Length of Arcs.

1) If the curve is given in Cartesian form $\left\{ \begin{array}{l} y = f(x) \\ \text{or } x = f(y) \end{array} \right.$

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \quad \text{or} \quad S = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

2) If the curve is given in Parametric form $\left\{ \begin{array}{l} x = f(t) \\ y = f(t) \end{array} \right\}$

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

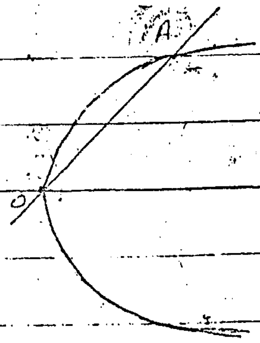
3) If the curve is given in Polar form $\left\{ \begin{array}{l} r = f(\theta) \\ \text{or } \theta = f(r) \end{array} \right.$

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \text{or} \quad S = \int_{r_1}^{r_2} \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, dr$$

Ex 7.6

Q1 (i) $y^2 = 4ax$ —(1)

$3y = 8x$ —(2) $\qquad y = \frac{8x}{3}$



Solving (1) & (2) to find pt of X_n.

$$\left(\frac{8x}{3}\right)^2 = 4ax.$$

$$\frac{64x^2}{9} = 4ax. \Rightarrow 64x^2 = 36ax. \Rightarrow 64x^2 - 36ax = 0$$

$4x(16x - 9a) = 0$

$4x = 0, \quad 16x - 9a = 0$

$x = 0, \quad x = \frac{9a}{16}$ Put in (2)

when $x = 0$ in (2) $y = 0$

when $x = \frac{9a}{16}$, $3y = 8\left(\frac{9a}{16}\right)$

$3y = \frac{9a}{2} \Rightarrow y = \frac{3a}{2}$

Hence pt of Xn are $(0,0)$, $\left(\frac{9a}{16}, \frac{3a}{2}\right)$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{\frac{9a}{16}} \sqrt{\frac{x+a}{x}}\, dx$$

$y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$

$\left(\frac{dy}{dx}\right)^2 = \frac{4a^2}{y^2} = \frac{4a^2}{4ax}$

$1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{a}{x}$

$= \frac{x+a}{x}$

Put $x = a\,Tan^2\theta$

$dx = a\,2\,Tan\theta\,Sec^2\theta\,d\theta$

$$\int_0^{\frac{9a}{16}} \sqrt{\frac{x+a}{x}}\, dx = \int \sqrt{\frac{a\,Tan^2\theta + a}{a\,Tan^2\theta}}\; 2a\,Tan\theta\,Sec^2\theta\,d\theta$$

$$= \int \sqrt{\frac{\cancel{a}(Tan^2\theta+1)}{\cancel{a}\,Tan^2\theta}}\; 2a\,Tan\theta\,Sec^2\theta\,d\theta.$$

$$= \int \sqrt{\frac{Sec^2\theta}{Tan^2\theta}}\; 2a\,Tan\theta\,Sec^2\theta\,d\theta.$$

$$= \int \frac{Sec\theta}{\cancel{Tan\theta}}\; 2a\,\cancel{Tan\theta}\,Sec^2\theta\,d\theta = 2a\int Sec^3\theta\,d\theta \quad \text{---} (3)$$

Consider $I = \int Sec^3\theta\,d\theta = \int \underset{II}{Sec^2\theta}\,\underset{I}{Sec\theta}\,d\theta.$ IBP

$= Sec\theta\,Tan\theta - \int (Sec\theta\,Tan\theta)\,Tan\theta\,d\theta.$

$= Sec\theta\,Tan\theta - \int Sec\theta\,Tan^2\theta\,d\theta$

$$I = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$$

$$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \int \sec\theta\,d\theta$$

$$2I = \sec\theta\tan\theta + \int \sec\theta\,d\theta$$

$$I = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| \quad \text{———} (4)$$

Using (4) in (3)

$$\int_0^{\frac{9a}{16}} \sqrt{\frac{x+a}{x}} = 2a\left|\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right|_0^{\frac{9a}{16}}$$

$$= a\left|\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right|_0^{\frac{9a}{16}}$$

$$= a\left|\sqrt{\frac{x+a}{a}}\sqrt{\frac{x}{a}} + \ln\left|\sqrt{\frac{a+x}{a}} + \sqrt{\frac{x}{a}}\right|\right|_0^{\frac{9a}{16}}$$

$\because x = a\tan^2\theta$

$\frac{x}{a} = \tan^2\theta$

$\sqrt{\frac{x}{a}} = \tan\theta$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \frac{x}{a} = \sec^2\theta$

$\sqrt{\frac{a+x}{a}} = \sec\theta$

$$= a\left(\sqrt{\frac{\frac{9a}{16}+a}{a}}\sqrt{\frac{\frac{9a}{16}}{a}} + \ln\left|\sqrt{\frac{\frac{9a}{16}+a}{a}} + \sqrt{\frac{\frac{9a}{16}}{a}}\right|\right)$$

$$= a\left(\sqrt{\frac{(\frac{9}{16}+1)\cancel{a}}{\cancel{a}}}\sqrt{\frac{9\cancel{a}}{16\cancel{a}}} + \ln\left|\sqrt{\frac{(\frac{9}{16}+1)\cancel{a}}{\cancel{a}}} + \sqrt{\frac{9\cancel{a}}{16\cancel{a}}}\right|\right)$$

$$= a\left(\sqrt{\frac{25}{16}}\sqrt{\frac{9}{16}} + \ln\left|\sqrt{\frac{25}{16}} + \sqrt{\frac{9}{16}}\right|\right)$$

$$= a\left(\frac{5}{4}\cdot\frac{3}{4} + \ln\left|\frac{5}{4} + \frac{3}{4}\right|\right) = a\left(\frac{15}{16} + \ln\left|\frac{8}{4}\right|\right)$$

$$= a\left[\frac{15}{16} + \ln 2\right] \text{ Ans.}$$

Easy Method of Q1.

$y^2 = 4ax$ —— ①

$3y = 8x$. —— ②

Solving ① & ② to find Pt of x_n.

$\left(\frac{8x}{3}\right)^2 = 4ax \Rightarrow 64x^2 = 36ax$

$64x^2 - 36ax = 0 \Rightarrow 4x(16x - 9a) = 0 \Rightarrow x = 0, \frac{9a}{16}$

Putting values of x in ② we get when $x = 0$, $y = 0$

when $x = \frac{9a}{16}$, $y = \frac{8}{3}\left(\frac{9a}{16}\right) = \frac{3a}{2}$

$$Pt \text{ of } X_n = (0, 0)\left(\frac{9a}{16}, \frac{3a}{2}\right)$$

$$S = \int_0^{\frac{3a}{2}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$y^2 = 4ax$

$2y \frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y}$

$\frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

$\left(\frac{dx}{dy}\right)^2 = \frac{y^2}{4a^2}$

$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2}$

$1 + \left(\frac{dx}{dy}\right)^2 = \frac{4a^2 + y^2}{4a^2}$

$$S = \int_0^{\frac{3a}{2}} \sqrt{\frac{4a^2 + y^2}{4a^2}}\, dy$$

$$= \frac{1}{2a} \int_0^{\frac{3a}{2}} \sqrt{(2a)^2 + y^2}\, dy$$

$$= \frac{1}{2a} \left| \frac{y}{2}\sqrt{4a^2 + y^2} + \frac{4a^2}{2} \ln\left[\frac{y + \sqrt{4a^2 + y^2}}{2a}\right] \right|_0^{3a/2}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\sqrt{4a^2 + \frac{9a^2}{4}} + \frac{4a^2}{2} \ln\left[\frac{\frac{3a}{2} + \sqrt{4a^2 + \frac{9a^2}{4}}}{2a}\right]\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\sqrt{\frac{25a^2}{4}} + 2a^2 \ln\left[\frac{\frac{3a}{2} + \sqrt{\frac{25a^2}{4}}}{2a}\right]\right]$$

$\int \sqrt{a^2 + y^2} = \frac{y}{2}\sqrt{a^2 + y^2} + \frac{a^2}{2} \ln\left(\frac{y + \sqrt{a^2 + y^2}}{a}\right)$

= sec θ tan θ − ∫ ....

$$= \frac{1}{2a}\left\{\frac{3a}{4}\left(\frac{5a}{2}\right) + 2a^2 \ln\left[\frac{\frac{3a}{2}+\frac{5a}{2}}{2a}\right]\right\}$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\ln\left(\frac{\frac{8a}{2}}{2a}\right)\right]$$

$$= \frac{15a^2}{16a} + \frac{2a^2}{2a}\ln\left(\frac{8\cancel{a}}{4\cancel{a}}\right)$$

$$= \frac{15a}{16} + a\ln(2) \qquad = a\left(\frac{15}{16}+\ln 2\right) \text{ Ans.}$$

---

(2) $y = c\cosh\frac{x}{c}$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^x \sqrt{1+\sinh^2\frac{x}{c}}\,dx$$

$$= \int_0^x \sqrt{\cosh^2\frac{x}{c}}\,dx$$

$$= \int_0^x \cosh\frac{x}{c}\,dx$$

$$= \left|\frac{\sinh\frac{x}{c}}{\frac{1}{c}}\right|_0^x$$

$$S = \left[c\sinh\frac{x}{c} - 0\right] = c\sinh\frac{x}{c} \text{ Ans.}$$

$$= 12(e^2 - 1) \text{ Ans.}$$

$y = c\cosh\frac{x}{c}$

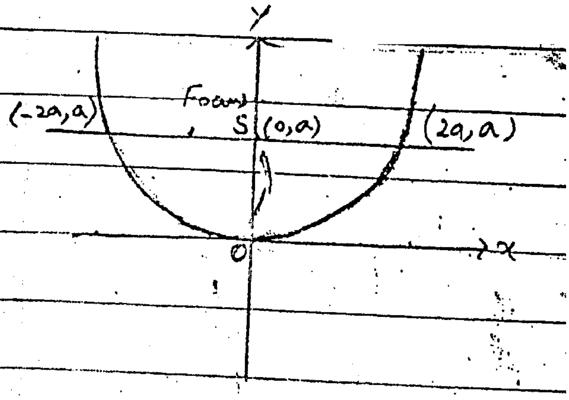
$\frac{dy}{dx} = \cancel{c}\sinh\frac{x}{c}\cdot\frac{1}{\cancel{c}}$

$\left(\frac{dy}{dx}\right)^2 = \sinh^2\frac{x}{c}$

$1+\left(\frac{dy}{dx}\right)^2 = 1+\sinh^2\frac{x}{c}$

we know $\cosh^2\frac{x}{c} - \sinh^2\frac{x}{c} = 1$

Q3 $x^2 = 4ay$



$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{2a} \sqrt{1+\frac{x^2}{4a^2}}\, dx = \int_0^{2a} \sqrt{\frac{4a^2+x^2}{4a^2}}\, dx$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{(2a)^2+x^2}\, dx = \frac{1}{2a}\left[\sqrt{(2a)^2+x^2}\cdot\frac{x}{2} + \frac{(2a)^2}{2}\ln\left|\frac{x+\sqrt{(2a)^2+x^2}}{2a}\right|\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\sqrt{(2a)^2+(2a)^2}\cdot\frac{2a}{2} + \frac{(2a)^2}{2}\ln\left|\frac{2a+\sqrt{(2a)^2+(2a)^2}}{2a}\right|\right] - \frac{1}{2a}\left[0+2a^2\ln\left|\frac{a+\sqrt{4a^2}}{2a}\right|\right]$$

$$= \frac{1}{2a}\left[\sqrt{8a}\;\; a + 2a^2\ln\left|\frac{2a+\sqrt{8a^2}}{2a}\right|\right] - \frac{1}{2a}\left[0+2a^2\ln 1\right]$$

$$= \frac{1}{2a}\left[2a\sqrt{2}\,a + 2a^2\ln\left|\frac{\cancel{2a}(1+\sqrt{2})}{\cancel{2a}}\right|\right] - \frac{1}{2a}\left[0+2a^2(0)\right]$$

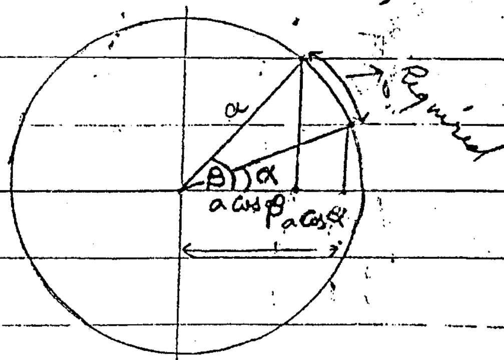
$$= \frac{1}{2a}\left[2a^2\sqrt{2} + 2a^2\ln(1+\sqrt{2})\right] - 0$$

$$= \frac{2a^2}{2a}\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$$

$$= a\left[\sqrt{2} + \ln(1+\sqrt{2})\right]$$

Q.5. $x^2+y^2=a^2$ $x=a\cos\alpha$, $x=a\cos\beta$

$$S=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$



Parametric Eqs of circle are

$x=a\cos\theta$ $y=a\sin\theta$

$\frac{dx}{d\theta}=-a\sin\theta$ $\frac{dy}{d\theta}=a\cos\theta$

$\left(\frac{dx}{d\theta}\right)^2=a^2\sin^2\theta$ $\left(\frac{dy}{d\theta}\right)^2=a^2\cos^2\theta$.

$$S=\int_{\alpha}^{\beta}\sqrt{a^2\sin^2\theta+a^2\cos^2\theta}\,d\theta=\int_{\alpha}^{\beta}\sqrt{a^2(\sin^2\theta+\cos^2\theta)}\,d\theta$$

$$=\int_{\alpha}^{\beta}a\,d\theta=a\Big|\theta\Big|_{\alpha}^{\beta}=a(\beta-\alpha)\text{ Ans.}$$

IInd Method $x^2+y^2=a^2 \Rightarrow y=\sqrt{a^2-x^2}$

$$S=\int_{a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$\frac{dy}{dx}=\frac{1}{2\sqrt{a^2-x^2}}(-2x)$

$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2-x^2}$

$$=\int_{a\cos\beta}^{a\cos\alpha}\sqrt{\frac{a^2}{a^2-x^2}}\,dx=a\int_{a\cos\beta}^{a\cos\alpha}\frac{dx}{\sqrt{a^2-x^2}}$$

$1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{a^2-x^2}=\frac{a^2\cancel{-x^2}\cancel{+x^2}}{a^2-x^2}$

$$=a\Big|-\cos^{-1}\frac{x}{a}\Big|_{a\cos\beta}^{a\cos\alpha}=a\left(-\cos^{-1}\frac{\cancel{a}\cos\alpha}{\cancel{a}}+\cos^{-1}\frac{\cancel{a}\cos\beta}{\cancel{a}}\right)$$

$$=a\left[-\cos^{-1}\cos\alpha+\cos^{-1}\cos\beta\right]=a(-\alpha+\beta)\text{ Ans.}$$

$$= \int_0^{2\pi} \sqrt{a^2(2-2\cos\theta)}\, d\theta$$

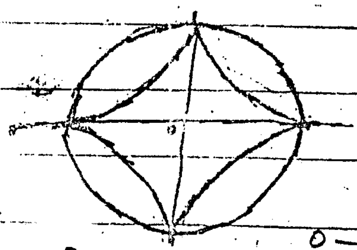
$$= \int_0^{2\pi} \sqrt{a^2 2(1-\cos\theta)}\, d\theta = \int_0^{2\pi} \sqrt{2a^2\, 2\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} 2a \sin\frac{\theta}{2}\, d\theta = 2a \left| \frac{-\cos\frac{\theta}{2}}{1/2} \right|_0^{2\pi}$$

$$= 4a\left[-\cos\frac{2\pi}{2} + \cos 0\right]$$

$$= 4a(-\cos\pi + 1) = 4a(-(-1)+1) = 8a$$

Q8. $x^{2/3} + y^{2/3} = a^{2/3}$



Parametric Eqs of Astroid

$x = a\cos^3\theta \qquad y = a\sin^3\theta$

$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) \qquad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$

$0 \to \frac{\pi}{2}$ One Quad

$$S = \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 3a\cos\theta\sin\theta\, d\theta = 3a\left|\frac{\sin^2\theta}{2}\right|_0^{\frac{\pi}{2}}$$

$$= 3a\left[\frac{\sin^2\frac{\pi}{2}}{2}\right] = 3a\left(\frac{1}{2}\right) = \frac{3a}{2}$$ Proved.

~~Exp~~ (ii) $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

$x=a\cos^3\theta \qquad y=b\sin^3\theta$

$\frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta) \qquad \frac{dy}{d\theta}=3b\sin^2\theta\cos\theta$

$\left(\frac{dx}{d\theta}\right)^2=9a^2\cos^4\theta\sin^2\theta \qquad \left(\frac{dy}{d\theta}\right)^2=9b^2\sin^4\theta\cos^2\theta$

$$S=\int_0^{\frac{\pi}{2}}\sqrt{9a^2\cos^4\theta\sin^2\theta+9b^2\sin^4\theta\cos^2\theta}\;d\theta$$

$$=\int_0^{\frac{\pi}{2}}\sqrt{9\cos^2\theta\sin^2\theta\left(a^2\cos^2\theta+b^2\sin^2\theta\right)}\;d\theta$$

$$=\int_0^{\frac{\pi}{2}}3\cos\theta\sin\theta\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\;d\theta$$

when $\theta=0,\ t=a$

Put $a^2\cos^2\theta+b^2\sin^2\theta=t^2$ $\qquad \theta=\frac{\pi}{2},\ t=b$

$\left[2a^2\cos\theta(-\sin\theta)+2b^2\sin\theta\cos\theta\right]d\theta=2t\,dt$

$2\sin\theta\cos\theta\left(b^2-a^2\right)d\theta=2t\,dt$

$\sin\theta\cos\theta\,d\theta=\frac{\cancel{2}t\,dt}{\cancel{2}(b^2-a^2)}$

$$\int_a^b\sqrt{t^2}\,\frac{3t\,dt}{b^2-a^2}=\int_a^b\frac{3t^2\,dt}{b^2-a^2}=\left|\frac{\cancel{3}t^3}{\cancel{3}(b^2-a^2)}\right|_a^b$$

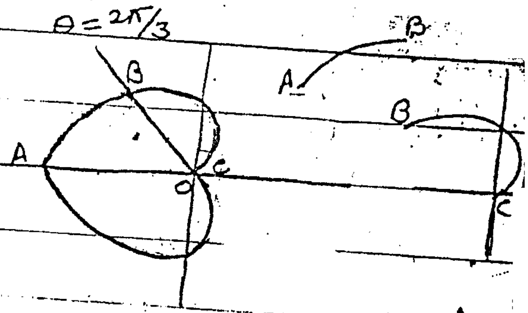
$$=\frac{b^3}{b^2-a^2}-\frac{a^3}{b^2-a^2}=\frac{b^3-a^3}{b^2-a^2}=\frac{\cancel{(b-a)}(b^2+ab+a^2)}{(b+a)\cancel{(b-a)}}$$

$$=\frac{b^2+ab+a^2}{b+a}\ \text{Ans.}$$

7·6-12

~~Imp~~

Q9. $r = a(1-\cos\theta)$



$\frac{dr}{d\theta} = a(\sin\theta)$

$S = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$0 \to \pi$ for upper half only

$= 2\int_0^{\pi} \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$

$= 2\int_0^{\pi} \sqrt{a^2(1+\cos^2\theta - 2\cos\theta + \sin^2\theta)}\, d\theta$

$= 2\int_0^{\pi} \sqrt{a^2(1+1-2\cos\theta)}\, d\theta$

$= 2\int_0^{\pi} \sqrt{a^2 2(1-\cos\theta)}\, d\theta. \quad = 2\int_0^{\pi} \sqrt{2a^2 2\sin^2\frac{\theta}{2}}\, d\theta$

$= 2\int_0^{\pi} 2a\sin\frac{\theta}{2} = 4a\left|\frac{-\cos\frac{\theta}{2}}{1/2}\right|_0^{\pi} = 8a\left[-\cos\frac{\pi}{2} + \cos 0\right]$

$= 8a(0+1) = 8a$ = Entire length of Cardiod

Now length of the arc from $\theta = 0$ to $\theta = \frac{2\pi}{3}$

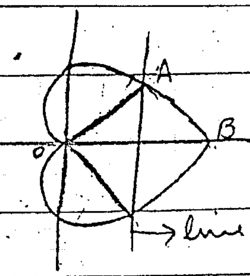
$S_2 = \int_0^{\frac{2\pi}{3}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\frac{2\pi}{3}} 2a\sin\frac{\theta}{2}\, d\theta$ ∵ solved above

$= 2a\left|\frac{-\cos\frac{\theta}{2}}{1/2}\right|_0^{\frac{2\pi}{3}} = 4a\left[-\cos\frac{2\pi}{3\cdot 2} + \cos 0\right]$

$= 4a\left[-\frac{1}{2} + 1\right] = 4a\left(\frac{1}{2}\right) = 2a$. Ans.

11. $r = a(1+\cos\theta)$ —— (i)

$4r\cos\theta = 3a$ —— (ii)



To find the upper & lower limits

from (ii) $r = \dfrac{3a}{4\cos\theta}$ Put in (i)

The curve is symmetrical about initial line. θ by $(-\theta)$ No change.

$$\frac{3a}{4\cos\theta} = a(1+\cos\theta)$$

$$3a = 4a\cos\theta(1+\cos\theta)$$

$$3a = 4a\cos\theta + 4a\cos^2\theta.$$

$$4\cos^2\theta + 4\cos\theta - 3 = 0.$$

$$\cos\theta = -\frac{3}{2}, \frac{1}{2}$$

Put $\cos\theta = t$

$\cancel{4t^2+4t-3=0}$

$4t^2+6t-2t-3=0$

$2t(2t+3)-1(2t+3)=0$

$(2t+3)(2t-1)=0.$

$t=-\frac{3}{2}$ $t=\frac{1}{2}$

$\cos\theta = -\frac{3}{2}, \frac{1}{2}$

$\cos\theta$ can never be $-\frac{3}{2}$ $\because$ limit of $\cos\theta$ is from -1 to 1.

$\therefore$ $\cos\theta = \frac{1}{2}$, $\Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}$

Length of arc from B to A is $S = \displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta.$

$$S = \int_0^{\pi/3} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$r = a(1+\cos\theta)$

$\frac{dr}{d\theta} = a(-\sin\theta)$

$\left(\frac{dr}{d\theta}\right)^2 = a^2\sin^2\theta$

$r^2+\left(\frac{dr}{d\theta}\right)^2 = a^2(1+\cos\theta)^2 + a^2\sin^2\theta$

$$= \int_0^{\pi/3} \sqrt{a^2(1+\cos^2\theta+2\cos\theta+\sin^2\theta)}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{a^2(1+1+2\cos\theta)}\, d\theta$$

$$= \int_0^{\pi/3} \sqrt{2a^2(1+\cos\theta)}\,d\theta = \int_0^{\pi/3} \sqrt{2a^2\,2\cos^2\frac{\theta}{2}}\,d\theta$$

$$= \int_0^{\pi/3} 2a\cos\frac{\theta}{2}\,d\theta = 2a \left| \frac{\sin\frac{\theta}{2}}{1/2} \right|_0^{\pi/3}$$

$$S_1 = 4a\left[\sin\frac{\pi}{6} - \sin 0\right] = 4a\left(\frac{1}{2} - 0\right) = 2a.$$

Now length of arc from A to O

$$S_2 = \int_{\pi/3}^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_{\pi/3}^{\pi} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= \int_{\pi/3}^{\pi} 2a\cos\frac{\theta}{2}\,d\theta \quad \text{as solved above}$$

$$= 2a \left| \frac{\sin\frac{\theta}{2}}{1/2} \right|_{\pi/3}^{\pi} = 4a\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right]$$

$$S_2 = 4a\left(1 - \frac{1}{2}\right) = 2a.$$

$S_1 = S_2$ Ratio $2a : 2a = 1:1$

This shows that line $4r\cos\theta = 3a$ bisect upper half of the curve at $\theta = \pi/3$. Since curve is symmetric therefore similarly line bisect lower half at $\theta = -\frac{\pi}{3}$

$\therefore$ $4r\cos\theta = 3a$ bisects the cardiod in ratio 1:1.

Q12 $x = (a+b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta$

$y = (a+b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta$

$$\frac{dx}{d\theta} = (a+b)(-\sin\theta) - \cancel{b}\left(-\sin\left(\frac{a+b}{b}\right)\theta\right)\left(\frac{a+b}{\cancel{b}}\right)$$

$$= (a+b)\left[\sin\left(\frac{a+b}{b}\right)\theta - \sin\theta\right]$$

$$\left(\frac{dx}{d\theta}\right)^2 = (a+b)^2\left[\sin\left(\frac{a+b}{b}\right)\theta - \sin\theta\right]^2$$

$$\left(\frac{dy}{d\theta}\right) = (a+b)\cos\theta - \cancel{b}\cos\left(\frac{a+b}{b}\right)\theta\left(\frac{a+b}{\cancel{b}}\right)$$

$$= (a+b)\left[\cos\theta - \cos\left(\frac{a+b}{b}\right)\theta\right]$$

$$\left(\frac{dy}{d\theta}\right)^2 = (a+b)^2\left[\cos\theta - \cos\left(\frac{a+b}{b}\right)\theta\right]^2$$

$$S = \int_{\frac{\pi b}{a}}^{\theta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_{\frac{\pi b}{a}}^{\theta} \sqrt{(a+b)^2\left[\sin\left(\frac{a+b}{b}\right)\theta - \sin\theta\right]^2 + (a+b)^2\left[\cos\theta - \cos\left(\frac{a+b}{b}\right)\theta\right]^2}\, d\theta$$

$$= \int_{\frac{\pi b}{a}}^{\theta} \sqrt{(a+b)^2\left(\sin^2\left(\frac{a+b}{b}\right)\theta + \sin^2\theta - 2\sin\left(\frac{a+b}{b}\right)\theta\sin\theta + \cos^2\theta + \cos^2\left(\frac{a+b}{b}\right)\theta - 2\cos\theta\cos\left(\frac{a+b}{b}\right)\theta\right.}$$

$$= \int_{\frac{\pi b}{a}}^{\theta} \sqrt{(a+b)^2\left[\left(\sin^2\left(\frac{a+b}{b}\right)\theta + \cos^2\left(\frac{a+b}{b}\right)\theta\right) + \left(\sin^2\theta + \cos^2\theta\right) - 2\sin\left(\frac{a+b}{b}\right)\theta\sin\theta - 2\cos\theta\cos\left(\frac{a+b}{b}\right)\theta\right.}$$

$$= (a+b)\int_{\pi b/a}^{\theta} \sqrt{1+1-2\left(\sin\left(\frac{a+b}{b}\right)\theta \sin\theta + \cos\left(\frac{a+b}{b}\right)\theta\cos\theta\right)}\,d\theta.$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$= (a+b)\int_{\pi b/a}^{\theta} \sqrt{2-2\left[\cos\left\{\left(\frac{a+b}{b}\right)\theta - \theta\right\}\right]}\,d\theta.$$

$$= (a+b)\int_{\pi b/a}^{\theta} \sqrt{2\left[1-\cos\left\{\frac{(a+b)\theta - b\theta}{b}\right\}\right]}\,d\theta$$

$$= (a+b)\int_{\pi b/a}^{\theta} \sqrt{2\left[1-\cos\left(\frac{a\theta}{b}\right)\right]}\,d\theta.$$

$$\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$= (a+b)\int_{\pi b/a}^{\theta} \sqrt{2\left(2\sin^2\left(\frac{a\theta}{2b}\right)\right)}\,d\theta \quad = (a+b)2\int_{\pi b/a}^{\theta} \sin\frac{a\theta}{2b}\,d\theta.$$

$$= 2(a+b)\left|-\frac{\cos\frac{a\theta}{2b}}{\frac{a}{2b}}\right|_{\pi b/a}^{\theta} \quad = 2(a+b)\frac{2b}{a}\left|-\cos\left(\frac{a\theta}{2b}\right)\right|_{\pi b/a}^{\theta}$$

$$= \frac{4b}{a}(a+b)\left(-\cos\frac{a\theta}{2b} + \cos\frac{\cancel{a}\pi \cancel{b}}{2\cancel{b}\cancel{a}}\right)$$

$$= \frac{4b(a+b)}{a}\left[-\cos\frac{a\theta}{2b} + \cos\frac{\pi}{2}\right] \qquad \cos\frac{\pi}{2} = 0$$

$$= -\frac{4b(a+b)}{a}\cos\frac{a\theta}{2b}$$

$$= \frac{4b(a+b)}{a}\cos\left(\frac{a\theta}{2b}\right) \quad \text{Ignoring -ve sign}$$

7·6–17

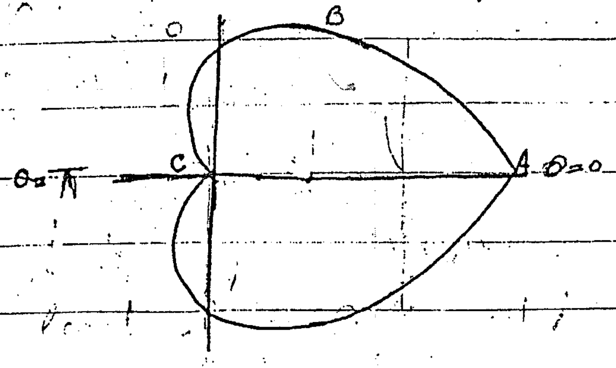
one loop if $a > b$ — $r = 3+2\cos\theta$ — $r = 3-2\cos\theta$

$r = a \pm b\sin\theta$ — Two loops if $a < b$ — $r = 2+3\sin\theta$ — $r = 2-3\sin\theta$

one loop if $a > b$ — $r = 3+2\sin\theta$ — $r = 3-2\sin\theta$

Q 13 $r = a + b\cos\theta$. $\frac{b}{a}$ is small i.e. $b < a$

It is symmetric about initial line $\because \theta = -\theta$ No change.



$r = 0 \therefore 0 = a + b\cos\theta$

$-a = b\cos\theta$

$-\frac{a}{b} = \cos\theta$

Minimum value of $\cos\theta$ $-1 = \cos\theta \Rightarrow \theta = \pi$

Total length of perimeter $= 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$ i.e. $2\widehat{ABC}$

$r = a + b\cos\theta$

$\frac{dr}{d\theta} = -b\sin\theta$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (a + b\cos\theta)^2 + b^2\sin^2\theta$$

$$= a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta$$

$$= a^2 + 2ab\cos\theta + b^2(\cos^2\theta + \sin^2\theta)$$

$$= a^2 + 2ab\cos\theta + b^2 \qquad \because \frac{b}{a} = K.$$

$$= a^2\left[1 + 2\frac{b\cos\theta}{a} + \frac{b^2}{a^2}\right] = a^2(1 + 2K\cos\theta + K^2)$$

$$2\int_0^{\pi} \sqrt{a^2(1 + 2K\cos\theta + K^2)}\, d\theta = 2a\int_0^{\pi} (1 + 2K\cos\theta + K^2)^{\frac{1}{2}}\, d\theta$$

Apply Binomial Series $(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2+ \cdots \infty$

$$= 2a\int_0^{\pi}\left[1+\frac{1}{2}(2K\cos\theta+K^2)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(2K\cos\theta+K^2)^2+ \cdots \infty\right.$$

neglecting $K^3, K^4 \cdots$ $\because K$ is small

$$= 2a\int_0^{\pi}\left(1+K\cos\theta+\frac{K^2}{2}-\frac{1}{8}(4K^2\cos^2\theta+K^4+4K^3\cos\theta)+ \cdots\right)$$

$$= 2a\int_0^{\pi}\left(1+K\cos\theta+\frac{K^2}{2}-\frac{K^2}{2}\cos^2\theta\right)d\theta.$$

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$$= 2a\int_0^{\pi}\left(1+K\cos\theta+\frac{K^2}{2}(1-\cos^2\theta)\right)d\theta.$$

$$= 2a\int_0^{\pi}\left(1+K\cos\theta+\frac{K^2}{2}\sin^2\theta\right)d\theta$$

$$= 2a\int_0^{\pi}d\theta+2a\int_0^{\pi}K\cos\theta\,d\theta+\frac{2aK^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta.$$

$$\int_0^{\pi}\sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left|\theta-\frac{\sin 2\theta}{2}\right|_0^{\pi}$$

$$= 2a|\theta|_0^{\pi}+2aK|\sin\theta|_0^{\pi}+\frac{aK^2}{2}\left|\theta-\frac{\sin 2\theta}{2}\right|_0^{\pi}$$

$$= 2a(\pi-0)+2aK(\sin\pi-\sin 0)+\frac{aK^2}{2}\left(\pi-\frac{\sin 2\pi}{2}-0+0\right)$$

$$= 2a\pi+2aK(0)+\frac{aK^2}{2}\pi$$

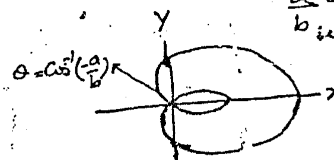
$$= 2a\pi+\frac{aK^2\pi}{2} = 2a\pi\left[1+\frac{aK^2\pi}{2\cdot 2a\pi}\right]$$

$$= 2a\pi\left(1+\frac{K^2}{4}\right) = 2a\pi\left(1+\frac{b^2}{4a^2}\right)$$ Proved

7·6-19

Q14 Prove that the difference between the lengths of the two loops of the limacon $r = a + b\cos\theta$ is $8a$ ($\frac{a}{b}$ is small)

$\frac{a}{b}$ is small i.e. $a < b$



Sol $r = a + b\cos\theta$ —— (1)

Put $r = 0 \Rightarrow \cos\theta = \frac{-a}{b}$

$\theta = \cos^{-1}\left(\frac{-a}{b}\right)$

(1) is symmetric about initial line. The upper half of outer loop varies from 0 to $\cos^{-1}\left(\frac{-a}{b}\right)$. The lower half of inner loop varies from $\cos^{-1}\left(\frac{-a}{b}\right)$ to π.

$\frac{dr}{d\theta} = -b\sin\theta$.

$$S = 2\left\{\int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta - \int_\theta^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta\right\} \text{ —— (II)}$$

(First find $\int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$, then find $\int_\theta^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$ and put values in eq (II))

$$\int_0^\theta \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^\theta \sqrt{(a+b\cos\theta)^2 + b^2\sin^2\theta}\,d\theta$$

$$= \int_0^\theta \sqrt{a^2 + b^2(\cos^2\theta + \sin^2\theta) + 2ab\cos\theta}\,d\theta$$

$$= \int_0^\theta \sqrt{b^2\left(\frac{a^2}{b^2} + \cancel{\frac{b^2}{b^2}} + \frac{2a\cancel{b}\cos\theta}{b^{\cancel{2}}}\right)}\,d\theta$$

× & ÷ by b^2

$$= \int_0^\theta b\left(1 + \frac{2a}{b}\cos\theta\right)^{\frac{1}{2}} d\theta$$

(neglecting $\frac{a^2}{b^2}$ since $\frac{a}{b}$ is small)

Apply B. Series $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

neglecting terms $\frac{a^2}{b^2}, \frac{a^3}{b^3} \cdots \because \frac{a}{b}$ is small

$$= b\int_0^\theta \left[1 + \left(\frac{1}{\cancel{2}}\right)\frac{\cancel{2}a}{b}\cos\theta + \cdots\right] d\theta$$

$$= b\int_0^\theta \left(1 + \frac{a}{b}\cos\theta\right) d\theta$$

$$= b\left|\theta + \frac{a}{b}\sin\theta\right|_0^\theta$$

$$= b\left[\theta + \frac{a}{b}\sin\theta\right] \text{ —— (III)}$$

Similarly $\int_\theta^\pi \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = b\left|\theta + \frac{a}{b}\sin\theta\right|_\theta^\pi$

$$= b\left[\pi + 0 - \theta - \frac{a}{b}\sin\theta\right] \text{ —— (IV)}$$

Put (III) & (IV) in (II)

$$S = 2b\left[\theta + \frac{a}{b}\sin\theta - \pi + \theta + \frac{a}{b}\sin\theta\right]$$

$$= 2b\left[2\theta - \pi + 2\frac{a}{b}\sin\theta\right]$$

$$= 2b\left[2\theta - \pi + 2\frac{a}{b}\left(\sqrt{1-\cos^2\theta}\right)\right]$$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin\theta = \sqrt{1-\cos^2\theta}$

7·6-20

$S = 2b\left[2\left(\frac{\pi}{2}+\frac{a}{b}\right) - \pi + 2\frac{a}{b}\cdot\sqrt{1-\frac{a^2}{b^2}}\right]$

$= 2b\left[2\left(\frac{\pi}{2}+\frac{a}{b}\right) - \pi + \frac{2a}{b}\sqrt{1}\right.$ Neglecting $\frac{a^2}{b^2}$ $\because \frac{a}{b}$ is small.

$= 2b\left[\cancel{\pi} + \frac{2a}{b} - \cancel{\pi} + \frac{2a}{b}\right.$

$= 2\cancel{b}\left[\frac{4a}{\cancel{b}}\right]$

$= 8a$ Ans

$\because \theta = \cos^{-1}\left(-\frac{a}{b}\right) = \frac{\pi}{2} + \frac{a}{b}$

$\cos^{-1}\left(-\frac{a}{b}\right) = \frac{\pi}{2} + \frac{a}{b}$

Proof $\cos\left(\frac{\pi}{2}+\frac{a}{b}\right)$

$= \cos\frac{\pi}{2}\cos\frac{a}{b} - \sin\frac{\pi}{2}\sin\frac{a}{b}$

$\cos\left(\frac{\pi}{2}+\frac{a}{b}\right) = 0 - \sin\frac{a}{b}$

$\cos\left(\frac{\pi}{2}+\frac{a}{b}\right) = -\frac{a}{b}$ ($\because \sin\theta = \theta$ when θ is small)

$\frac{\pi}{2} + \frac{a}{b} = \cos^{-1}\left(-\frac{a}{b}\right)$

Q3 Find length of loop of curve $3ay^2 = x(a-x)^2$

Sol Put y by $-y$, curve is unchanged so it is symmetric about x-axis.

Put $y=0 \Rightarrow 0 = x(a-x)^2 \Rightarrow x = 0, a$

$S = 2\int_0^a \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$= 2\int_0^a \sqrt{1+\frac{(a-3x)^2}{12ax}}\,dx$

$= 2\int_0^a \sqrt{\frac{12ax+a^2+9x^2-6ax}{12ax}}\,dx$

$= 2\int_0^a \sqrt{\frac{(3x+a)^2}{12ax}}\,dx$

$= \frac{2}{\sqrt{12a}}\int_0^a \frac{(3x+a)}{\sqrt{x}}\,dx$

$= \frac{1}{\sqrt{3a}}\int_0^a \left(3\sqrt{x} + a x^{-\frac{1}{2}}\right)dx$

$= \frac{1}{\sqrt{3a}}\left|3\frac{x^{3/2}}{3/2} + a\frac{x^{\frac{1}{2}}}{1/2}\right|_0^a = \frac{1}{\sqrt{3a}}\left[2a^{3/2} + 2a\,a^{\frac{1}{2}} - 0 - 0\right]$

$\frac{1}{\sqrt{3a}}\left(4a^{3/2}\right) = \frac{4}{\sqrt{3}}a^{\frac{3}{2}-\frac{1}{2}} = \boxed{\frac{4a}{\sqrt{3}}}$ Ans

$3ay^2 = x(a-x)^2$

$\sqrt{3a}\,y = \sqrt{x}(a-x)$

$\sqrt{3a}\,y = \sqrt{x}\,a - x^{3/2}$

$\sqrt{3a}\frac{dy}{dx} = \frac{a}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$

$\sqrt{3a}\frac{dy}{dx} = \frac{a-3x}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{(a-3x)}{\sqrt{3a}\cdot 2\sqrt{x}}$

$\left(\frac{dy}{dx}\right)^2 = \frac{(a-3x)^2}{12ax}$

Q10 $r = \sin^2\frac{\theta}{2}$ from $(0,0)$ to $(1,\pi)$

$\frac{dr}{d\theta} = \cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{1}{\cancel{2}}$

$\left(\frac{dr}{d\theta}\right)^2 = \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}$

$$S = \int_0^{\pi}\sqrt{r^2+(dr/d\theta)^2}\,d\theta = \int_0^{\pi}\sqrt{\left(\sin^2\frac{\theta}{2}\right)^2 + \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\,d\theta$$

$$= \int_0^{\pi}\sqrt{\sin^4\frac{\theta}{2} + \sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\theta}{2}\right)}\,d\theta = \int_0^{\pi}\sqrt{\cancel{\sin^4\frac{\theta}{2}} + \sin^2\frac{\theta}{2} - \cancel{\sin^4\frac{\theta}{2}}}\,d\theta$$

$$= \int_0^{\pi}\sin\frac{\theta}{2}\,d\theta = \left|\frac{-\cos\frac{\theta}{2}}{1/2}\right|_0^{\pi} = \left|-2\cos\frac{\theta}{2}\right|_0^{\pi}$$

$$= \left(-2\cos\frac{\pi}{2} + 2\cos\frac{0}{2}\right) = (0+2) = 2 \text{ Ans.}$$

Q15 $a = r\theta$ $r = \frac{a}{\theta}$ $\frac{dr}{d\theta} = \frac{-a}{\theta^2}$

$$S = \int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

when $r = a$ $\theta = \frac{a}{a} = 1$

when $r = 2a$ $\theta = \frac{a}{2a} = \frac{1}{2}$

$$S = \int_1^{1/2}\sqrt{\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}}\,d\theta$$

$$= \int_1^{1/2}\sqrt{\frac{a^2\theta^2+a^2}{\theta^4}}\,d\theta = \int_1^{1/2}\sqrt{\frac{a^2(\theta^2+1)}{\theta^4}}\,d\theta$$

$$= a\int_1^{\frac{1}{2}}\frac{\sqrt{\theta^2+1}}{\theta^2}\cdot\frac{\sqrt{\theta^2+1}}{\sqrt{\theta^2+1}}\,d\theta = a\int\frac{\theta^2+1}{\theta^2\sqrt{\theta^2+1}}\,d\theta$$

IInd Method

$$S = \int_a^{2a}\sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}\,dr$$

$$= \int_a^{2a}\frac{\sqrt{r^2+a^2}}{r}\,dr$$ Put $r = a\tan\alpha$, $dr = a\sec^2\alpha$

$= a\int\sec^2\alpha\,\operatorname{cosec}\alpha\,d\alpha$ IB

$\star\ \frac{\sqrt5-1}{2(\sqrt2-1)} = \frac{\sqrt5-1}{2(\sqrt2-1)}\cdot\frac{\sqrt5+1}{\sqrt5+1}\cdot\frac{\sqrt2+1}{\sqrt2+1}$. Rationalizing 7·6-22

$$= \frac{1}{2}\frac{(5-1)(\sqrt2+1)}{\sqrt5+1\ (2-1)} = \frac{4}{2}\frac{(\sqrt2+1)}{\sqrt5+1} = \frac{2(\sqrt2+1)}{\sqrt5+1} = \frac{2\sqrt2+2}{\sqrt5+1} = \frac{\sqrt8+2}{\sqrt5+1}$$

$$= a\int_1^{\frac12}\frac{\theta^2\,d\theta}{\theta^2\sqrt{\theta^2+1}} + a\int_1^{\frac12}\frac{d\theta}{\theta^2\sqrt{\theta^2+1}}$$

$$\int\frac{\sqrt{a^2+r^2}}{r}\,dr = a\sqrt{1+\cot^2\alpha}\,\tan\alpha + a\ln\left|\sqrt{1+\cot^2\alpha}-\cot\alpha\right|$$

$$\int_a^{2a}\frac{\sqrt{a^2+r^2}}{r}\,dr = \left|a\sqrt{1+\left(\frac{a}{r}\right)^2}\,\frac{r}{a} + a\ln\left|\sqrt{1+\left(\frac{a}{r}\right)^2}-\frac{a}{r}\right|\right|_a^{2a}$$

$$= a\int_1^{\frac12}\frac{d\theta}{\sqrt{\theta^2+1}} + a\int_1^{\frac12}\frac{d\theta}{\theta^2\sqrt{\theta^2+1}} \quad (1)$$

$$= \left|a\frac{\sqrt{r^2+a^2}}{r}\frac{r}{a} + a\ln\left|\frac{\sqrt{r^2+a^2}-a}{r}\right|\right|_a^{2a}$$

$$= \left[a\sqrt5 + a\ln\left|\frac{a\sqrt5-a}{2a}\right|\right] - \left[a\sqrt2 + a\ln\left|\frac{a\sqrt2-a}{a}\right|\right]$$

$$= a(\sqrt5-\sqrt2) + a\ln(\sqrt5-1) - a\ln(\sqrt2-1)$$

$$= a\left[\sqrt5-\sqrt2+\ln\frac{\sqrt5-1}{2(\sqrt2-1)}\right] = a\left[\sqrt5-\sqrt2+\ln\frac{2+\sqrt8}{1+\sqrt5}\right]$$

Now $a\int_1^{\frac12}\frac{d\theta}{\sqrt{\theta^2+1}} = a\left|\ln\left(\theta+\sqrt{\theta^2+1}\right)\right|_1^{\frac12}$

$$= a\left[\ln\left(\tfrac12+\sqrt{\tfrac14+1}\right) - \ln\left(1+\sqrt{1+1}\right)\right] = a\left[\ln\left(\tfrac12+\tfrac{\sqrt5}{2}\right) - \ln(1+\sqrt2)\right]$$

See above $\star$

$$= a\left[\ln\left(\frac{1+\sqrt5}{2}\right) - \ln(1+\sqrt2)\right] = a\ln\left[\frac{\frac{1+\sqrt5}{2}}{(1+\sqrt2)}\right] = a\ln\left[\frac{1+\sqrt5}{2(1+\sqrt2)}\right]$$

Consider $a\int_1^{\frac12}\frac{d\theta}{\theta^2\sqrt{\theta^2+1}}$ Put $\theta = \frac1t$ $d\theta = -\frac{1}{t^2}dt$ when $\theta=1,\ t=1$; $\theta=\frac12,\ t=2$

$$= a\int_1^2\frac{\left(-\frac{1}{t^2}\right)dt}{\left(\frac{1}{t^2}\right)\sqrt{\frac{1}{t^2}+1}} = a\int_1^2\frac{-dt}{\frac{\sqrt{1+t^2}}{t}} = a\int_1^2\frac{-t\,dt}{\sqrt{1+t^2}} = \frac{-a}{2}\int_1^2(1+t^2)^{-\frac12}2t\,dt$$

$$= \frac{-a}{2}\left|\frac{(1+t^2)^{\frac12}}{\frac12}\right|_1^2 = -\frac{a}{2\cdot\frac12}\left((1+4)^{\frac12}-(1+1)^{\frac12}\right) = -a\left[\sqrt5-\sqrt2\right] \quad (2)$$

$\therefore$ (1) becomes $S = a\ln\left[\frac{1+\sqrt5}{2(1+\sqrt2)}\right] - a\left[\sqrt5-\sqrt2\right]$

$$= a\ln\left[\frac{2(1+\sqrt2)}{1+\sqrt5}\right]^{-1} - a(\sqrt5-\sqrt2) = -a\ln\left[\frac{2(1+\sqrt2)}{1+\sqrt5}\right] - a(\sqrt5-\sqrt2)$$

$$= -\left[a\ln\frac{2+\sqrt8}{1+\sqrt5} + a(\sqrt5-\sqrt2)\right]$$ we take absolute value

$$\therefore S = +a\left[\ln\frac{2+\sqrt8}{1+\sqrt5} + a(\sqrt5-\sqrt2)\right]$$

7 · 6 − 23

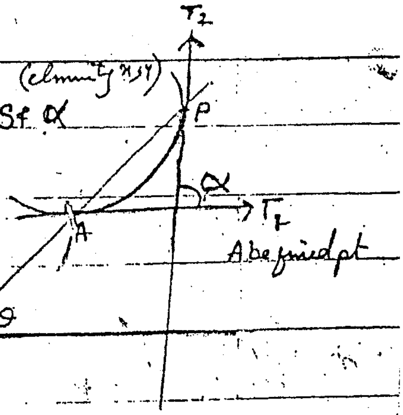
V. Imp

(eliminating x, y)

Intrinsic Eq of a Curve is the relation between S & α

where S = the Arc length measured from some fixed pt A on the curve to any pt $P(x,y)$

α = the angle through which tangent line turns from the fixed pt A to general pt $P(x,y)$



1. **Working Rule** $\tan\alpha = \dfrac{dy}{dx}$ —— (i) $\quad S = \displaystyle\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ —— (ii) Eliminate x, y from (i) + (ii)

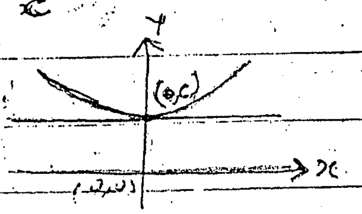
Note We usually choose the fixed pt. A on the curve such that tangent line at 'A' becomes $\parallel$ to x-axis

Imp

Q16. Intrinsic Eq of Catenary $y = c\cosh\frac{x}{c}$ is $S = c\tan\alpha$

$$y = c\cosh\frac{x}{c}$$

(Eliminate x, y & get a relation in S & α)



$$\frac{dy}{dx} = \cancel{c}\sinh\frac{x}{c}\cdot\frac{1}{\cancel{c}}$$

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

Taking the vertex $(0, c)$ as fixed pt.

But $\dfrac{dy}{dx} = \tan\alpha \quad \therefore \boxed{\sinh\frac{x}{c} = \tan\alpha}$ —— (i)

$$S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^x \sqrt{1+\sinh^2\frac{x}{c}} = \int_0^x \sqrt{\cosh^2\frac{x}{c}}$$

$\cosh^2 x - \sinh^2 x = 1$

$$= \int_0^x \cosh\frac{x}{c} = \left|\frac{\sinh\frac{x}{c}}{1/c}\right|_0^x = \left(c\sinh\frac{x}{c} - c\sinh\frac{0}{c}\right)$$

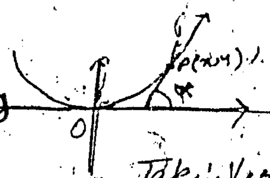
$\sinh(0) = 0$

$$S = c\sinh\frac{x}{c}$$

$$S = c\tan\alpha$$

proved

7.6-24



Taking vertical as axis, O as mid pt.

Q No 17. $x^2 = 4ay$

$\frac{x^2}{4a} = y \Rightarrow \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}.$

Also $\tan\alpha = \frac{dy}{dx} = \frac{x}{2a}$ $\therefore \frac{x}{2a} = \tan\alpha$

$x = 2a\tan\alpha.$

$$S = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^x \sqrt{1+\frac{x^2}{4a^2}}\, dx = \int_0^x \frac{\sqrt{4a^2+x^2}}{2a} = \frac{1}{2a}\int_0^x \sqrt{(2a)^2+x^2}\, dx$$

$$= \frac{1}{2a}\left|\frac{x}{2}\sqrt{(2a)^2+x^2} + \frac{(2a)^2}{2}\ln\left(\frac{x+\sqrt{(2a)^2+x^2}}{2a}\right)\right|_0^x$$

$\because \int\sqrt{x^2+a^2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right|$

$$= \frac{1}{2a}\left(\frac{x}{2}\sqrt{4a^2+x^2} + \frac{4a^2}{2}\ln\left(\frac{x+\sqrt{4a^2+x^2}}{2a}\right) - 0 - \frac{4a^2}{2}\ln\left(\frac{2a}{2a}\right)\right)$$

$\because \ln(1) = 0$

$$S = \frac{x}{4a}\sqrt{4a^2+x^2} + a\ln\left(\frac{x+\sqrt{4a^2+x^2}}{2a}\right)$$

$\because x = 2a\tan\alpha$

$$S = \frac{2a\tan\alpha}{4a}\sqrt{4a^2+4a^2\tan^2\alpha} + a\ln\left(\frac{2a\tan\alpha+\sqrt{4a^2+4a^2\tan^2\alpha}}{2a}\right)$$

$$= \frac{\tan\alpha}{2}\sqrt{4a^2(1+\tan^2\alpha)} + a\ln\left(\frac{2a\tan\alpha+\sqrt{4a^2(1+\tan^2\alpha)}}{2a}\right)$$

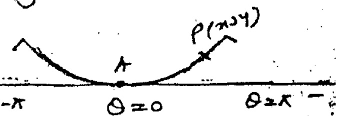
$$= \frac{\tan\alpha\,\cancel{2}a\sqrt{\sec^2\alpha}}{\cancel{2}} + a\ln\left[\frac{2a\tan\alpha+2a\sqrt{\sec^2\alpha}}{2a}\right]$$

$$= \tan\alpha\; a\sec\alpha + a\ln\left[\frac{\cancel{2a}(\tan\alpha+\sec\alpha)}{\cancel{2a}}\right]$$

$$= a\tan\alpha\sec\alpha + a\ln(\tan\alpha+\sec\alpha)$$

proved.

7.6-25



Q19. $x = a(\theta + \sin\theta)$ $y = a(1 - \cos\theta)$. Taking $\theta = 0$ as fixed pt

$\dfrac{dx}{d\theta} = a(1+\cos\theta)$ $\dfrac{dy}{d\theta} = a\sin\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\,\frac{d\theta}{dx} = a\sin\theta \cdot \frac{1}{a(1+\cos\theta)} = \frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2}$$

$$\frac{dy}{dx} = \tan\theta/2$$

Also $\dfrac{dy}{dx} = \tan\alpha$

$\therefore \boxed{\alpha = \frac{\theta}{2}}$ — (I)

$$S = \int_0^\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2[1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2(1+(\cos^2\theta+\sin^2\theta)+2\cos\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2(2+2\cos\theta)}\, d\theta = \int_0^\theta \sqrt{a^2 2(1+\cos\theta)}\, d\theta$$

$$= \int_0^\theta \sqrt{2a^2\left(2\cos^2\frac{\theta}{2}\right)}\, d\theta = \int_0^\theta 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 2a\left|\frac{\sin\theta/2}{1/2}\right|_0^\theta = 2a\left[2\sin\frac{\theta}{2} - 0\right] = 4a\sin\frac{\theta}{2}$$ — (II)

Put (I) in (II)

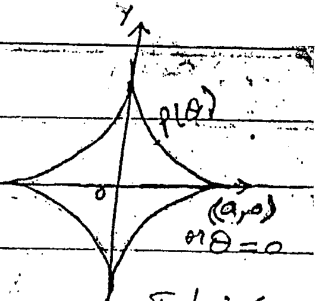
$$S = 4a\sin\frac{\theta}{2}$$

$$S = 4a\sin\alpha$$

7.6-26

Imp.

18 $\quad x^{2/3}+y^{2/3}=a^{2/3}$



Parametric Eqs are $\left.\begin{aligned} x&=a\cos^3\theta \\ y&=a\sin^3\theta \end{aligned}\right]$, $\quad \frac{dx}{d\theta}=3a\cos^2\theta(-\sin\theta)$, $\quad \frac{dy}{d\theta}=3a\sin^2\theta(\cos\theta)$

$$\tan\alpha=\frac{dy}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$$

Taking (a,0) as fixed pt. or θ=0 as fixed pt

$$\therefore \tan\alpha=-\tan\theta=\tan(-\theta)\implies\boxed{\alpha=-\theta}$$

$$S=\int_0^\theta\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta=\int_0^\theta\sqrt{9a^2\cos^4\theta\sin^2\theta+9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$=\int_0^\theta\sqrt{9a^2\sin^2\theta\cos^2\theta(\cos^2\theta+\sin^2\theta)}=\int_0^\theta 3a\sin\theta\cos\theta\,d\theta$$

$$S=\left|\frac{3a\sin^2\theta}{2}\right|_0^\theta=\left[\frac{3a}{2}\sin^2\theta-0\right]=\frac{3a}{2}\sin^2\theta$$

$$S=\frac{3a}{2}\sin^2(-\alpha)=\frac{3a}{2}\left[\sin(-\alpha)\right]^2=\frac{3a}{2}\sin^2\alpha$$

OR $\quad \frac{dy}{dx}=\frac{\sin\theta}{-\cos\theta}=-\tan\theta=\tan(\pi-\theta)$

but $\frac{dy}{dx}=\tan\alpha$

$$\therefore \tan\alpha=\tan(\pi-\theta)$$

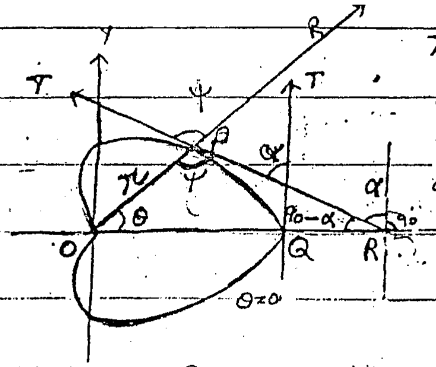
$$=\pi-\theta\implies\theta=\pi-\alpha$$

$$\therefore S=\frac{3a}{2}\sin^2\theta=\frac{3a}{2}\sin^2(\pi-\alpha)=\frac{3a}{2}\sin^2\alpha$$

Ans.

$r = a(1+\cos\theta)$

$\frac{dr}{d\theta} = a(-\sin\theta)$



Take Q ($\theta = 0$) as fixed pt ~~where tangent line is not~~ $\parallel$ to x-axis (initial) line as was in all previous Qs. $P(r,\theta)$ is any pt on the curve

$$S = \int_0^\theta \sqrt{r^2 + \frac{dr}{d\theta}}\, d\theta$$

$$\int_0^\theta \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\, d\theta = \int_0^\theta \sqrt{a^2(1+\cos^2\theta + 2\cos\theta + \sin^2\theta}\, d\theta$$

$$= \int_0^\theta \sqrt{a^2(1+1+2\cos\theta)}\, d\theta = \int_0^\theta \sqrt{a^2 2(1+\cos\theta)} = \int_0^\theta \sqrt{a^2 2\left(2\cos^2\frac{\theta}{2}\right)}\, d\theta$$

$$= \int_0^\theta 2a\cos\frac{\theta}{2}\, d\theta = \left| 2a\frac{\sin\frac{\theta}{2}}{\frac{1}{2}} \right|_0^\theta = \left| 4a\sin\frac{\theta}{2} \right|_0^\theta$$

$$S = 4a\sin\frac{\theta}{2} \quad \text{—— (1)} \quad \text{No fixed value of } \theta/2$$

$$\tan\psi = \frac{r}{\frac{dr}{d\theta}} = \frac{a(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\tan\psi = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right) \Rightarrow \boxed{\psi = \frac{\pi}{2} + \frac{\theta}{2}}$$

From fig. in $\triangle OPR$ $\quad \theta + \psi + 90 - \alpha = 180$

$$\theta + \psi = 180 - 90 + \alpha$$

$$\theta + \frac{\pi}{2} + \frac{\theta}{2} = 90 + \alpha$$

$$\frac{3\theta}{2} + \cancel{\frac{\pi}{2}} = \cancel{90} + \alpha \Rightarrow \frac{3\theta}{2} = \alpha \Rightarrow \boxed{\frac{\theta}{2} = \frac{\alpha}{3}}$$

Put in (1)

From (1) $S = 4a\sin\frac{\theta}{2}$

$$= 4a\sin\frac{\alpha}{3} \quad \text{proved}$$