ① at $x = 0$ $\quad y = 0.$

From ① at $x = \pm\sqrt{\frac{3}{2}}\,a$ $\quad y = \sqrt{\frac{3}{2}}\frac{a}{a}\sqrt{a^2 - \frac{3a^2}{2}}$

$$Y = \frac{\sqrt{3}}{\sqrt{2}}\, a\sqrt{\frac{2-3}{2}}$$

$\frac{d}{dx}$ $\quad Y = \frac{a\sqrt{3}}{2}\left(\sqrt{-1}\right) = \frac{a\sqrt{3}}{2}\, i$

∵ imaginary so neglect it

The only pt of inflection is $(0, 0)$

$\therefore$ (2) $\left(\frac{-16}{\sqrt{3}}, \frac{32}{3\sqrt{3}}\right)$

7·2-22

Q17 $f(x) = ax^3 + bx^2$ ——— ①

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since $(1,6)$ is a pt of Inflection.

$f''(1) = 6a + 2b$

Put $f''(1) = 0$. $6a + 2b = 0$

$2b = -6a$

$\boxed{b = -3a}$ ——— ②

and $f(1) = 6 = a + b$ ——— ③ from ①

from ② & ③ $6 = a + (-3a)$

$6 = -2a$ $\Rightarrow \boxed{a = -3}$ Put in ②

$b = -3(-3) = \boxed{9}$

Q18 $y = 3x^5 - 40x^3 + 3x - 20$ ——— ①

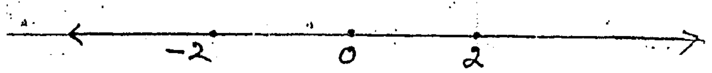
$\frac{dy}{dx} = 15x^4 - 120x^2 + 3$

$\frac{d^2y}{dx^2} = 60x^3 - 240x. = 60x(x^2 - 4)$

$\frac{d^2y}{dx^2} = 0.$ $60x(x^2 - 4) = 0.$ $x = 0,$ $x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$



$x = 0, \pm 2.$

Consider the intervals $]-\infty,-2[\,;\,]-2,0[\,;\,]0,2[\,;\,]2,\infty[$

i) for $x \in]-\infty,-2[$ Let x be -5.

$y'' = 60x(x^2-4) = 60(-5)(25-4) = -ve$

y'' is $-ve$ so curve faces downward in $]-\infty,-2[$

ii) for $x \in]-2,0[$ Let x be -1

$y'' = 60x(x^2-4) = 60(-1)(1-4) = +ve$

y'' is $+ve$ so curve faces upward in $]-2,0[$

iii) for $x \in]0,2[$ Let x be 1

$y'' = 60x(x^2-4) = 60(1)(1-4) = -ve$

y'' is $-ve$ So curve faces downward in $]0,2[$

iv) for $x \in]2,\infty[$ Let x be 4.

$y'' = 60x(x^2-4) = 60(4)(16-4) = +ve$

y'' is $+ve$ So curve faces upward in $]2,\infty[$

Now for pts of Inflection.

$\frac{d^3y}{dx^3} = 180x^2 - 240$ $\quad \therefore \frac{d^2y}{dx^2} = 60x^3 - \ldots$

at $x=0$ $\frac{d^3y}{dx^3} = -240 \neq$ $\therefore$ Pt of Inflection is at $x=0$

at $x=2$ $\frac{d^3y}{dx^3} = 720-240 \neq 0$ $\therefore$ Pt of Inflection is at $x=2$.

at $x=-2$ $\frac{d^3y}{dx^3} = 720-240 \neq 0$ $\therefore$ Pt of Inflection is at $x=2$.

Put $x=0$ in (1) we get $y=-20$. So $(0,-20)$ is Pt of Inflection

Put $x=2$ in (1) we get $y = 96-320+6-20 = -238$ So $(2,-238)$ is Pt of Infl

Put $x=-2$ in (1) we get $y = -96+320-6-20 = 198$ So $(-2,198)$ is Pt of Inflection

$\alpha\left(\frac{[illegible]}{\sqrt{3}}, \frac{20}{\sqrt{3}}\right)$

Q No. 19 $y = (x^2+4x+5)e^{-x}$ ①

$$\frac{dy}{dx} = (2x+4)e^{-x} - (x^2+4x+5)e^{-x}.$$

$$\frac{d^2y}{dx^2} = 2e^{-x} - e^{-x}(2x+4) + e^{-x}(x^2+4x+5) - (2x+4)e^{-x}$$

$$\frac{d^2y}{dx^2} = e^{-x}\left[2-2x-4+x^2+4x+5-2x-4\right] = e^{-x}\left[x^2-1\right]$$

$\therefore \frac{d^2y}{dx^2} = 0$. $e^{-x} \neq 0$. So. $x^2-1=0$ $x = \pm 1$

$$\longleftarrow -1 \quad 1 \longrightarrow$$

Consider the intervals $]-\infty, -1[$; $]-1, 1[$; $]1, \infty[$

(i) for $x \in]-\infty, -1[$ Let x be -3

$$\frac{d^2y}{dx^2} = e^{-x}[x^2-1] = e^{+3}[9-1] = +ve$$

$\therefore$ curve faces upward

(ii) for $x \in]-1, 1[$ Let x be 0.

$\frac{d^2y}{dx^2} = e^{-0}[0-1] = -ve$ $\therefore$ curve faces downward.

(iii) for $x \in]1, \infty[$ Let x be 4.

$\frac{d^2y}{dx^2} = e^{-4}[16-1] = +ve$ $\therefore$ curve faces upward.

Now for pt of Inflection.

$$\frac{d^3y}{dx^3} = e^{-x}[2x] - e^{-x}[x^2-1] = e^{-x}[2x-x^2+1]$$

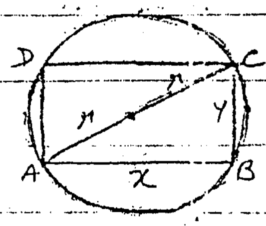
At $x=1$ $\quad \frac{d^3y}{dx^3} = e^{-1}[2-1+1] \neq 0$. Pt of Inflection

At $x=-1$ $\quad \frac{d^3y}{dx^3} = e^{1}[-2-1+1] \neq 0$. Pt of Inflection is at $x=-1$

Put $x=1$ in ① $\quad y=(1+4+5)e^{-1} = \frac{10}{e}$, $\therefore (1, \frac{10}{e})$ is pt of Inflection

Put $x=-1$ in ① $\quad y=(1-4+5)e^{1} = 2e$, $\therefore (-1, 2e)$ is pt of Inflection

Q20. Let x, y be the dimensions of the rectangle inscribed in a circle of radius r, then by geometry $x^2+y^2=(2r)^2=4r^2$



$$|AB|^2+|BC|^2=|AC|^2$$
$$x^2+y^2=(2r)^2$$

$$y^2 = 4r^2 - x^2$$

$$y = \sqrt{4r^2-x^2} \quad \text{——— ①}$$

Area of the rectangle $= xy$

$\therefore$ Area $A = x\sqrt{4r^2-x^2}$ ——— ② using ①.

$$\frac{dA}{dx} = \sqrt{4r^2-x^2} + x\left(\frac{1}{2}\right)\frac{(-2x)}{\sqrt{4r^2-x^2}}$$

$$\frac{dA}{dx} = \sqrt{4r^2-x^2} - \frac{x^2}{\sqrt{4r^2-x^2}} = \frac{4r^2-x^2-x^2}{\sqrt{4r^2-x^2}} = \frac{4r^2-2x^2}{\sqrt{4r^2-x^2}}$$

Put $\frac{dA}{dx}=0$ (for relative extrema).

$$\frac{4r^2-2x^2}{\sqrt{4r^2-x^2}} = 0 \implies 4r^2-2x^2=0 \implies x^2 = \frac{4}{2}r^2$$

$$x = \pm\sqrt{2}\, r$$

$$\frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

$$\frac{d^2A}{dx^2} = \frac{\sqrt{4r^2-x^2}\,[-4x] - [4r^2-2x^2]\left[\frac{1}{\cancel{2}\sqrt{4r^2-x^2}}(-\cancel{2}x)\right]}{\left(\sqrt{4r^2-x^2}\right)^2}$$

$$= \frac{-4x(4r^2-x^2) + x(4r^2-2x^2)}{\sqrt{(4r^2-x^2)}\,(4r^2-x^2)}$$

At $x = \sqrt{2}\,r$. $\frac{d^2A}{dx^2} = \frac{-4\sqrt{2}\,r\,(4r^2-2r^2) + \sqrt{2}r\,(4r^2 - 2(2r^2))}{\sqrt{4r^2-2r^2}\,(4r^2-2r^2)}$

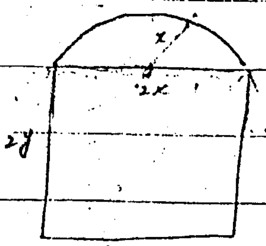
$$= \frac{-4(\sqrt{2}\,r)(2r^2) + 0}{(\sqrt{2}\,r)(2r^2)} = -4 < 0$$

$\therefore$ A is max for $x = \sqrt{2}\,r$.

from ① $y = \sqrt{4r^2 - 2r^2} = \sqrt{2}\,r$.

Thus the rectangle of Max area is a square of side $\sqrt{2}\,r$

Q21. Let $2x$, $2y$ be the dimensions of the rectangle then radius of the semi-circle $= x$.



Perimeter $m = 2x + 2x + 2y + 2y + \frac{1}{2}(2\pi x)$

$m = 4x + 4y + \pi x$.

Circumference of $\odot = 2\pi r$
Area of $\odot = \pi r^2$

$$y = \frac{m - 4x - \pi x}{4} \quad \text{①}$$

Area of the window $= (2x)(2y) + \frac{\pi x^2}{2}$.

$$A = 4xy + \frac{\pi x^2}{2} = 4x\left(\frac{m - 4x - \pi x}{4}\right) + \frac{\pi x^2}{2}$$ using ①.

$A = mx - 4x^2 - \pi x^2 + \frac{\pi x^2}{2}$

$A = mx - 4x^2 - \frac{\pi x^2}{2}$

$\frac{dA}{dx} = m - 8x - \frac{2\pi x}{2} = m - 8x - \pi x$.

Put $\frac{dA}{dx} = 0$ For relative extrema.

$m - 8x - \pi x = 0 \Rightarrow m - (8+\pi)x = 0$.

$x = \frac{m}{8+\pi}$

$\frac{d^2A}{dx^2} = -8 - \pi$

At $x = \frac{m}{8+\pi}$ $\frac{d^2A}{dx^2}$ is -ve < 0 $\therefore$ A is max for $x = \frac{m}{8+\pi}$

from ① $Y = \frac{m - 4x - \pi x}{4} = \frac{m - (4+\pi)x}{4}$

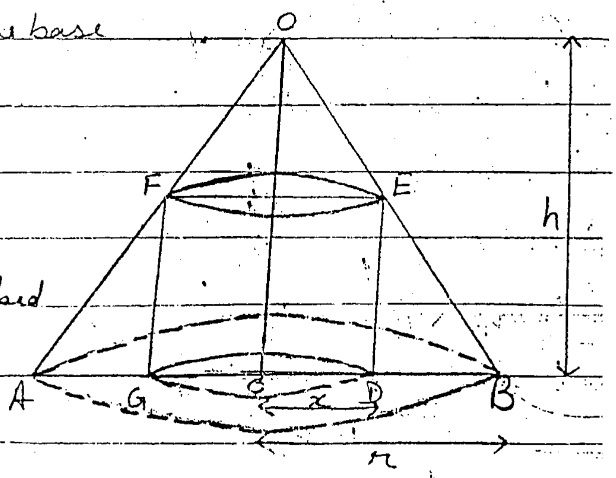
$= \frac{m - (4+\pi)\left(\frac{m}{8+\pi}\right)}{4} = \frac{m(8+\pi) - (4+\pi)m}{4(8+\pi)}$

$Y = \frac{8m + \cancel{\pi m} - 4m - \cancel{\pi m}}{4(8+\pi)} = \frac{4m}{4(8+\pi)} = \frac{m}{8+\pi}$

Dimensions of the window of max area $= 2x, 2y$

$= \frac{2m}{8+\pi}, \frac{2m}{8+\pi}$

Q22. Let 'r' be the radius of the base of the cone & 'h' be its height



$\left.\begin{array}{l} CB = AC = r. \\ OC = h \end{array}\right\}$ given

DEFG is the cylinder inscribed in the cone. Radius of base of cylinder is 'x' $CD = x$

Height of the cylinder $ED = ?$

Δs OCB & EDB are similar.

$$\frac{ED}{DB} = \frac{OC}{CB} \qquad \because \frac{ED}{DB} = \frac{OC}{CB} = \tan\theta$$

$$ED = \frac{(OC)(DB)}{CB} \Rightarrow ED = \frac{h(r-x)}{r}$$

S = Curved Surface Area of cylinder $= 2\pi \times \text{radius} \times \text{height}$

$= 2\pi \times CD \times ED$

$$\therefore S = 2\pi x \left(\frac{h(r-x)}{r}\right) = \frac{2\pi h}{r}(rx - x^2)$$

$$\therefore \frac{dS}{dx} = \frac{2\pi h}{r}(r - 2x)$$

Put $\frac{dS}{dx} = 0 \Rightarrow \frac{2\pi h}{r}(r-2x) = 0 \Rightarrow r - 2x = 0$

$\Rightarrow x = \frac{r}{2}$

$$\frac{d^2S}{dx^2} = \frac{2\pi h}{r}(0-2) = -\frac{4\pi h}{r} < 0$$

At $x = \frac{r}{2}$, $\frac{d^2S}{dx^2}$ is < 0 Hence S is max at $x = \frac{r}{2}$.

Q23 Let ABCD be a cylinder inscribed in sphere of radius 'r'.

Let $\angle BAC = \theta$ $\quad AC = 2r$.

From ΔABC $\quad \dfrac{AB}{AC} = \cos\theta$.

$$AB = AC\cos\theta = 2r\cos\theta.$$

$$\frac{BC}{AC} = \sin\theta \Rightarrow BC = 2r\sin\theta.$$ BC is height of cylinder.

Radius of the base of the cylinder $= \dfrac{AB}{2} = \dfrac{2r\cos\theta}{2} = r\cos\theta$

Total Surface Area of Cylinder = Curved Surface Area + Circular Area of lower base + Circular Area of top

$$= 2\pi(\text{radius})(\text{height}) + \pi(\text{radius})^2 + \pi(\text{radius})^2.$$

$$= 2\pi(r\cos\theta)(2r\sin\theta) + \pi(r\cos\theta)^2 + \pi(r\cos\theta)^2$$

$$= 4\pi r^2\cos\theta\sin\theta + 2\pi r^2\cos^2\theta.$$

$$= 2\pi r^2(2\sin\theta\cos\theta) + 2\pi r^2\cos^2\theta$$

$$S = 2 r^2[\sin 2\theta + \cos^2\theta] \quad \text{(1)}$$

$$\frac{dS}{d\theta} = 2\pi r^2(\cos 2\theta(2) + 2\cos\theta(-\sin\theta))$$

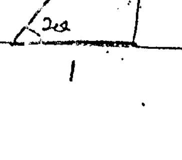
$$= 2\pi r^2(2\cos 2\theta - \sin 2\theta) \quad \text{(2)}$$

Put $\dfrac{dS}{d\theta} = 0$ For relative Extrema.

$$2\pi r^2(2\cos 2\theta - \sin 2\theta) = 0.$$

$$2\cos 2\theta - \sin 2\theta = 0.$$

$$2 = \tan 2\theta$$



$$\sin 2\theta = \frac{2}{\sqrt{5}} \qquad \cos 2\theta = \frac{1}{\sqrt{5}}$$

$$\frac{d^2S}{d\theta^2} = 2\pi r^2(2(-\sin 2\theta)2 - \cos 2\theta(2)) \quad \text{Diff (2)}$$

$$\frac{d^2S}{d\theta^2} = 2\pi r^2(-4\sin 2\theta - 2\cos 2\theta) = -4\pi r^2(2\sin 2\theta + \cos 2\theta)$$

At $\left.\begin{array}{l}\sin 2\theta = \frac{2}{\sqrt{5}} \\ \cos 2\theta = \frac{1}{\sqrt{5}}\end{array}\right\}$ $\quad \frac{d^2S}{d\theta^2} = -4\pi r^2\left(2\,\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}\right) < 0$

$\therefore$ S is max (greatest) at $\sin 2\theta = \frac{2}{\sqrt{5}}$ & $\cos 2\theta = \frac{1}{\sqrt{5}}$

Now from (1) $\quad S_{Max} = 2\pi r^2 \sin 2\theta + 2\pi r^2 \cos^2\theta$

$$= 2\pi r^2 \sin 2\theta + \cancel{2}\pi r^2\left(\frac{1+\cos 2\theta}{\cancel{2}}\right)$$

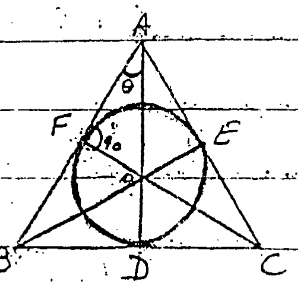
$$= 2\pi r^2\left(\frac{2}{\sqrt{5}}\right) + \pi r^2\left(1 + \frac{1}{\sqrt{5}}\right)$$

$$= \pi r^2\left(\frac{4}{\sqrt{5}} + \frac{\sqrt{5}+1}{\sqrt{5}}\right)$$

Greatest Surface $= S_{Max} = \pi r^2\left(\frac{5+\sqrt{5}}{\sqrt{5}}\right) = \pi r^2(\sqrt{5}+1)$

(1989)

Q.24. Let r be the radius of the circle inscribed in an isoceles $\triangle$



$|AF| = ?$

$\triangle AFO$ is rt$\triangle$ $\quad OF = r$, $OA = x$

$|AF|^2 + |OF|^2 = |AO|^2$

$|AF|^2 + r^2 = x^2$

$|AF| = \sqrt{x^2 - r^2}$

Also in rt $\triangle$ ADB $\angle BAD = \theta$

$$\frac{|BD|}{|AD|} = \frac{|OF|}{|AF|}$$ $\because \frac{|BD|}{|AD|} = \tan\theta = \frac{|OF|}{|AF|}$

$$|BD| = |AD| \frac{|OF|}{|AF|}$$

$$|BD| = (x+r)\frac{r}{\sqrt{x^2-r^2}}$$

Perimeter $= \cancel{AB}+AC+BC$ $\because AB = AC$ $\because$ Isoceles $\triangle$

$$= 2AB + BC$$

$$\cancel{= 2AB + 2BD}$$

$$= 2[BF+AF] + 2BF \qquad BF = BD$$

$$P = 4BF + 2AF$$

$$P = 4(r+x)\frac{r}{\sqrt{x^2-r^2}} + 2\sqrt{x^2-r^2} \qquad \because BF = BD = \frac{(r+x)r}{\sqrt{x^2-r^2}}$$

$$= \frac{4(r+x)r + 2(x^2-r^2)}{\sqrt{x^2-r^2}} = \frac{4r^2+4xr+2x^2-2r^2}{\sqrt{x^2-r^2}}$$

$$= \frac{2r^2+2x^2+4xr}{\sqrt{x^2-r^2}} = \frac{2(x^2+r^2+2xr)}{\sqrt{x+r}\sqrt{x-r}}$$

$$P = \frac{2(x+r)^2}{\sqrt{x+r}\sqrt{x-r}} = \frac{2(x+r)^{3/2}}{(x-r)^{1/2}} \quad (1)$$

$$\frac{dP}{dx} = \frac{(x-r)^{\frac{1}{2}}\, \cancel{2}\left(\frac{3}{2}\right)(x+r)^{\frac{1}{2}} - \cancel{2}(x+r)^{3/2}\frac{1}{\cancel{2}(x-r)^{1/2}}}{(x-r)}$$

$$\frac{3(x-r)(x+r)^{\frac{1}{2}} - (x+r)^{3/2}}{(x-r)^{3/2}}$$

$$= \frac{(x+r)^{\frac{1}{2}}}{(x-r)^{3/2}}\left[3(x-r) - (x+r)\right]$$

$$\frac{dP}{dx} = \frac{(x+r)^{1/2}}{(x-r)^{3/2}}(2x-4r)$$

$\therefore$ Put $\frac{dP}{dx} = 0$. $\quad \frac{(x+r)^{\frac{1}{2}}(2x-4r)}{(x-r)^{3/2}} = 0$

$$(x+r)^{\frac{1}{2}}(2x-4r) = 0.$$

$\Rightarrow \sqrt{x+r} = 0$. $\quad$ or $\quad 2x - 4r = 0$

$\boxed{x = 2r.}$

$\Rightarrow x + r = 0$

$\Rightarrow x = -r$. Not possible

$\because$ radius is non-negative

$$\frac{d^2P}{dx^2} = \frac{(x-r)^{3/2}\left[\frac{1}{2}(x+r)^{-\frac{1}{2}}(2x-4r) + (x+r)^{\frac{1}{2}}(2x)\right] - \left[(x+r)^{\frac{1}{2}}(2x-4r)\right]\frac{3}{2}(x-r)^{\frac{1}{2}}}{(x-r)^3}$$

at $x = 2r$. $\quad \frac{d^2P}{dx^2} = \frac{(r)^{3/2}\left[0 + (3r)^{1/2}(4r)\right] - 0}{r^3} > 0$

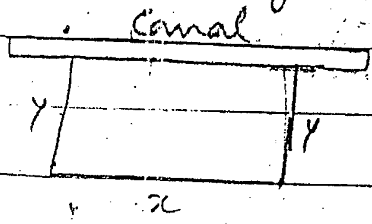
$\therefore$ Perimeter is least (min) for $x = 2r$.

$$P_{least} = \frac{2(2r+r)^{3/2}}{(2r-r)^{\frac{1}{2}}} = \frac{2(3r)^{3/2}}{r^{1/2}} = 2(3)^{3/2} r^{\frac{3}{2}-\frac{1}{2}} = 2\cdot 3\cdot\sqrt{3}\, r$$

$$= 6\sqrt{3}\, r \text{ Ans.}$$

Q26. Let x, y be the dimensions of the rectangular fi[eld]

s.t $x + 2y = 1000m$



$x = 1000 - 2y$ —— ①

Area 'A' $= xy = (1000 - 2y)y$

$A = 1000y - 2y^2$

$\frac{dA}{dy} = 1000 - 4y$

Put $\frac{dA}{dy} = 0 \Rightarrow 1000 - 4y = 0$

$\frac{1000}{4} = y \Rightarrow y = 250m$

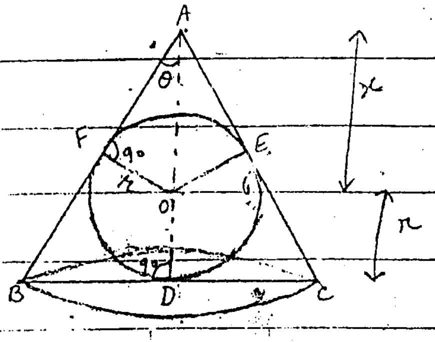
and $x = 1000 - 2(250)$

$x = 500m$

$\frac{d^2A}{dy^2} = -4 < 0$

$\Rightarrow$ Area is max

Dimensions are $500 \times 250m$



Q25 Let r be the radius of the sphere and ABC be a cone circumscribing the sphere s.t. its vertex A is at a distance x from the centre. Let θ be the semi-vertical angle of the cone of radius $BD = DC$.

Height of Cone $= AD = AO + OD = x + r$

$BD = ?$ In $\Delta\, AFO$ $\quad |AF|^2 + |OF|^2 = |AO|^2$

$$|AF|^2 = |AO|^2 - |OF|^2$$

$$|AF|^2 = x^2 - r^2$$

$$AF = \sqrt{x^2 - r^2}$$

Also $\dfrac{|BD|}{|AD|} = \dfrac{|OF|}{|AF|}$ $\qquad \because \dfrac{BD}{AD} = \dfrac{OF}{AF} = \tan\theta$

$$BD = |AD|\frac{|OF|}{|AF|}$$

$$= (x+r)\frac{r}{\sqrt{x^2-r^2}} = \frac{\sqrt{x+r}}{\sqrt{x-r}} \cdot r$$

V = Volume of the cone $= \frac{1}{3}\pi(\text{radius of cone})^2(\text{height})$

$$= \frac{1}{3}\pi |BD|^2 |AD|$$

$$V = \frac{1}{3}\pi\left[\frac{r\sqrt{x+r}}{\sqrt{x-r}}\right]^2(x+r) = \frac{1}{3}\pi r^2\frac{(x+r)^2}{(x-r)} \quad \text{--- (1)}$$

$$\frac{dv}{dx} = \frac{1}{3}\pi r^2\left[\frac{(x-r)\,2(x+r) - (x+r)^2}{(x-r)^2}\right]$$

$$= \frac{1}{3}\pi r^2 \frac{(x+r)[2x-2r-x-r]}{(x-r)^2}$$

$$= \frac{1}{3}\pi r^2 \frac{(x+r)(x-3r)}{(x-r)^2}$$

Put $\frac{dv}{dx} = 0$. $\Rightarrow \frac{1}{3}\pi r^2 \frac{(x+r)(x-3r)}{(x-r)^2} = 0$

$\Rightarrow (x+r)(x-3r) = 0$

$x + r = 0$ $\quad$ $x - 3r = 0$

$x = -r$ (Radius is never –ive) $\quad$ $\boxed{x = 3r}$

$$\frac{d^2v}{dx^2} = \frac{1}{3}\pi r^2 \left(\frac{(x-r)^2\{(x+r)+(x-3r)\} - \{(x+r)(x-3r)\}2(x-r)}{(x-r)^4} \right)$$

$$\frac{d^2v}{dx^2} = \frac{1}{3}\pi r^2 \frac{(3r-r)^2\{(3r+r)+0\} - 0}{(3r-r)^2} > 0$$

∴ volume is lea at $x = 3r$.

Radius of the co $= BD = \frac{r\sqrt{x+r}}{\sqrt{x-r}} = \frac{r\sqrt{3r+r}}{\sqrt{3r-r}} = \frac{2r\sqrt{r}}{\sqrt{2r}}$

radius $= \sqrt{2}\, r$.

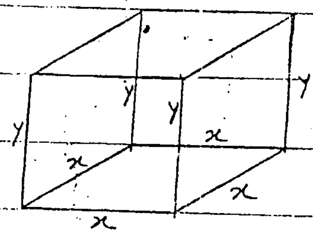
Altitude of the cone $= AD = x + r = 3r + r = 4r$

Semi Vertical angle $\theta = ?$ $\quad$ From ΔOAF

$$\sin\theta = \frac{OF}{OA} = \frac{r}{x} = \frac{r}{3r} = \frac{1}{3}$$

$$\theta = \sin^{-1}\left(\frac{1}{3}\right) \text{ Ans.}$$

Q27. Let x be the side of the square base of open box, 'y' be its height



$$\text{volume} = (x)(x)(y)$$

$$1296 = x^2 y.$$

$$\frac{1296}{x^2} = y \quad \text{——— (1)}$$

$$\text{Area of base} = x^2 \text{ sq cm.}$$

$$C_1 = \text{Cost of base} = 3x^2. \quad \text{——— (2)}$$

$$\text{Area of the sides} = 4xy$$

$$C_2 = \text{Cost of sides} = 2(4xy) = 8xy \quad \text{——— (3)}$$

$$\text{Total Cost} = C_1 + C_2 = 3x^2 + 8xy.$$

$$C = 3x^2 + 8x\left(\frac{1296}{x^2}\right) = 3x^2 + \frac{10368}{x}$$

$$\frac{dc}{dx} = 6x - \frac{10368}{x^2}$$

Put $\frac{dc}{dx} = 0$ $\quad 6x - \frac{10368}{x^2} = 0.$

$$\Rightarrow \frac{6x^3 - 10368}{x^2} = 0 \Rightarrow 6x^3 - 10368 = 0$$

$$\Rightarrow x^3 = \frac{10368}{6} = 1728 = (12)^3$$

$$\Rightarrow x = 12.$$

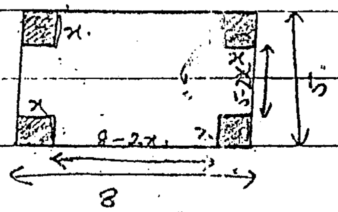
$$\frac{d^2c}{dx^2} = 6 + \frac{2(10368)}{x^3}$$

at $x = 12$ $\quad \frac{d^2c}{dx^2} = 6 + \frac{2(10368)}{(12)^3} > 0$ $\quad \therefore$ Cost is min for $x = 12$

$\therefore\; y = \dfrac{1296}{(12)^2} = \dfrac{1296}{144} = 9$ cms.

Side of the base $= 12$ cm & height of the box $= 9$ cm

Q28. Let x be the height of square;



Dimension of sheet is 8 by 5

Length $= 8-2x$.

Breadth $= 5-2x$.

height $= x$.

Volume $=$ Length $\times$ Breadth $\times$ height

$V = (8-2x)(5-2x)x$.

$V = (40-26x+4x^2)x = 4x^3-26x^2+40x$.

$\dfrac{dv}{dx} = 12x^2-52x+40$

$\dfrac{dv}{dx} = 0$. $\quad 12x^2-52x+40=0$.

$4(3x^2-13x+10)=0$

$3x^2-13x+10=0$.

$x = \dfrac{13 \pm \sqrt{169-4\cdot3\cdot10}}{6} = \dfrac{13 \pm \sqrt{169-120}}{6}$

$x = \dfrac{13 \pm \sqrt{49}}{6} = \dfrac{13 \pm 7}{6} = \dfrac{13+7}{6} = \dfrac{10}{3}$ & $\dfrac{13-7}{6} = 1$

$\therefore x = \dfrac{10}{3}, 1$

$\dfrac{d^2v}{dx^2} = 24x-52$.

At $x=\frac{10}{3}$ $\frac{d^2v}{dx^2}=24\left(\frac{10}{3}\right)-52 > 0$ Volume is Min

At $x=1$ $\frac{d^2v}{dx^2}=24(1)-52 < 0$ Volume is Max at $x=1$

Fair	Passenger	Revenue
200 Paisa	1200	200×1200
(200−1) Paisa	1200+10	$(200-1)\times(1200+10)$
(200−2) Paisa	1200+2(10)	$(200-2)\times(1200+2(10))$
(200−3) Paisa	1200+3(10)	$(200-3)\times(1200+3(10))$
(200−x) Paisa	1200+x(10)	$(200-x)\times(1200+x(10))$
$\left(2-\frac{x}{100}\right)$ Rs		$\left(2-\frac{x}{100}\right)\times(1200+x(10))$
		$2400+20x-12x-\frac{x^2}{10}$
		$2400+8x-\frac{x^2}{10}$

$R=2400+8x-\frac{x^2}{10}$

$\frac{dR}{dx}=8-\frac{2x}{10}=8-\frac{x}{5}$

$\frac{dR}{dx}=8-\frac{x}{5}$

$0=8-\frac{x}{5}$

$\frac{x}{5}=8 \Rightarrow x=40$

$\frac{d^2R}{dx^2}=-\frac{1}{5}<0$ ∴ R is maximum for $x=40$

For max Revenue Fare should be $2-\frac{40}{100}$

$=\frac{200-40}{100}$

$=\frac{160}{100}=1.60$ Rs.

Q30 Present | Wt of cattle | Profit/quintal | Total Profit
Present | 200 quintals | 500 Rs | $(200)(500)$
after 1 week | $(200+5)$ | $(500-10)$ Rs | $(200+5)(500-10)$
after 2 week | $(200+2(5))$ | $(500-2(10))$ Rs | $(200+2(5))(500-2(10))$
after 3 week | $(200+3(5))$ | $(500-3(10))$ Rs | $(200+3(5))(500-3(10))$

after 'x' week | $(200+x(5))$ quintal | $(500-x(10))$ Rs | $(200+x(5))(500-x(10))$

$= 100000 - 2000x + 2500x - 50x^2$

$100000 + 500x - 50x^2$

$$P = 100000 + 500x - 50x^2$$

$$\frac{dP}{dx} = 500 - 100x \quad \text{——(1)}$$

$$0 = 500 - 100x$$

$$x = \frac{500}{100} = 5$$

diff (1)

$$\frac{d^2P}{dx^2} = -100 < 0$$

$\therefore$ at $x = 5$ i.e after 5 weeks cattle should be sold to get maximum profit.