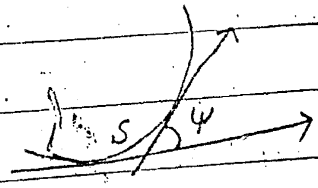
Curvature and Radius of Curvature 7.7

Rate of change of ψ with respect to S is called curvature K.



$$K = \left|\frac{d\psi}{dS}\right|$$

S is the arc length

ψ is the inclination of the tangent to the curve.

The reciprocal of curvature is called Radius of Curvature denoted by ρ

$$K = \frac{1}{\rho} \quad \text{or} \quad \rho = \frac{1}{K}$$

1) When the eq of the curve is given in cartesian form

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \qquad K = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

2) When the curve is given in Implicit form.

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{\left|(f_y)^2(f_{xx}) - 2f_x f_y f_{xy} + (f_x)^2 f_{yy}\right|} \qquad K = \frac{1}{\rho}$$

3) When the curve is given in Parametric form.

$$\rho = \frac{\left[(f')^2 + (g')^2\right]^{3/2}}{|f'g'' - g'f''|} \qquad \rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\cdot\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right|}$$

$$K = \frac{|f'g'' - g'f''|}{\left[(f')^2 + (g')^2\right]^{3/2}}$$

When the eq of curve is given in Polar form.

$$\rho = \frac{\left(r^2+(r')^2\right)^{3/2}}{\left(r^2+2(r')^2-rr''\right)} \quad ; \quad r' = \frac{dr}{d\theta}, \quad r'' = \frac{d^2r}{d\theta^2}$$

$$K = \frac{1}{\rho}$$

Also,

$$\rho = r\frac{dr}{dp} \quad , \quad \text{where } p = r\sin\phi$$

$$K = \frac{1}{r}\frac{dp}{dr}.$$

Ex 7.7

(1) $y = c\cosh\frac{x}{c}$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$\frac{dy}{dx} = \cancel{c}\sinh\frac{x}{c}\cdot\frac{1}{\cancel{c}}$$

$$\frac{d^2y}{dx^2} = \cosh\frac{x}{c}\,\frac{1}{c}$$

$$= \frac{\left[1+\sinh^2\frac{x}{c}\right]^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$\because \cosh^2\frac{x}{c} - \sinh^2\frac{x}{c} = 1$$

$$= \frac{\left[\cosh^{\cancel{2}}\frac{x}{c}\right]^{3/\cancel{2}}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\left(\cosh\frac{x}{c}\right)^3}{\frac{1}{c}\cosh\frac{x}{c}} = c\left(\cosh\frac{x}{c}\right)^2$$

but $\frac{y}{c} = \cosh\frac{x}{c}$

$\therefore\ \rho = c\,\frac{y^2}{c^2} = \frac{y^2}{c}$ Ans.

(2) $x = a(\cos t + t\sin t)$ $\quad y = a(\sin t - t\cos t)$

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right|}$$

$\frac{dx}{dt} = a(\cancel{-\sin t} + \cancel{\sin t} + t\cos t)$

$= at\cos t$

$\frac{d^2x}{dt^2} = a\cos t + at(-\sin t)$

$= a(\cos t - t\sin t)$

$\frac{dy}{dt} = a(\cancel{\cos t} - \cancel{\cos t} - t(-\sin t))$

$= at\sin t$

$\frac{d^2y}{dt^2} = a\sin t + at\cos t.$

$= a(\sin t + t\cos t)$

$$= \frac{\left[(at\cos t)^2 + (at\sin t)^2\right]^{3/2}}{\left|(at\cos t)a(\sin t + t\cos t) - (at\sin t)\cdot a(\cos t - t\sin t)\right|}$$

$$= \frac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{\left|\cancel{a^2t\cos t\sin t} + a^2t^2\cos^2 t - \cancel{a^2t\sin t\cos t} + a^2t^2\sin^2 t\right|}$$

$$= \frac{\left[a^2t^2(\cos^2 t + \sin^2 t)\right]^{3/2}}{\left|a^2t^2(\cos^2 t + \sin^2 t)\right|}$$

$$= \frac{\left[(at)^2\right]^{3/2}}{\left|(at)^2\right|} = \frac{(at)^3}{\left|(at)^2\right|} = at. \text{ Ans.}$$

$x = a\cos^3\theta$

$y = a\sin^3\theta$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\cdot\frac{d^2x}{d\theta^2}\right|} \quad (1)$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$\frac{d^2x}{d\theta^2} = -3a\left[2\cos\theta(-\sin\theta)(\sin\theta) + \cos^2\theta\cos\theta\right]$$

$$= -3a\cos\theta\left[-2\sin^2\theta + \cos^2\theta\right]$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

$$\frac{d^2y}{d\theta^2} = 3a(2\sin\theta\cos\theta\cos\theta + \sin^2\theta(-\sin\theta))$$

$$= 3a\sin\theta(2\cos^2\theta - \sin^2\theta)$$

Putting values in (1)

$$\rho = \frac{\left[(3a\cos^2\theta(-\sin\theta))^2 + \left[3a\sin^2\theta\cos\theta\right]^2\right]^{3/2}}{\left|3a\cos^2\theta(-\sin\theta)\,3a\sin\theta(2\cos^2\theta - \sin^2\theta) - 3a\sin^2\theta\cos\theta\left[-3a\cos\theta(-2\sin^2\theta + \cos^2\theta)\right]\right|}$$

$$= \frac{\left[9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta\right]^{3/2}}{\left|-9a^2\sin^2\theta\cos^2\theta(2\cos^2\theta - \sin^2\theta) + 9a^2\sin^2\theta\cos^2\theta(-2\sin^2\theta + \cos^2\theta)\right|}$$

$$= \frac{\left[9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)\right]^{3/2}}{\left|9a^2\sin^2\theta\cos^2\theta\left[-2\cos^2\theta + \sin^2\theta - 2\sin^2\theta + \cos^2\theta\right]\right|}$$

$$= \frac{(9a^2\cos^2\theta\sin^2\theta)^{3/2}}{\left|9a^2\sin^2\theta\cos^2\theta\left[-2(\cos^2\theta+\sin^2\theta)+(\sin^2\theta+\cos^2\theta)\right]\right|}$$

$$= \frac{\left[(3a\cos\theta\sin\theta)^2\right]^{3/2}}{\left|(3a\sin\theta\cos\theta)^2\left[-2+1\right]\right|}$$

$$= \frac{(3a\cos\theta\sin\theta)^3}{\left|-(3a\sin\theta\cos\theta)^2\right|}$$

$$= 3a\cos\theta\sin\theta.$$

$$= 3a\left(\frac{x}{a}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{1}{3}}$$

$\because x = a\cos^3\theta$

$\therefore \left(\frac{x}{a}\right)^{\frac{1}{3}} = \cos\theta$

& $y = a\sin^3\theta$

$\therefore \left(\frac{y}{a}\right)^{\frac{1}{3}} = \sin\theta.$

$$= 3a\,\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\,\frac{y^{\frac{1}{3}}}{a^{\frac{1}{3}}}$$

$$\therefore = 3a\,\frac{x^{\frac{1}{3}}y^{\frac{1}{3}}}{a^{2/3}}$$

$$= 3a^{1-\frac{2}{3}}\,x^{\frac{1}{3}}\,y^{\frac{1}{3}}$$

$$= 3a^{\frac{1}{3}}x^{\frac{1}{3}}y^{\frac{1}{3}}$$

$$= 3(axy)^{\frac{1}{3}}$$

Parametric Form can be changed into Cartesian form by $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$.

and then using $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

$r = 2\cos 2\theta \quad at\ \theta = \frac{\pi}{12}$

$\frac{dr}{d\theta} = 2(-\sin 2\theta)2. \quad = -4\sin 2\theta.$

$\frac{d^2r}{d\theta^2} = -4\cos 2\theta(2) = -8\cos 2\theta.$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

At $\theta = \frac{\pi}{12}$

$r = 2\cos 2\left(\frac{\pi}{12}\right) = 2\cos\frac{\pi}{6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$

$\frac{dr}{d\theta} = -4\sin 2\left(\frac{\pi}{12}\right) = -4\sin\frac{\pi}{6} = -4\left(\frac{1}{2}\right) = -2.$

$\frac{d^2r}{d\theta^2} = -8\cos 2\left(\frac{\pi}{12}\right) = -8\cos\frac{\pi}{6} = -8\frac{\sqrt{3}}{2} = -4\sqrt{3}$

$$\rho = \frac{\left[3 + (-2)^2\right]^{3/2}}{3 + 2(-2)^2 - \sqrt{3}(-4\sqrt{3})}$$

$$= \frac{(3+4)^{3/2}}{3+8+12} = \frac{(7)^{3/2}}{23} = \frac{7\sqrt{7}}{23} \quad \text{Ans.}$$

(6) $r\theta = a$

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$r = \frac{a}{\theta}$

$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$

$\frac{d^2r}{d\theta^2} = +\frac{2a}{\theta^3}$

$$\rho = \frac{\left[\frac{a^2}{\theta^2}+\left(\frac{-a}{\theta^2}\right)^2\right]^{3/2}}{\frac{a^2}{\theta^2}+2\left(\frac{-a}{\theta^2}\right)^2-\left(\frac{a}{\theta}\right)\left(\frac{2a}{\theta^3}\right)}$$

$$\rho = \frac{\left[\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right]^{3/2}}{\frac{a^2}{\theta^2}+\cancel{\frac{2a^2}{\theta^4}}-\cancel{\frac{2a^2}{\theta^4}}} = \frac{\left[\frac{a^2\theta^2+a^2}{\theta^4}\right]^{3/2}}{\frac{a^2}{\theta^2}}$$

$$\rho = \frac{\frac{\left[a^2(\theta^2+1)\right]^{3/2}}{(\theta^4)^{3/2}}}{\frac{a^2}{\theta^2}} = \frac{\frac{(a^2)^{3/2}(\theta^2+1)^{3/2}}{\theta^6}}{\frac{a^2}{\theta^2}}$$

$$= \frac{a^3(\theta^2+1)^{3/2}\,\theta^2}{\theta^6\,a^2} = \frac{a(\theta^2+1)^{3/2}}{\theta^4} \text{ Ans.}$$

New Method Difficult

First find $\tan\phi = \frac{r}{\frac{dr}{d\theta}}$

then use $p = r\sin\phi$

then use $\rho = r\frac{dr}{dp}$

(7) $r^n = a^n \sin n\theta$

$n r^{n-1}\frac{dr}{d\theta} = a^n \cos n\theta\,(n)$

$r^n \cdot r^{-1}\frac{dr}{d\theta} = a^n \cos n\theta \frac{\cancel{n}}{\cancel{n}}$

$$\frac{r^n}{r}\frac{dr}{d\theta} = a^n \cos n\theta.$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a^n}{r^n}\cos n\theta.$$

$$\frac{r\,d\theta}{dr} = \frac{r^n}{a^n \cos n\theta}$$

$$\frac{r\,d\theta}{dr} = \frac{a^n \sin n\theta}{a^n \cos n\theta} \qquad \because r^n = a^n \sin n\theta$$

$$\tan\phi = \tan n\theta. \qquad \because \frac{r\,d\theta}{dr} = \tan\phi$$

$$\phi = n\theta$$

We know $p = r\sin\phi$

so $p = r\sin n\theta$.

But $\sin(n\theta) = \frac{r^n}{a^n}$ $\qquad \because r^n = a^n \sin n\theta$

$$\therefore p = r\,\frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}$$

$$\frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$$

$$\frac{dr}{dp} = \frac{a^n}{(n+1)r^n}$$

$$r\frac{dr}{dp} = \frac{r\,a^n}{(n+1)r^n}$$

$$\rho = \frac{a^n}{(n+1)r^{n-1}} \qquad \because \rho = r\frac{dr}{dp}$$

other Method, easy

(7) $r^n = a^n \sin n\theta$

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

$$n r^{n-1}\frac{dr}{d\theta} = a^n \cos n\theta (n)$$

$$\frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^{n-1}}\,\frac{\cancel{n}}{\cancel{n}}$$

x+÷ by r

$$\frac{dr}{d\theta} = \frac{r\, a^n \cos n\theta}{r \cdot r^{n-1}} = \frac{r\,\cancel{a^n}\cos n\theta}{\cancel{a^n}\sin n\theta} \qquad \because r^n = a^n \sin n\theta$$

$$\frac{dr}{d\theta} = r\cot n\theta.$$

$$\frac{d^2r}{d\theta^2} = r\left(-\operatorname{cosec}^2 n\theta\,(n)\right) + \frac{dr}{d\theta}\cot n\theta.$$

$$= -nr\operatorname{cosec}^2 n\theta + (r\cot n\theta)(\cot n\theta)$$

$$\frac{d^2r}{d\theta^2} = r\cot^2 n\theta - nr\operatorname{cosec}^2 n\theta$$

$$\rho = \frac{\left[r^2+r^2\cot^2 n\theta\right]^{3/2}}{r^2+2(r^2\cot^2 n\theta) - r\left\{r\cot^2 n\theta - nr\operatorname{cosec}^2 n\theta\right\}}$$

$$= \frac{\left[r^2(1+\cot^2 n\theta)\right]^{3/2}}{r^2+2r^2\cot^2 n\theta - r^2\cot^2 n\theta + nr^2\operatorname{cosec}^2 n\theta}$$

$$= \frac{r^3(\operatorname{cosec}^2 n\theta)^{3/2}}{r^2+r^2\cot^2 n\theta + nr^2\operatorname{cosec}^2 n\theta}.$$

$$= \frac{r^3 \operatorname{Cosec}^3 n\theta}{r^2(1+\cot^2 n\theta) + nr^2\operatorname{Cosec}^2 n\theta}$$

$$= \frac{r^3 \operatorname{Cosec}^3 n\theta}{r^2\operatorname{Cosec}^2 n\theta + nr^2\operatorname{Cosec}^2 n\theta} \qquad \because \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

$$= \frac{r^3 \operatorname{Cosec}^3 n\theta}{r^2\operatorname{Cosec}^2 n\theta(1+n)} = \frac{r \operatorname{Cosec} n\theta}{(n+1)}$$

$$= \frac{r}{n+1} \cdot \frac{1}{\operatorname{Sin}(n\theta)} = \frac{r}{n+1} \cdot \frac{r^n}{a^n} \qquad \because r^n = a^n \operatorname{Sin}(n\theta)$$

$$= \frac{a^n}{(n+1)\, r^{n-1}} \quad \text{Ans.}$$

(8) Easy $r(1+\operatorname{Cos}\theta) = a$. $\qquad r = \dfrac{a}{(1+\cos\theta)}$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}} \quad \text{(1)}$$

$$\frac{dr}{d\theta} = \frac{-a(-\sin\theta)}{(1+\operatorname{Cos}\theta)^2} = \frac{a\operatorname{Sin}\theta}{(1+\operatorname{Cos}\theta)^2}$$

$$\frac{d^2r}{d\theta^2} = \frac{(1+\operatorname{Cos}\theta)^2 a\operatorname{Cos}\theta - a\operatorname{Sin}\theta\, 2(1+\operatorname{Cos}\theta)(-\operatorname{Sin}\theta)}{(1+\operatorname{Cos}\theta)^4}$$

$$= \frac{a(1+\operatorname{Cos}\theta)\left[(1+\operatorname{Cos}\theta)\operatorname{Cos}\theta + 2\operatorname{Sin}^2\theta\right]}{(1+\operatorname{Cos}\theta)^4}$$

$$= \frac{a\left[(1+\operatorname{Cos}\theta)\operatorname{Cos}\theta + 2(1-\operatorname{Cos}^2\theta)\right]}{(1+\operatorname{Cos}\theta)^3}$$

$$= \frac{a\,\cancel{(1+\operatorname{Cos}\theta)}\left[\operatorname{Cos}\theta + 2(1-\operatorname{Cos}\theta)\right]}{(1+\operatorname{Cos}\theta)^{\cancel{3}\,2}}$$

$$= \frac{a[\cos\theta + 2 - 2\cos\theta]}{(1+\cos\theta)^2}$$

$$\frac{d^2r}{d\theta^2} = \frac{a(2-\cos\theta)}{(1+\cos\theta)^2}$$

~~from (1)~~

$$\rho = \frac{\left[r^2 + \frac{a^2\sin^2\theta}{(1+\cos\theta)^4}\right]^{3/2}}{r^2 + 2\frac{a^2\sin^2\theta}{(1+\cos\theta)^4} - r\frac{a(2-\cos\theta)}{(1+\cos\theta)^2}}$$

$$= \frac{\left[r^2(1+\cos\theta)^4 + a^2\sin^2\theta\right]^{3/2} \Big/ (1+\cos\theta)^{\cancel{4}\times\frac{3}{\cancel{2}}}}{r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - r(1+\cos\theta)^2 a(2-\cos\theta) \Big/ (1+\cos\theta)^4}$$

$$= \frac{\dfrac{\left(r^2(1+\cos\theta)^4 + a^2\sin^2\theta\right)^{3/2}}{(1+\cos\theta)^6}}{\dfrac{r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - r(1+\cos\theta)^2 a(2-\cos\theta)}{(1+\cos\theta)^4}}$$

$$\rho = \frac{\left(r^2(1+\cos\theta)^4 + a^2\sin^2\theta\right)^{3/2}}{(1+\cos\theta)^2\left\{r^2(1+\cos\theta)^4 + 2a^2\sin^2\theta - r(1+\cos\theta)^2 a(2-\cos\theta)\right\}} \quad (11)$$

Since $r(1+\cos\theta) = a$.

$\therefore \quad 1+\cos\theta = \frac{a}{r}$

$\cos\theta = \frac{a}{r} - 1 = \frac{a-r}{r}$

$\cos^2\theta = \left(\frac{a-r}{r}\right)^2$

$\because 1-\cos^2\theta = \sin^2\theta$.

& $1-\cos^2\theta = 1 - \left(\frac{a-r}{r}\right)^2$

$\therefore \sin^2\theta = \frac{r^2 - (a^2 + r^2 - 2ar)}{r^2}$

$= \frac{\cancel{r^2} - a^2 - \cancel{r^2} + 2ar}{r^2}$

$\sin^2\theta = \frac{2ar - a^2}{r^2}$

~~from (11)~~

$$\therefore \quad \rho = \frac{\left[r^2\left(\frac{a}{r}\right)^4 + a^2\left(\frac{2ar-a^2}{r^2}\right)\right]^{3/2}}{\left(\frac{a}{r}\right)^2\left[r^2\frac{a^4}{r^4} + 2a^2\left(\frac{2ar-a^2}{r^2}\right) - r\left(\frac{a}{r}\right)^2 a\left(\frac{2-(a-r)}{r}\right)\right]}$$

$$\rho = \frac{\left[\frac{a^4}{r^2} + a^2\frac{(2ar-a^2)}{r^2}\right]^{3/2}}{\frac{a^2}{r^2}\left[\frac{a^4}{r^2} + \frac{4a^3r-2a^4}{r^2} - \frac{\cancel{r}a^2}{r^2}a\left(\frac{2r-a+r}{\cancel{r}}\right)\right]}$$

$$\rho = \frac{\left(\cancel{a^4} + 2a^3r - \cancel{a^4}\right)^{3/2}}{r^3\frac{a^2}{r^2}\left[\frac{\cancel{a^4} + 4a^3r - \cancel{2a^4} - a^3 2r + \cancel{a^4} - a^3 r}{r^2}\right]}$$

$$\rho = \frac{(2a^3r)^{3/2} \cdot r^4}{r^3 a^2 (4a^3r - 3a^3r)}$$

$$\rho = \frac{2^{3/2} a^{9/2} r^{\frac{3}{2}+1}}{a^2 \, a^3 r} = \frac{2 a^{\frac{9}{2}} r^{\frac{5}{2}}}{a^5 r}$$

$$= 2^{\frac{3}{2}} a^{\frac{9}{2}-5} r^{\frac{5}{2}-1}$$

$$= 2^{\frac{3}{2}} a^{-\frac{1}{2}} r^{3/2} \qquad = \frac{2\sqrt{2}\, r^{\frac{3}{2}}}{\sqrt{a}} \text{ Ans}$$

Difficult

(8) $r(1+\cos\theta)=a$

$\tan\phi = \frac{r}{dr/d\theta}$

$p = r\sin\phi$

$\rho = r\frac{dr}{dp}$

$\frac{dr}{d\theta}(1+\cos\theta) + r(-\sin\theta) = 0$

$\frac{dr}{d\theta}(1+\cos\theta) = r\sin\theta$

$\frac{r}{\frac{dr}{d\theta}} = \frac{\cancel{r}(1+\cos\theta)}{\cancel{r}\sin\theta}$

$\frac{r}{dr/d\theta} \quad \frac{1+\cos\theta}{\sin\theta} = \frac{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}$

$\tan\phi = \cot\frac{\theta}{2}$

$\tan\phi = \tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$

$\phi = \frac{\pi}{2}-\frac{\theta}{2}$

We know $p = r\sin\phi$

$\therefore p = r\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = r\cos\frac{\theta}{2}$

Squaring $p^2 = r^2\cos^2\frac{\theta}{2} = r^2\left(\frac{1+\cos\theta}{2}\right)$

$p^2 = \frac{r^2}{2}\left(\frac{a}{r}\right) = \frac{ar}{2}$ $\quad \therefore r(1+\cos\theta)=a$

$p = \frac{\sqrt{ar}}{\sqrt{2}}$

$\frac{dp}{dr} = \frac{\sqrt{a}}{\sqrt{2}}\cdot\frac{1}{2\sqrt{r}} = \frac{\sqrt{a}}{2^{3/2}\sqrt{r}}$

$\frac{dr}{dp} = \frac{2^{\frac{3}{2}}r^{\frac{1}{2}}}{a^{\frac{1}{2}}}$

$$\rho = r\frac{dr}{dp} = \frac{r \cdot r^{\frac{1}{2}}\, 2^{3/2}}{\sqrt{a}} = \frac{r^{\frac{3}{2}}\, 2^{3/2}}{\sqrt{a}} \quad \text{Ans}$$

Q9. $x^2 y = a(x^2+y^2)$ Implicit

$$\rho = \frac{(f_x^2+f_y^2)^{3/2}}{f_y^2(f_{xx}) + f_x^2 f_{yy} - 2 f_x f_y f_{xy}}$$

Here $f(x,y) = a(x^2+y^2) - x^2 y$.

$f_x = a(2x) - 2xy$ $\qquad f_y = a(2y) - x^2$

$f_{xy} = -2x$

$f_{xx} = a(2) - 2y$ $\qquad f_{yy} = a(2)$

At $(2a, 2a)$

$f_x = a[2(2a)] - 2(2a)(2a) = 4a^2 - 8a^2 = -4a^2$

$f_y = a\,2(2a) - (2a)^2 = \cancel{4a^2} - \cancel{4a^2} = 0$

$f_{xy} = -2(2a) = -4a.$

$f_{xx} = 2a - 2(2a) = 2a - 4a = -2a.$

$f_{yy} = 2a.$

Therefore
$$\rho = \frac{[(-4a^2)^2 + 0]^{3/2}}{0 + (-4a^2)^2(2a) - 2(-4a^2)(0)(-4a)} = \frac{(16a^4)^{3/2}}{16a^4(2a)}$$

$$= \frac{(4a^2)^{\cancel{2}\times\frac{3}{\cancel{2}}}}{32a^5} = \frac{64a^6}{32a^5} = 2a. \text{ Ans.}$$

(b) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

We use parametric eqs. $x = a\cos\theta$
$y = b\sin\theta$.

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left(\frac{dx}{d\theta}\right)\left(\frac{d^2y}{d\theta^2}\right)-\left(\frac{dy}{d\theta}\right)\left(\frac{d^2x}{d\theta^2}\right)}$$

$\dfrac{dx}{d\theta} = a(-\sin\theta)$

$\dfrac{d^2x}{d\theta^2} = -a\cos\theta$.

$\dfrac{dy}{d\theta} = b\cos\theta$

$\dfrac{d^2y}{d\theta^2} = -b\sin\theta$.

$$\rho = \frac{\left[(-a\sin\theta)^2+(b\cos\theta)^2\right]^{3/2}}{(-a\sin\theta)(-b\sin\theta)-(b\cos\theta)(-a\cos\theta)}$$

$$\rho = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta+ab\cos^2\theta}.$$

$$\rho = \frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab(1)} \quad \text{——— (1)}$$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}$

$\dfrac{dy}{dx} = \dfrac{b\cos\theta}{a(-\sin\theta)}$

Eq of Tangent.

$y-y_1 = m(x-x_1)$

$y-(b\sin\theta) = -\dfrac{b\cos\theta}{a\sin\theta}(x-a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$.

$bx\cos\theta + ay\sin\theta = ab\cos^2\theta + ab\sin^2\theta$.

$bx\cos\theta + ay\sin\theta = ab(\cos^2\theta+\sin^2\theta)$

7-7-16

$bx\cos\theta + ay\sin\theta = ab$

$bx\cos\theta + ay\sin\theta - ab = 0$

Origin $(0,0)$

$$p = \frac{|bx\cos\theta + ay\sin\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{|b(0)\cos\theta + a(0)\sin\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$p = \frac{|ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \Rightarrow p^2 = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$a^2\sin^2\theta + b^2\cos^2\theta = \frac{a^2b^2}{p^2} \quad \text{Put in } (1)$$

$$\rho = \frac{\left(\frac{a^2b^2}{p^2}\right)^{3/2}}{ab} = \frac{a^3b^3}{p^3 ab} = \boxed{\frac{a^2b}{p}}$$

If CQ is semidiameter conjugate to CP, then coordinates of Q are $(-a\sin\theta, b\cos\theta)$ $C(0,0)$ is centre of ellipse

$$|CQ| = \sqrt{(-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$|CQ|^2 = a^2\sin^2\theta + b^2\cos^2\theta \quad \text{Put in } (1)$$

$$\rho = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} = \frac{\left[|CQ|^2\right]^{3/2}}{ab} = \frac{|CQ|^3}{ab}$$

7-7-17

① $y^2 = 4ax$

$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{\overset{2}{\cancel{4}}a}{\cancel{2}y} = \frac{2a}{y}$

$\frac{d^2y}{dx^2} = \frac{-2a}{y^2}\left(\frac{dy}{dx}\right) = \frac{-2a}{y^2}\left(\frac{2a}{y}\right) = \frac{-4a^2}{y^3}$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\left(\frac{2a}{y}\right)^2\right)^{3/2}}{-\frac{4a^2}{y^3}} = \frac{\left(\frac{y^2+4a^2}{y^2}\right)^{3/2}}{\frac{-4a^2}{y^3}}$$

$$= \frac{(y^2+4a^2)^{3/2}}{\cancel{y^3}} \cdot \left(\frac{\cancel{y^3}}{-4a^2}\right) = \frac{(4ax+4a^2)^{3/2}}{-4a^2} = \frac{[4a(x+a)]^{3/2}}{-4a^2}$$

$$\rho = \frac{(2^2)^{3/2} a^{3/2}(x+a)^{3/2}}{-4a^2} = \frac{2^3 a^{3/2}(x+a)^{3/2}}{-4a^2} = \frac{8(x+a)^{3/2}}{-4a^{2-\frac{3}{2}}}$$

$$\rho = \frac{2(x+a)^{3/2}}{-\sqrt{a}}$$

$$\rho^2 = \frac{4(x+a)^3}{a} \quad \text{---} ①$$

$P: (x,y)$

$S: (a,0)$ focus of Parabola

$|SP| = \sqrt{(x-a)^2+(y-0)^2} = \sqrt{x^2+a^2-2ax+y^2} \quad \because y^2 = 4ax$

$|SP| = \sqrt{x^2+a^2-2ax+4ax} = \sqrt{x^2+a^2+2ax} = \sqrt{(x+a)^2} = x+a$

from ① $\rho^2 = \frac{4|SP|^3}{a}$, $\rho^2 \propto |SP|^3$, $\frac{4}{a}$ is const.

we may use here $P(at^2, 2at)$ but then $\rho^2 = \frac{4}{a}(at^2+a)^3$

IInd Method

$f(x,y) = 4ax - y^2$

$f_x = 4a$, $f_y = -2y$

$f_{xx} = 0$, $f_{yy} = -2$, $f_{xy} = 0$

$$\rho = \frac{(f_x^2+f_y^2)^{3/2}}{f_y^2 f_{xx} - 2f_xf_yf_{xy} + f_x^2 f_{yy}}$$

$$= \frac{(16a^2+4y^2)^{3/2}}{4y^2(0) - 2(4a)(-2y)(0) + (16a^2)(-2)}$$

$$\rho = \frac{[16a^2+4(4ax)]^{3/2}}{-32a^2}$$

$$\rho = \frac{(16a)^{3/2}[a+x]^{3/2}}{-32a^2} = \frac{(4^2)^{3/2} a^{\frac{3}{2}}(a+x)^{3/2}}{-32a^2}$$

$$\rho = \frac{-2(a+x)^{3/2}}{\sqrt{a}}$$

7·7-18

Imp.

(12) $r = a(1+\cos\theta)$ $\qquad \frac{r}{a} = 1+\cos\theta$ $\qquad \rho = r\frac{dr}{dp}$

$\frac{dr}{d\theta} = -a\sin\theta$ $\qquad p = r\sin\psi$

$\frac{d^2r}{d\theta^2} = -a\cos\theta.$ $\qquad \tan\psi = \frac{r}{\frac{dr}{d\theta}}$

$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{\cancel{a}(1+\cos\theta)}{-\cancel{a}\sin\theta}$

$= \frac{\cancel{2}\cos^2\frac{\theta}{2}}{-\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$\tan\psi = -\cot\frac{\theta}{2}$

$\tan\psi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$\psi = \frac{\pi}{2}+\frac{\theta}{2}$

We know $p = r\sin\psi$

$p = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$

$p = r\cos\frac{\theta}{2}$

Squaring

$p^2 = r^2\cos^2\frac{\theta}{2}$

$p^2 = r^2\frac{1}{2}\left(2\cos^2\frac{\theta}{2}\right) = \frac{r^2}{2}(1+\cos\theta)$

$p^2 = \frac{r^2}{2}\left(\frac{r}{a}\right) = \frac{r^3}{2a}$

$p = \frac{r^{3/2}}{\sqrt{2}\sqrt{a}}$

$\frac{dp}{dr} = \frac{1}{\sqrt{2}\sqrt{a}}\left(\frac{3}{2}r^{\frac{3}{2}-1}\right) = \frac{3}{2\sqrt{2}\sqrt{a}}r^{\frac{1}{2}}$

IInd Method $\qquad$ Easy

$\frac{dr}{d\theta} = -a\sin\theta$

$\frac{d^2r}{d\theta^2} = -a\cos\theta$

$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left[r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)\right]}$

$= \frac{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta - a(1+\cos\theta)(-a\cos\theta)}$

$= \frac{\left[a^2(1+\cos^2\theta+2\cos\theta+\sin^2\theta)\right]^{3/2}}{a^2\left[1+\cos^2\theta+2\cos\theta+2\sin^2\theta+\cos\theta+\cos^2\theta\right]}$

$= \frac{a^3(1+1+2\cos\theta)^{3/2}}{a^2(1+2\cos^2+2\sin^2\theta+3\cos\theta)}$

$= \frac{a(2+2\cos\theta)^{3/2}}{(1+2+3\cos\theta)}$

$= \frac{a2^{3/2}(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$

7-7-19

$$\frac{dr}{dp} = \frac{2\sqrt{2}\,a}{3\sqrt{r}}$$

$$\rho_1 = \frac{r\,dr}{dp} = r\,\frac{2\sqrt{2}\sqrt{a}}{3\sqrt{r}}$$

$$\rho = r^{\frac{1}{2}}\frac{2\sqrt{2}\sqrt{a}}{3}$$

sq.

$$\rho^2 = \frac{r\,4(2)(a)}{9} = \frac{8ar}{9}$$

$$\frac{\rho^2}{r} = \frac{8a}{9} = \text{Constant}$$

$$\rho = \frac{a\,2\sqrt{2}}{3}(1+\cos\theta)^{\frac{3}{2}-1}$$

$$\rho = \frac{a\,2\sqrt{2}}{3}(1+\cos\theta)^{\frac{1}{2}}$$

squaring

$$\rho^2 = \frac{a^2\,8\,(1+\cos\theta)}{9}$$ ← (Remember) for Q17, 18

Same

$$\rho^2 = a^2\frac{8}{9}\,\frac{r}{a}$$

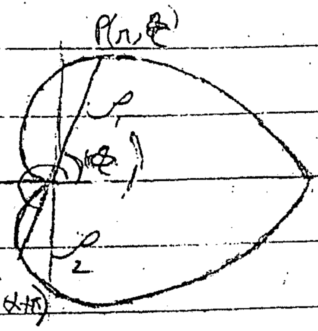
$$\rho^2 = a\frac{8}{9}\,r$$

$$\frac{\rho^2}{r} = \frac{8a}{9} = \text{Const.}$$

Q17 Let PQ be a chord through pole 'O'

then $P = (r, \alpha)$ & $Q = (r', \alpha+\pi)$

$$r = a(1+\cos\theta)$$



$$\rho^2 = \frac{8a^2}{9}(1+\cos\theta)$$ (As I solved above in Exam solve completely)

$$\rho_1^2 = \frac{8a^2}{9}(1+\cos\alpha) \quad \text{at } P(r,\alpha)$$

&

$$\rho_2^2 = \frac{8a^2}{9}(1+\cos(\pi+\alpha)) \quad \text{at } Q(r', \alpha+\pi)$$

$$\rho_2^2 = \frac{8a^2}{9}(1-\cos\alpha)$$

$$\rho_1^2+\rho_2^2 = \frac{8a^2}{9}(1+\cos\alpha) + \frac{8a^2}{9}(1-\cos\alpha)$$

$$= \frac{8a^2}{9}\{1+\cos\alpha+1-\cos\alpha\} = \frac{8a^2}{9}(2) = \frac{16a^2}{9}$$

Q18 $r = a(1+\cos\theta)$ — ①

$\therefore \rho^2 = \frac{8a^2}{9}(1+\cos\theta)$ (As I solved in Q12 In exam solve completely)

$\frac{dr}{d\theta} = -a\sin\theta \qquad \frac{d^2r}{d\theta^2} = -a\cos\theta.$

$\tan\psi = \frac{r}{dr/d\theta} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\theta/2}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\theta/2}{-\sin\frac{\theta}{2}}$

$\tan\psi = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right) \Rightarrow \psi = \frac{\pi}{2}+\frac{\theta}{2}.$

Let α be the angle which tangent makes then $\alpha = \theta+\psi$

$\therefore \alpha = \theta + \frac{\pi}{2} + \frac{\theta}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$ — ⑪

Now tangent is ∥ to initial line when $\alpha = 0, \pi$

when $\alpha = 0$ use ⑪ $\quad 0 = \frac{3\theta}{2} + \frac{\pi}{2} \Rightarrow \frac{3\theta}{2} = -\frac{\pi}{2} \Rightarrow \boxed{\theta = -\frac{\pi}{3}}$

when $\alpha = \pi$ using ⑪ $\quad \pi = \frac{3\theta}{2} + \frac{\pi}{2} \Rightarrow \frac{3\theta}{2} = \pi - \frac{\pi}{2} \Rightarrow \boxed{\theta = \frac{\pi}{3}}$

from ① At $\theta = -\frac{\pi}{3}$ $\quad \rho^2 = \frac{8a^2}{9}\left(1+\cos\left(-\frac{\pi}{3}\right)\right)$

$= \frac{8a^2}{9}\left(1+\cos\frac{\pi}{3}\right) = \frac{8a^2}{9}\left(1+\frac{1}{2}\right) = \frac{8a^2}{9}\left(\frac{3}{2}\right) = \frac{4a^2}{3}$

$\rho^2 = \frac{4a^2}{3} \Rightarrow \boxed{\rho = \frac{2a}{\sqrt{3}}}$

At $\theta = \frac{\pi}{3}$ $\quad \rho^2 = \frac{8a^2}{9}\left(1+\cos\frac{\pi}{3}\right) = \frac{8a^2}{9}\left(1+\frac{1}{2}\right) = \frac{8a^2}{9}\left(\frac{3}{2}\right) = \frac{4a^2}{3}.$

$\therefore \boxed{\rho = 2a/\sqrt{3}}$

7-7-21

Q. 13. $\rho = r\frac{dr}{dp}$ To Prove.

Don't use $\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ it becomes very difficult

Proof. We know $p = r\sin\psi$

$$\frac{dp}{dr} = 1.\sin\psi + r\cos\psi\frac{d\psi}{dr} \quad \text{—— (1)}$$

Since $\tan\psi = r\frac{d\theta}{dr}$ $\therefore \sin\psi = \frac{r\,d\theta}{ds}$

& $\cos\psi = \frac{dr}{ds}$

$$\Rightarrow \tan\psi = \frac{r\frac{d\theta}{ds}}{dr/ds}$$

$\therefore$ (1) becomes $\frac{dp}{dr} = 1.\frac{r\,d\theta}{ds} + r\frac{dr}{ds}\frac{d\psi}{dr}$

$$\frac{dp}{dr} = r\frac{d\theta}{ds} + r\frac{d\psi}{ds}$$

$$= r\frac{d}{ds}(\theta+\psi)$$

$$= r\frac{d\alpha}{ds} \qquad \because \alpha = \theta+\psi$$

$$\frac{dp}{dr} = r\left(\frac{1}{\rho}\right) \qquad \because \frac{d\alpha}{ds} = K = \frac{1}{\rho}$$

$$\rho = r\frac{dr}{dp} \quad \text{Proved.}$$

7.7-22

(14) $p^2 = ar$

$$\frac{2p\,dp}{dr} = a$$

$$\frac{dp}{dr} = \frac{a}{2p} \Rightarrow \frac{dr}{dp} = \frac{2p}{a}$$

$$\rho = r\left(\frac{dr}{dp}\right) = r\left(\frac{2p}{a}\right) = \frac{p^2}{a}\left(\frac{2p}{a}\right) \quad \because r = \frac{p^2}{a}$$

$$\rho = \frac{2p^3}{a^2} \quad \text{Ans.}$$

(15) $\frac{1}{p^2} = \frac{A}{r^2} + B$

$$\frac{-2}{p^3}\frac{dp}{dr} = -\frac{2A}{r^3} + 0$$

$$\frac{dp}{dr} = \cancel{-2}\frac{A}{r^3}\left(\frac{p^3}{\cancel{-2}}\right) = \frac{Ap^3}{r^3}$$

$$\frac{dr}{dp} = \frac{r^3}{Ap^3}$$

$$\rho = r\frac{dr}{dp} = r \cdot \frac{r^3}{Ap^3} = \frac{r^4}{Ap^3} \quad \text{Ans. in the Book}$$

$$\frac{1}{p^2} - B = \frac{A}{r^2}$$

$$\rho = \frac{(Ap^2)^2}{\{1-p^2B\}^2}\,\frac{1}{Ap^3} = \frac{A^2p^4}{Ap^3}\,\frac{1}{(1-p^2B)^2}$$

$$\frac{1-p^2B}{p^2} = \frac{A}{r^2}$$

$$\rho = \frac{Ap}{(1-p^2B)^2} \quad \text{Ans (I think this is the right one)}$$

$$r^2 = \frac{Ap^2}{1-p^2B}$$

Q18 $p^2(a^2+b^2-r^2)=a^2b^2$

$$2p\frac{dp}{dr}(a^2+b^2-r^2)+p^2(0+0-2r)=0$$

$$2p\frac{dp}{dr}(a^2+b^2-r^2)-2p^2r=0$$

$$2p\frac{dp}{dr}(a^2+b^2-r^2)=2p^2r$$

$$\frac{dp}{dr}(a^2+b^2-r^2)=\frac{2p^2r}{2p}=pr$$

$$\frac{dp}{dr}=\frac{pr}{(a^2+b^2-r^2)} \Rightarrow \frac{dr}{dp}=\frac{(a^2+b^2-r^2)}{pr}$$

$$\rho=r\left(\frac{dr}{dp}\right)=r\frac{(a^2+b^2-r^2)}{pr}$$

$$\rho=\frac{1}{p}(a^2+b^2-r^2)=\frac{1}{p}\left(\frac{a^2b^2}{p^2}\right) \qquad \because p^2(a^2+b^2-r^2)=a^2b^2$$

$$\rho=\frac{a^2b^2}{p^3} \text{ Ans.}$$

Q19 $y=ax^2+bx+c$

$$\frac{dy}{dx}=2ax+b$$

$$\frac{d^2y}{dx^2}=2a$$

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}=\frac{\left[1+(2ax+b)^2\right]^{3/2}}{2a}$$

$$y=ax^2+bx+c$$

$$\frac{y}{a}=x^2+\frac{b}{a}x+\frac{c}{a}$$

$$\frac{y}{a}=x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}$$

$$\therefore \frac{y}{a}-\frac{c}{a}+\frac{b^2}{4a^2}=\left(x+\frac{b}{2a}\right)^2$$

$$\left[\frac{4ay-4ac+b^2}{4a^2}\right]=\left(x+\frac{b}{2a}\right)^2$$

$$\text{Vertex}\left(\frac{-b}{2a},\frac{4ac-b^2}{4a}\right)\Leftarrow\frac{1}{a}\left[y-\left(\frac{4ac-b^2}{4a}\right)\right]=\left(x+\frac{b}{2a}\right)^2\Leftarrow\frac{y}{a}-\left(\frac{4ac-b^2}{4a^2}\right)=\left(x+\frac{b}{2a}\right)^2$$

For extreme values of ρ, Put $\frac{d\rho}{dx} = 0$

$$\frac{d\rho}{dx} = \frac{\frac{3}{\cancel{2}}\left(1+(2ax+b)^2\right)^{\frac{1}{2}}\cancel{2}(2ax+b)2a}{2a} = 3(2ax+b)\left(1+(2ax+b)^2\right)^{\frac{1}{2}}$$

$$0 = 3(2ax+b)\left[1+(2ax+b)^2\right]^{\frac{1}{2}}$$

either $(2ax+b) = 0$ or $\left[1+(2ax+b)^2\right]^{\frac{1}{2}}$

$\Rightarrow 2ax+b = 0.$

$$\boxed{x = \frac{-b}{2a}}$$

$\left[1+(2ax+b)^2\right]^{\frac{1}{2}} = 0$

squaring $\left(\left[1+(2ax+b)^2\right]^{\frac{1}{2}}\right)^2 = 0$

$(2ax+b)^2 = -1$

$2ax+b = \pm i$

Discard being Imaginary.

$$\frac{d^2\rho}{dx^2} = 3(2a)\left[1+(2ax+b)^2\right]^{\frac{1}{2}} + 3(2ax+b)\frac{1}{\cancel{2}}\frac{\cancel{2}(2ax+b)2a}{\sqrt{1+(2ax+b)^2}}$$

$$\frac{d^2\rho}{dx^2} = \frac{6a\left[1+(2ax+b)^2\right] + 6a(2ax+b)^2}{\sqrt{1+(2ax+b)^2}}$$

At $x = \frac{-b}{2a}$ i.e. $2ax+b = 0$

$$\frac{d^2\rho}{dx^2} = \frac{6a[1+0]+6a(0)}{\sqrt{1+0}} = 6a > 0$$ Hence ρ is minimum at $x = \frac{-b}{2a}$

For Pt of Vertex $y = ax^2+bx+c \Rightarrow y = a\left(\frac{-b}{2a}\right)^2 + b\left(\frac{-b}{2a}\right)+c$

$\Rightarrow y = \frac{ab^2}{4a^2} - \frac{b^2}{2a} + c \Rightarrow y = \frac{b^2}{4a} - \frac{b^2}{2a} + c \Rightarrow y = \frac{b^2-2b^2+4ac}{4a}$

$y = \frac{-b^2+4ac}{4a}$ So Vertex of Parabola is $\left(\frac{-b}{2a}, \frac{-b^2+4ac}{4a}\right)$

(10) $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

Curvature at any pt (x, y) of the curve is $K = \dfrac{|d^2y/dx^2|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$

$$K = \frac{\left|-\frac{1}{x^2}\right|}{\left(1+\frac{1}{x^2}\right)^{3/2}} = \frac{\frac{1}{x^2}}{\frac{(x^2+1)^{3/2}}{x^3}} = \frac{1 \cdot x^{\cancel{3}}}{x^2(x^2+1)^{3/2}}$$

$$K = \frac{x}{(x^2+1)^{3/2}}$$

For entrem values Put $\frac{dK}{dx} = 0$.

$$\frac{dK}{dx} = \frac{(x^2+1)^{3/2}(1) - (x)\frac{3}{\cancel{2}}(x^2+1)^{\frac{1}{2}}\cancel{2}x}{\left[(x^2+1)^{\frac{3}{2}}\right]^2} = \frac{(x^2+1)^{3/2} - 3x^2(x^2+1)^{\frac{1}{2}}}{(x^2+1)^3}$$

$$\frac{dK}{dx} = \frac{(x^2+1)^{\frac{1}{2}}\left[x^2+1-3x^2\right]}{(x^2+1)^3} = \frac{1-2x^2}{(x^2+1)^{5/2}}$$

$$0 = \frac{1-2x^2}{(x^2+1)^{5/2}} \Rightarrow 1-2x^2 = 0 \Rightarrow x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

$$\frac{d^2K}{dx^2} = \frac{(x^2+1)^{5/2}(-4x) - (1-2x^2)\frac{5}{\cancel{2}}(x^2+1)^{\frac{3}{2}}\cancel{2}x}{\left[(x^2+1)^{\frac{5}{2}}\right]^2}$$

$$-(x^2+1)^{3/2}\frac{\{(x^2+1)(-4x)-5x(1-2x^2)\}}{(x^2+1)^5}$$

$$\frac{d^2K}{dx^2}=\frac{\{(x^2+1)(-4x)-5x(1-2x^2)\}}{(x^2+1)^{7/2}}$$

At $x=\frac{1}{\sqrt2}$ $\quad \frac{d^2K}{dx^2}=\frac{\{(\frac{1}{2}+1)(-4\frac{1}{\sqrt2})-5\frac{1}{\sqrt2}(1-\cancel{2}\frac{1}{\cancel{2}})\}}{(\frac{1}{2}+1)^{7/2}}$

$$=\frac{(\frac{3}{\cancel{2}})(-\cancel{2}\sqrt2)-0}{(\frac{3}{2})^{7/2}}, \qquad =\frac{-3\sqrt2}{(\frac{3}{2})^{7/2}}$$

Hence ρ is Max at $x=\frac{1}{\sqrt2}$

For $x=\frac{1}{\sqrt2}$ $\quad y=\ln x$

$$=\ln\left(\frac{1}{\sqrt2}\right)=\ln(2)^{-\frac{1}{2}}=(-1)\ln 2^{\frac{1}{2}}$$

$$y \;\; (-1)\ln\sqrt2$$

Hence required pt is $\left(\frac{1}{\sqrt2},-\ln\sqrt2\right)$

Prove $\rho = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$

Radius of Curvature for Cartesian Curve.

Let $y = f(x)$ be the eq of the curve and ψ be the inclination of the tangent to the curve at $x = a$.

then $\tan\psi = \frac{dy}{dx}$

Diff w.r.t. 'S'

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$

$$\therefore \frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{\sec^2\psi \frac{ds}{dx}}$$

$ds^2 = (dx)^2 + (dy)^2$

$\frac{ds^2}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$

$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\left|\frac{d\psi}{ds}\right| = \frac{\left|d^2y/dx^2\right|}{(1+\tan^2\psi)\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}} = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

curvature

$$\rho = \frac{1}{K} = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

radius of curvature

Prove that $\rho = \dfrac{[(f')^2+(g')^2]^{3/2}}{|f'g''-g'f''|}$ Radius of Curvature for Parametric Eqs

Proof

Let $x = f(t)$ $y = g(t)$ be the parametric Eqs.

$\dfrac{dx}{dt} = f'(t)$ $\dfrac{dy}{dt} = g'(t)$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = g'(t)\cdot\frac{1}{f'(t)} = \frac{g'}{f'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{g'}{f'}\right) = \frac{d}{dt}\left(\frac{g'}{f'}\right)\frac{dt}{dx}$$

$$y'' = \frac{d^2y}{dx^2} = \left(\frac{f'g''-g'f''}{(f')^2}\right)\frac{1}{f'} = \frac{f'g''-g'f''}{(f')^3}$$

$$K = \frac{|y''|}{[1+(y')^2]^{3/2}} = \frac{|f'g''-g'f''|}{(f')^3\left\{1+\left(\frac{g'}{f'}\right)^2\right\}^{3/2}}$$

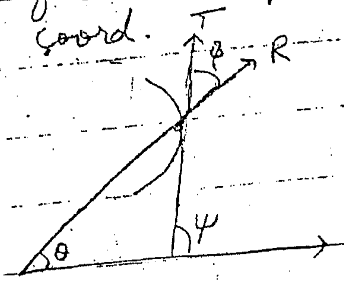
$$K = \frac{|f'g''-g'f''|}{(f')^3\left\{\frac{(f')^2+(g')^2}{[(f')^2]^{3/2}}\right\}^{3/2}} = \frac{|f'g''-g'f''|}{\cancel{(f')^3}\left\{\frac{(f')^2+(g')^2}{\cancel{(f')^3}}\right\}^{3/2}}$$

$$= \frac{f'g''-g'f''}{\{(f')^2+(g')^2]^{3/2}}$$

$\rho = \dfrac{1}{K}$

$$\therefore \rho = \frac{[(f')^2+(g')^2]^{3/2}}{|f'g''-g'f''|}$$ proved

Prove that $\rho = \dfrac{\left[r^2+(r')^2\right]^{3/2}}{r^2+2(r')^2-rr''}$ Radius of Curvature for curves in polar coord.



Proof Let $r=f(\theta)$ be the curve and ψ be the angle of the tangent at $P(r,\theta)$

From fig $\psi = \theta+\phi$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left[1+\frac{d\phi}{d\theta}\right] \qquad (1)$$

Now find $\frac{d\phi}{d\theta}$.

$$\tan\phi = \frac{r}{dr/d\theta}$$

Diff w.r.t 'θ' $\sec^2\phi \dfrac{d\phi}{d\theta} = \dfrac{\left(\frac{dr}{d\theta}\right)\left(\frac{dr}{d\theta}\right) - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\sec^2\phi\left(\frac{dr}{d\theta}\right)^2} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{(1+\tan^2\phi)\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]\left(\frac{dr}{d\theta}\right)^2}$$ Put in (1)

$$\frac{d\psi}{ds} = \frac{d\theta}{ds}\left[1 + \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}\right]$$

$$K = \frac{d\theta}{ds}\left[\frac{\left(\frac{dr}{d\theta}\right)^2 + r^2 + \left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}\right]$$

$$K = \frac{2\left(\frac{dr}{d\theta}\right)^2 + r^2 - r\frac{d^2r}{d\theta^2}}{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\left(\frac{ds}{d\theta}\right)}$$

$$\because \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$

$$\therefore K = \frac{2(r')^2 + r^2 - r\,r''}{\left[(r')^2 + r^2\right]\left[\sqrt{r^2 + (r')^2}\right]}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$\sqrt{r^2 + (r')^2}$$

$$K = \frac{r^2 + 2(r')^2 - r\,r''}{\left[r^2 + (r')^2\right]^{3/2}}$$

$$\rho = \frac{1}{K} = \frac{\left[r^2 + (r')^2\right]^{3/2}}{r^2 + 2(r')^2 - r\,r''}$$