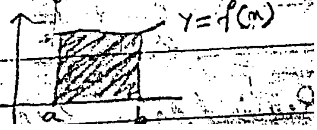
Quadrature

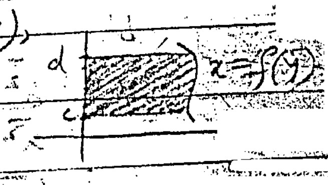
The process of finding the area of a plane region.

(1) Area of the region bounded by the curve, $y = f(x)$ the x-axis, $x = a$ and $x = b$



$$A = \int_a^b y\,dx.$$

(2) Area of the region bounded by the curve $x = f(y)$, y-axis, and $y = c$ and $y = d$



$$A = \int_c^d x\,dy$$

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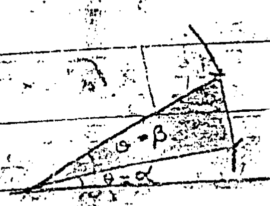
(3) Parametric Form $x = f(t)$ $y = g(t)$

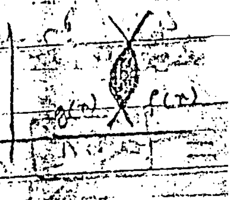
we shall use $\int y\,dx = \int_{t_1}^{t_2} g(t)\, f'(t)\,dt$

(4) Area bounded by the curve $r = f(\theta)$ and from $\theta = \alpha$ to $\theta = \beta$



$$\int_\alpha^\beta \frac{1}{2} r^2\,d\theta.$$

(5) Area between two curve



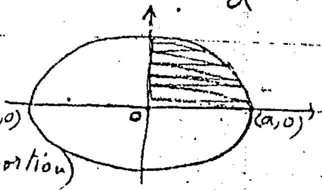
$$\int_a^b [g(x) - f(x)]\,dx.$$

Imp.

Ex 7.5

Q No 1 Area of the region bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{—①}$$



Required Area = 4 (Area of shaded portion)

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$A=\int_a^b y\,dx \qquad =4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$$

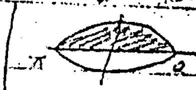
$$=\frac{4b}{a}\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right|_0^a$$

$$=\frac{4b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2}+c\sin^{-1}\left(\frac{a}{a}\right)-\left(0+\frac{a^2}{2}\sin^{-1}(0)\right)\right]$$

$$=\frac{4b}{a}\left[0+\frac{a^2}{2}\left(\frac{\pi}{2}\right)-0-0\right]=\frac{4ba^2\pi}{a\,4}$$

$$=\boxed{ab\pi} \text{ Ans (Area of Ellipse)}$$

II nd Method

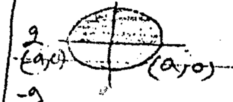


$x=a\cos\theta$, $dx=-a\sin\theta\,d\theta$

$y=b\sin\theta$

$$2\int_\pi^0 b\sin\theta(-a\sin\theta)\,d\theta$$

III nd Method



$$2\int_a^{-a} y\,dx$$

IV th Method

$$4\int_0^a y\,dx$$

$x=a\cos\theta$

$y=b\sin\theta$

$dx=-a\sin\theta\,d\theta$

Q2 Area of the region bounded by $y=\ln x$, x-axis, $x=a$, $x=b$

$$A=\int_a^b y\,dx = \int_a^b \ln x\,dx.$$

Integration by parts.

Area $= \int_a^b \underset{II}{1} \cdot \underset{I}{\ln x}\, dx$

$= \left| \ln x \,(x) \right|_a^b - \int_a^b \frac{1}{x}(x)\, dx$

$= b \ln b - a \ln a - \int_a^b dx$

$= b \ln b - a \ln a - |x|_a^b$

$= b \ln b - a \ln a - (b - a)$

$\Rightarrow \quad A = b \ln b - a \ln a - b + a.$

Q.3. Area of the region bounded by $xy = c^2$, x-axis, $x = a$, $x = b$

Area $= \int_a^b y\, dx = \int_a^b \frac{c^2}{x}\, dx$

$= c^2 \int_a^b \frac{dx}{x} = c^2 \left| \ln x \right|_a^b$

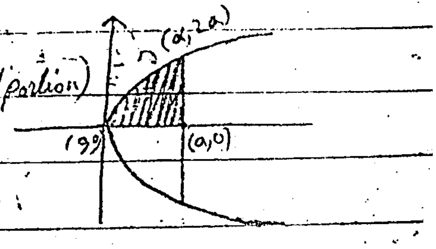
$= c^2 (\ln b - \ln a) = c^2 \ln\left(\frac{b}{a}\right)$ Ans.

Imp

Q4. Area of the region bounded by parabola $y^2 = 4ax$ & latus rectum

$y^2 = 4ax$

$y = \pm 2\sqrt{ax}$ Required Area = 2(shaded portion)



Area $= \int_a^b y\, dx = 2\int_0^a 2\sqrt{ax}\, dx$

$= 4\int_0^a a^{\frac{1}{2}} x^{\frac{1}{2}}\, dx = 4a^{\frac{1}{2}} \left| \frac{x^{3/2}}{3/2} \right|_0^a$

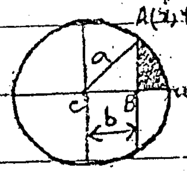
$= 4\sqrt{a}\left(\frac{a^{3/2}}{3/2} - 0\right) = \frac{8}{3} a^2$ Ans.

Q.5. Area of the region bounded by $y = c\cosh\left(\frac{x}{c}\right)$, x-axis, $x=a$, $x=b$

$$\int_a^b y\,dx = \int_a^b c\cosh\left(\frac{x}{c}\right)dx$$

$$= c\left|\frac{\sinh\frac{x}{c}}{1/c}\right|_a^b = c^2\left[\sinh\left(\frac{b}{c}\right) - \sinh\left(\frac{a}{c}\right)\right]$$

Q6. Taking centre of disc as origin and the chord ∥ to y-axis, then



a is radius, c is centre $(0,0)$

$(x-h)^2+(y-k)^2 = r^2$

$x^2+y^2 = a^2$

curve $x^2+y^2=a^2$ Required Area $= 2$(Shaded)

$y = \pm\sqrt{a^2-x^2}$

Also $x^2+y^2=a^2$

$$A = \int_a^b y\,dx = 2\int_b^a \sqrt{a^2-x^2}\,dx$$

$$= 2\left|\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right|_b^a$$

$$= 2\left(\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{\cancel{a}}{\cancel{a}}\right) - \frac{b}{2}\sqrt{a^2-b^2} - \frac{a^2}{2}\sin^{-1}\frac{b}{a}\right)$$

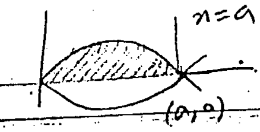
$$= 0 + \cancel{2}\frac{a^2}{\cancel{2}}\left(\frac{\pi}{2}\right) - \cancel{2}\frac{b}{\cancel{2}}\sqrt{a^2-b^2} - \cancel{2}\frac{a^2}{\cancel{2}}\sin^{-1}\frac{b}{a}$$

$$A = a^2\frac{\pi}{2} - b\sqrt{a^2-b^2} - a^2\sin^{-1}\left(\frac{b}{a}\right)$$ Ans.

Q7. $3ay^2 = x(x-a)^2$ ∵ boundary conditions are not given

Put $y = -y$: No change ∴ Curve is symmetric w.r.t. x-axis

Put $y=0$ $x(x-a)^2 = 0$ $\Rightarrow$ $x=0$ $x=a$



$3ay^2 = x(x-a)^2 \Rightarrow y^2 = \frac{x(x-a)^2}{3a}$

$y = \sqrt{\frac{x(x-a)^2}{3a}} \Rightarrow y = \frac{\sqrt{x}}{\sqrt{3a}}(x-a)$

Area $= \int_a^b y\,dx = 2\int_0^a \frac{\sqrt{x}}{\sqrt{3a}}(x-a)\,dx$

$= \frac{2}{\sqrt{3a}}\int_0^a \left(x^{3/2} - a x^{\frac{1}{2}}\right)dx$

$= \frac{2}{\sqrt{3a}}\left| \frac{x^{5/2}}{5/2} - a\frac{x^{3/2}}{3/2}\right|_0^a$

$= \frac{2}{\sqrt{3a}}\left(\frac{2a^{5/2}}{5} - 2a\frac{a^{3/2}}{3} - 0 - 0\right) = \frac{4}{\sqrt{3a}}\left(\frac{a^{5/2}}{5} - \frac{a^{5/2}}{3}\right)$

$= \frac{4a^{5/2}}{\sqrt{3a}}\left(\frac{1}{5} - \frac{1}{3}\right) = \frac{4a^{5/2}}{\sqrt{3a}}\left(\frac{3-5}{15}\right)$

$= \frac{-8a^{5/2}a^{-\frac{1}{2}}}{\sqrt{3}(15)} = \frac{-8a^2}{15\sqrt{3}}$

But Area is always +ive $\therefore$ Area $= \frac{8a^2}{15\sqrt{3}}$

Q8. $x^2(x^2+y^2) = a^2(y^2-x^2)$

Put $y=-y$ No change so curve is symmetric about x-axis

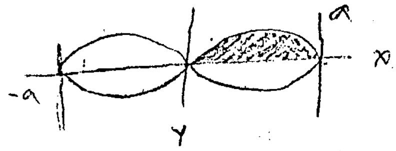
Put $x=-x$ No change so curve is symmetric about y-axis

Asymptote ∥ to y-axis $\{x - (0)\}$

$x^4 + x^2y^2 - a^2y^2 + a^2x^2 = 0$

$x^4 + y^2(x^2-a^2) + a^2x^2 = 0$

7.5 – 6



Coeff of highest power of y = 0: $x^2 - a^2 = 0$

$x = \pm a$

$x^4 + x^2y^2 - a^2y^2 + a^2x^2 = 0$

$y^2(x^2 - a^2) = -x^4 - a^2x^2$

$y^2 = \dfrac{-x^2(x^2+a^2)}{(x^2-a^2)}$

$y^2 = \dfrac{x^2(x^2+a^2)}{a^2-x^2}$

$y = \dfrac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}}$

$$\text{Area} = \int_a^b y\,dx = 4\int_0^a \frac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}}\,dx$$

$\times$ & $\div$ by $\sqrt{a^2+x^2}$

$$= 4\int_0^a \frac{x\sqrt{x^2+a^2}}{\sqrt{a^2-x^2}} \cdot \frac{\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}\,dx$$

$$= 4\int_0^a \frac{x(x^2+a^2)}{\sqrt{a^4-x^4}}\,dx$$

Also we can use $x^2 = a^2\cos\theta$ or $x^2 = a^2\sin\theta$. $2x\,dx = -a^2\sin\theta\,d\theta$, $0, \pi/2$.

$$= \int_0 \frac{x^3\,dx}{\sqrt{a^4-x^4}} + 4\int_0 \frac{xa^2}{\sqrt{a^4-x^4}}\,dx$$

Put $x^2 = z$
$2x\,dx = dz$
when $x = 0$, $z = 0$
when $x = a$, $z = a^2$

$$= -\int_0^a (a^4-x^4)^{-\frac{1}{2}}(-4x^3)\,dx + 2\int_0^{a^2} \frac{a^2\,dz}{\sqrt{(a^2)^2-z^2}}$$

$$= -\left|\frac{(a^4-x^4)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^a + 2a^2\int_0^{a^2} \frac{dz}{\sqrt{(a^2)^2-z^2}}$$

$$= -\left[2\left\{(a^4-a^4)^{\frac{1}{2}} - (a^4)^{\frac{1}{2}}\right\}\right] + 2a^2\left|\sin^{-1}\frac{z}{a^2}\right|_0^{a^2}$$

$$= -2\{0 - a^2\} + 2a^2\left(\sin^{-1}\left(\frac{a^2}{a^2}\right) - \sin^{-1}(0)\right)$$ by conditions given

$$A = 2a^2 + 2a^2\left(\frac{\pi}{2} - 0\right)$$

$$A = 2a^2 + \frac{2a^2\pi}{2} = a^2(2+\pi) \text{ Ans.}$$

Q9. $xy^2 = 4(2-x)$

$y^2 = \dfrac{4(2-x)}{x} \Rightarrow y = \dfrac{2\sqrt{2-x}}{\sqrt{x}}$

The curve is symmetric about x-axis (No change. (y by $-y$))

Put $y=0$. $\dfrac{2\sqrt{2-x}}{\sqrt{x}} = 0 \Rightarrow \sqrt{2-x} = 0$

$\Rightarrow x = 2$

Asymptote || to y-axis $x = 0$.

$y = \pm 2\sqrt{\dfrac{2-x}{x}}$ — Required Area = 2(Shaded Area)

$A = 2\int_0^2 2\sqrt{\dfrac{2-x}{x}}\, dx.$

Put $x = 2\sin^2\theta$

$dx = 4\sin\theta\cos\theta\, d\theta$

when $x=0$ $\theta = 0$

when $x=2$ $\theta = \frac{\pi}{2}$

$= 4\int_0^{\pi/2} \dfrac{\sqrt{2-2\sin^2\theta}}{\sqrt{2\sin^2\theta}}\, 4\sin\theta\cos\theta\, d\theta.$

$= 16\int_0^{\pi/2} \dfrac{\cancel{\sqrt{2}}\cos\theta}{\cancel{\sqrt{2}}\sin\theta}\cdot(\sin\theta\cos\theta)\, d\theta$

$= 16\int_0^{\pi/2} \cos^2\theta\, d\theta. \quad = 16\int_0^{\pi/2} \left(\dfrac{1+\cos 2\theta}{2}\right) d\theta$

$= 8\int_0^{\pi/2} d\theta + 8\int_0^{\pi/2} \cos 2\theta\, d\theta.$

$= 8\left|\theta\right|_0^{\pi/2} + 8\left|+\dfrac{\sin 2\theta}{2}\right|_0^{\pi/2}$

$= 8\left(\dfrac{\pi}{2}\right) + 8\left(\dfrac{+\sin 2\frac{\pi}{2}}{2}\right)$

$= 4\pi - 0 = 4\pi$ Ans.

Q10. $x^2y^2 = a^2(y^2-x^2)$

Asymptotes // to y-axis —— $x^2y^2 - a^2y^2 + a^2x^2 = 0$

$y^2(x^2-a^2) + a^2x^2 = 0$

$x^2 - a^2 = 0 \Rightarrow x = \pm a$

The curve is symmetric about x-axis & y-axis ($\because$ even powers of x & y)

y is real for $-a < x < a$

$y^2(x^2-a^2) + a^2x^2 = 0$

$y^2 = \frac{-a^2x^2}{(x^2-a^2)}$

$= \frac{a^2x^2}{(a^2-x^2)}$

$y = \pm \frac{ax}{\sqrt{a^2-x^2}}$

Required area $= 4\int_0^a \frac{ax}{\sqrt{a^2-x^2}}\,dx$

$= -4a\int_0^a (a^2-x^2)^{-\frac{1}{2}}(-x\,dx)$

$= -2a\int_0^a (a^2-x^2)^{-\frac{1}{2}}(-2x\,dx)$

$= -2a\left|\frac{(a^2-x^2)^{\frac{1}{2}}}{1/2}\right|_0^a$

$= -4a\left[(a^2-a^2)^{\frac{1}{2}} - (a^2-0)^{\frac{1}{2}}\right] = -4a\cdot\left(0-(a^2)^{\frac{1}{2}}\right)$

$= 4a^2$ Ans.

Q11. $ay^2 = x^2(a-x)$

The curve is symmetric only about x-axis ($\because$ even power of y)

Put $y=0 \Rightarrow x^2(a-x) = 0$ $x = 0, a$.

Area $= 2\int_0^a \frac{x\sqrt{a-x}}{\sqrt{a}}\,dx$

$y^2 = \frac{x^2(a-x)}{a}$

$y = \frac{x\sqrt{a-x}}{\sqrt{a}}$

Put $x = a\sin^2\theta$ $dx = 2a\sin\theta\cos\theta\,d\theta$

when $x = 0$ $\sin^2\theta = 0$ $\theta = 0$

when $x = a$ $\sin^2\theta = 1$ $\theta = 90$

$$\text{Area} = 2\int_0^{\pi/2} \frac{a\sin^2\theta\sqrt{a - a\sin^2\theta}}{\sqrt{a}} \times 2a\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \frac{\sin^2\theta\,\cancel{\sqrt{a}}\sqrt{1-\sin^2\theta}}{\cancel{\sqrt{a}}}\,\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \sin^3\theta\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$$

Using Walli Formula $\int_0^{\pi/2} \sin^p x\cos^q x = \frac{(p-1)(p-3)\cdots 2.1\,(q-1)(q-3)\cdots 2}{(p+q)(p+q-2)(p+q-4)\cdots}$

$$= 4a^2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta = 4a^2\left(\frac{(3-1)\cdots(2-1)}{(3+2)(3+2-2)(3+2-4)}\right)$$

$$4a^2\left(\frac{2}{(5)(3)(1)}\right) = \frac{8a^2}{15}.$$

Q. $a^2x^2 = y^3(2a-y)$

Change y by $-y$, Curve is changed So not symmetric about x-axis.

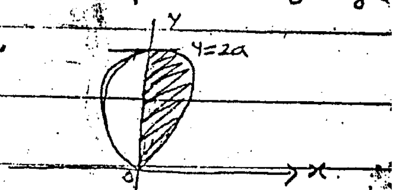
Change x by $-x$, Curve is not changed, So symmetric about y-axis.

For Y-Intercept Put $x = 0$ $y^3(2a-y) = 0$

$y = 0, 2a.$

Area $= \int_a^b x\,dy$

Curve passes through origin



Required Area = 2 (Shaded Portion)

$$A = 2\int_0^{2a} \frac{y^{3/2}}{a}\sqrt{(2a-y)}\;dy$$

{Shaded portion is bounded by the curve, y-axis, y = 0, y = 2a.}

$$a^2x^2 = y^3(2a-y)$$

$$x^2 = \frac{y^3}{a^2}(2a-y)$$

$$x = \pm\frac{y^{3/2}}{a}\sqrt{2a-y}$$

Put $y = 2a\sin^2\theta$

$dy = 4a\sin\theta\cos\theta\,d\theta$.

when $y = 0$ $\quad 0 = \sin\theta \Rightarrow \theta = 0$.

when $y = 2a$ $\quad 1 = \sin\theta \Rightarrow \theta = \pi/2$.

$$A = 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{a}\sqrt{2a - 2a\sin^2\theta}\,\{4a\sin\theta\cos\theta\,d\theta\}$$

x is real for $0 < y < 2a$

$$= \frac{2}{a}\int_0^{\pi/2} 2^{3/2} a^{3/2}\sin^3\theta\,\sqrt{2a}\,\sqrt{1-\sin^2\theta}\,(4a\sin\theta\cos\theta\,d\theta)$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\,\sqrt{\cos^2\theta}\,\cos\theta\,d\theta.$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta.$$

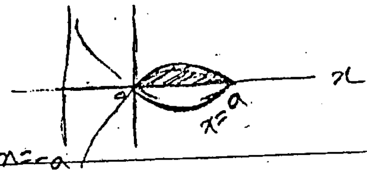
Wallis Formula.

$$\int_0^{\pi/2}\sin^p x\cos^q x\,dx = \frac{(p-1)(p-3)\cdots(q-1)(q-3)\cdots}{(p+q)(p+q-2)(p+q-4)\cdots}\frac{\pi}{2}$$

$$= 32a^2\left(\frac{(4-1)(4-3)\cdots(2-1)}{(4+2)(4+2-2)(4+2-4)\cdots}\pi/2\right)$$

$$= 32a^2\left(\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\pi/2\right) = \frac{32a^2(3)}{6\cdot4\cdot2}\frac{\pi}{2} = \frac{\cancel{96}a^2\pi}{\cancel{96}} = a^2\pi$$

& $a^2\pi$ is the area of circular disc of radius a.



Q13. Area of the region bounded by the loop of the curve $y^2(a+x)=x^2(a-x)$

The curve is symmetric about x-axis. (Even power of y)

x-Intercept Put $y=0$ $x^2(a-x)=0$ $x=0, a$.

So Area $=2\int_0^a x\frac{\sqrt{a-x}}{\sqrt{a+x}}\,dx$. $y^2=\frac{x^2(a-x)}{a+x}$

$y = x\sqrt{\frac{a-x}{a+x}}$.

Put $x=a\cos\theta$

$\therefore dx = -a\sin\theta\, d\theta$.

$x=0 \Rightarrow \cos\theta=0$ $\theta=\frac{\pi}{2}$

$x=a \Rightarrow \cos\theta=\frac{a}{a}=1$ $\theta=0$.

$$\text{Area} = 2\int_{\pi/2}^{0} a\cos\theta\,\frac{\cancel{\sqrt{a}}\sqrt{1-\cos\theta}}{\cancel{\sqrt{a}}\sqrt{1+\cos\theta}}\,(-a\sin\theta)\,d\theta$$

$\cos\theta = 2\cos^2\frac{\theta}{2}-1$

$1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$\cos\theta = 1-2\sin^2\frac{\theta}{2}$

$1-\cos\theta = 2\sin^2\frac{\theta}{2}$

$$=-2\int_{\pi/2}^{0} a^2\cos\theta\sin\theta\left(\frac{\cancel{\sqrt{2}}\sin\frac{\theta}{2}}{\cancel{\sqrt{2}}\cos\frac{\theta}{2}}\right)d\theta$$

$$=-2a^2\int_{\pi/2}^{0}\left(1-2\sin^2\frac{\theta}{2}\right)\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\frac{\sin\theta/2}{\cos\theta/2}$$

$$=-2a^2\int_{\pi/2}^{0}\left(2\sin^2\frac{\theta}{2}-4\sin^4\frac{\theta}{2}\right)d\theta$$

$$=-4a^2\int_{\pi/2}^{0}\left(\sin^2\frac{\theta}{2}-2\sin^4\frac{\theta}{2}\right)d\theta = -4a^2\int_{\pi/2}^{0}\sin^2\frac{\theta}{2}\,d\theta+8a^2\int_{\pi/2}^{0}\sin^4\frac{\theta}{2}\,d\theta$$

$$=+4a^2\int_{0}^{\pi/2}\sin^2\frac{\theta}{2}\,d\theta-8a^2\int_{0}^{\pi/2}\sin^4\frac{\theta}{2}\,d\theta.$$

$$4a^2\int_0^{\pi/2}\left(\frac{1-\cos\theta}{2}\right)d\theta \quad -8a^2\int_0^{\pi/2}\left(\frac{1-\cos\theta}{2}\right)^2 d\theta$$

$$=\frac{4a^2}{2}\Big|\theta-\sin\theta\Big|_0^{\pi/2} \quad - \quad 8a^2\int_0^{\pi/2}\left(\frac{1+\cos^2\theta-2\cos\theta}{4}\right)d\theta$$

$$=2a^2\left(\frac{\pi}{2}-\sin\frac{\pi}{2}-0-0\right)-2a^2\int_0^{\pi/2}\left(1+\left(\frac{1+\cos2\theta}{2}\right)-2\cos\theta\right)d\theta$$

$$=\cancel{2}a^2\frac{\pi}{\cancel{2}}-2a^2\sin\left(\frac{\pi}{2}\right) \quad -\frac{2a^2}{2}\int_0^{\pi/2}(2+1+\cos2\theta-4\cos\theta)\,d\theta$$

$$=a^2\pi-2a^2(1) \quad -a^2\int_0^{\pi/2}(3+\cos2\theta-4\cos\theta)\,d\theta$$

$$=a^2\pi-2a^2 \quad -a^2\Big|3\theta+\frac{\sin2\theta}{2}-4\sin\theta\Big|_0^{\pi/2}$$

$$=a^2\pi-2a^2 \quad -a^2\left[3\frac{\pi}{2}+0-4\right]$$

$$=a^2\pi-2a^2-3a^2\frac{\pi}{2}+4a^2=\frac{2a^2\pi-4a^2-3a^2\pi+8a^2}{2}$$

$$=\frac{-a^2\pi+4a^2}{2} \quad =\boxed{\frac{a^2}{2}(4-\pi)}$$

Now area between the curve & asymptote

Asymptote ∥ to y-axis $\quad x+a=0 \quad x=-a$

$$\text{Area}=2\int_{-a}^{0}x\sqrt{\frac{a-x}{a+x}}\,dx \qquad \text{Put } x=a\cos\theta \quad dx=-a\sin\theta\,d\theta$$

$$x=-a,\ \theta=\pi \qquad x=0,\ \theta=\frac{\pi}{2}$$

$$=2\int_{\pi}^{\pi/2}a\cos\theta\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\,(-a\sin\theta\,d\theta)$$

$$=2a^2\int_{\pi/2}^{\pi}\cos\theta\sin\theta\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,d\theta \quad =2a^2\Big|\theta-\sin\theta\Big|_{\pi/2}^{\pi}-a^2\Big|3\theta+\frac{\sin2\theta}{2}-4\sin\theta\Big|_{\pi/2}^{\pi}$$

$$= 2a^2\left[\pi - \sin\pi - \frac{\pi}{2} + \sin\frac{\pi}{2}\right] - a^2\left[3\pi + \frac{\sin 2\pi}{2} - 4\sin\pi - \frac{3\pi}{2} - \frac{\sin 2(\frac{\pi}{2})}{2} + 4\sin(\tfrac{\pi}{2})\right]$$

$$= 2a^2\left[\pi - 0 - \frac{\pi}{2} + 1\right] - a^2\left[3\pi + 0 - 0 - \frac{3\pi}{2} - 0 + 4\right]$$

$$= 2a^2\left[\frac{\pi}{2} + 1\right] - a^2\left(3\pi - \frac{3\pi}{2} + 4\right)$$

$$= \frac{2a^2\pi}{2} + 2a^2 - a^2\left(\frac{3\pi}{2} + 4\right)$$

$$= a^2\pi + 2a^2 - \frac{a^2 3\pi}{2} - 4a^2$$

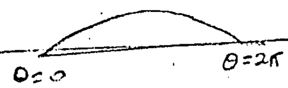
$$= \frac{2a^2\pi + 4a^2 - a^2 3\pi - 8a^2}{2}$$

$$= \frac{-a^2\pi - 4a^2}{2}$$

Area $= \frac{-a^2}{2}(\pi + 4)$

Q14 $\quad x = a(\theta - \sin\theta) \qquad y = a(1 - \cos\theta)$

$dx = a(1 - \cos\theta)\,d\theta$



$$A = \int_0^{2\pi} a(1-\cos\theta)\,a(1-\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}(1-\cos\theta)^2 \qquad = a^2\int_0^{2\pi}(1 + \cos^2\theta - 2\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}\left(1 + \frac{1+\cos 2\theta}{2} - 2\cos\theta\right)d\theta$$

$$= a^2\int_0^{2\pi}\left(\frac{2+1+\cos 2\theta - 4\cos\theta}{2}\right)d\theta$$

$$\frac{a^2}{2}\int_0^{2\pi}(3+\cos 2\theta - 4\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\Big|3\theta + \frac{\sin 2\theta}{2} - 4\sin\theta\Big|_0^{2\pi}$$

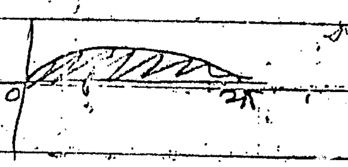
$$= \frac{a^2}{2}\left[3(2\pi) + \frac{\sin 2(2\pi)}{2} - 4\sin(2\pi) - 0 + 0 - 0\right]$$

$$= \frac{a^2}{2}6\pi = \boxed{3a^2\pi}$$

Q15 $x = a(\theta + \sin\theta)$

$y = a(1-\cos\theta)$

$dx = a(1+\cos\theta)\,d\theta.$



$$\int_0^{2\pi} y\,dx = \int_0^{2\pi} a(1-\cos\theta)\,a(1+\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi} 1-\cos^2\theta\,d\theta = a^2\int_0 \sin^2\theta\,d\theta.$$

$$= a^2\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta - \frac{a^2}{2}\int\cos 2\theta\,d\theta.$$

$$= \frac{a^2}{2}\Big|\theta\Big|_0^{2\pi} - \frac{a^2}{2}\Big|\frac{\sin 2\theta}{2}\Big|_0^{2\pi}$$

$$= \frac{a^2}{\cancel{2}}(\cancel{2}\pi) - \frac{a^2}{2}\left(\frac{\sin 2(2\pi)}{2} - 0\right)$$

Area $= a^2\pi - 0 = a^2\pi.$

Q15 Area of the region bounded by the loop of the curve

$x^4 + y^4 = 2a^2 xy.$

Put $x = r\cos\theta.$ $y = r\sin\theta.$

$r^4\cos^4\theta + r^4\sin^4\theta = 2a^2 r\cos\theta\, r\sin\theta$

$r^4(\cos^4\theta + \sin^4\theta) = 2a^2 r^2 \cos\theta\sin\theta$

$$\frac{r^4}{r^2} = \frac{2a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}$$

$$\Rightarrow r^2 = \frac{2a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}$$

Put $r = 0 \Rightarrow 2a^2\sin\theta\cos\theta = 0$

$\sin\theta\cos\theta = 0$

$\sin\theta = 0$ $\cos\theta = 0$

$\theta = 0$ $\theta = \pi/2$

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

$$\Rightarrow = \frac{1}{2}\int_0^{\pi/2} \frac{2a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}\,d\theta$$

$$= \frac{2a^2}{2}\int_0^{\pi/2} \frac{\frac{\sin\theta\cos\theta}{\cos^4\theta}}{\frac{\cos^4\theta + \sin^4\theta}{\cos^4\theta}}$$

Divide N & D by $\cos^4\theta$.

$$= a^2\int_0^{\pi/2} \frac{\tan\theta\sec^2\theta}{1 + \tan^4\theta}\,d\theta$$

Put $\tan^2\theta = u$.

$2\tan\theta\sec^2\theta\, d\theta = du$. when $\theta = 0$ $u = 0$

$\tan\theta\sec^2\theta\, d\theta = \dfrac{du}{2}$. when $\theta = \frac{\pi}{2}$ $u = \infty$

$$\text{Area} = a^2\int_0^\infty \frac{du/2}{1+u^2}$$

$$= \frac{a^2}{2}\int_0^\infty \frac{du}{1+u^2} = \frac{a^2}{2}\Big|\tan^{-1}u\Big|_0^\infty$$

$$= \frac{a^2}{2}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi a^2}{4}.$$

Q16. $(x+y)^2(x^2+y^2) = 2a^2xy$

Put $x = r\cos\theta$ $y = r\sin\theta$.

$(r\cos\theta + r\sin\theta)^2(r^2\cos^2\theta + r^2\sin^2\theta) = 2a^2 r\cos\theta\, r\sin\theta$.

$(r^2\cos^2\theta + r^2\sin^2\theta + 2r^2\sin\theta\cos\theta)(r^2)(\cos^2\theta + \sin^2\theta) = 2a^2r^2\cos\theta\sin\theta$.

$r^2(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta) = 2a^2\dfrac{\cancel{r^2}}{\cancel{r^2}}\cos\theta\sin\theta$

$r^2(1 + 2\sin\theta\cos\theta) = 2a^2\sin\theta\cos\theta$.

$$r^2 = \frac{2a^2\sin\theta\cos\theta}{1+2\sin\theta\cos\theta}$$

Put $r = 0$ $\Rightarrow$ $2a^2\sin\theta\cos\theta = 0$.

7.5-17

$\sin\theta\cos\theta = 0$

$\sin\theta = 0$ $\qquad\qquad \cos\theta = 0$

$\theta = 0$ $\qquad\qquad \theta = \frac{\pi}{2}$

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{a^2\, 2\sin\theta\cos\theta}{1+2\sin\theta\cos\theta}\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{\sin 2\theta}{1+\sin 2\theta}\, d\theta = a^2\int_0^{\pi/2} \frac{\sin 2\theta + 1 - 1}{\sin 2\theta + 1}\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(1 - \frac{1}{\sin 2\theta + 1}\right) d\theta$$

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$$= \frac{a^2}{2}\int_0^{\pi/2} d\theta - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\sin 2\theta}$$

$\sin\theta = \cos(90-\theta)$

$$= \frac{a^2}{2}\left|\theta\right|_0^{\frac{\pi}{2}} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\cos(90-2\theta)}$$

$\cos 2\theta = 2\cos^2\theta - 1$

$1+\cos 2\theta = 2\cos^2\theta$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} - 0\right) - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{2\cos^2\frac{(90-2\theta)}{2}}$$

$$= \frac{a^2\pi}{4} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{2\cos^2(45-\theta)}$$

$\int \frac{d\theta}{\sec^2\theta} = \tan$

$$= \frac{a^2\pi}{4} - \frac{a^2}{4}\int_0^{\pi/2} \sec^2(45-\theta)\, d\theta$$

$$= \frac{a^2\pi}{4} - \frac{a^2}{4}\left|\frac{\tan(45-\theta)}{-1}\right|_0^{\pi/2}$$

$$= \frac{a^2\pi}{4} + \frac{a^2}{4}\left[\tan\left(\frac{\pi}{4}-\frac{\pi}{2}\right) - \tan\left(\frac{\pi}{4}-0\right)\right].$$

$$= \frac{a^2\pi}{4} + \frac{a^2}{4}\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{a^2\pi}{4} + \frac{a^2}{4}\left[-\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{a^2\pi}{4} + \frac{a^2}{4}\left(-2\tan\frac{\pi}{4}\right) \qquad \because \tan\frac{\pi}{4} = 1$$

$$= \frac{a^2\pi}{4} - \frac{a^2}{2} \qquad = \frac{a^2}{2}\left(\frac{\pi}{2}-1\right)$$

Q17 Area of the region bounded by the circle $r=a$ & $r = a\cos 5\theta$.

Area of the circle $= \pi a^2$ — Required Area.

Symmetric is about initial line (θ by $-\theta$)

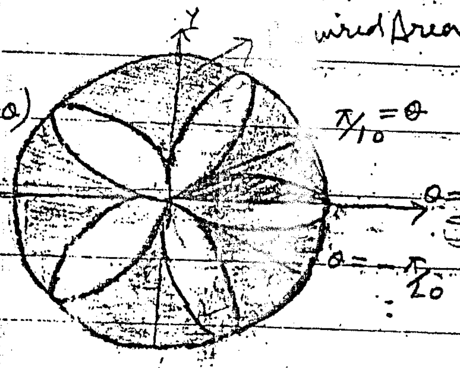
It h 5 loops.

For limits of loops Put $r=0$.

$0 = a\cos 5\theta$.

$\cos 5\theta = 0 \Rightarrow 5\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{10}$



Area of One loop $= 2\int_0^{\pi/10} \frac{r^2 d\theta}{2} = 2\cdot\frac{1}{2}\int_0^{\pi/10} a^2\cos^2 5\theta\, d\theta.$

$$= a^2 \int_0^{\pi/10} \cos^2 5\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{1+\cos 2\theta}{2} = \cos^2\theta$

$\frac{1+\cos 10\theta}{2} = \cos^2 5\theta$

$$= a^2 \int_0^{\frac{\pi}{10}} \left(\frac{1+\cos 10\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{10}} d\theta + \frac{a^2}{2}\int_0^{\frac{\pi}{10}} \cos 10\theta \, d\theta.$$

$$= \frac{a^2}{2}\left|\theta\right|_0^{\frac{\pi}{10}} + \frac{a^2}{2}\left|\frac{\sin 10\theta}{10}\right|_0^{\frac{\pi}{10}}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{10}\right) + \frac{a^2}{20}\left(\sin \cancel{10}\left(\frac{\pi}{\cancel{10}}\right) - \sin 10(0)\right)$$

Area of one loop $= \frac{a^2\pi}{20} + 0 - 0.$

Area of 5 loops $= 5 \times \frac{a^2\pi}{20} = \frac{a^2\pi}{4}$

~~Area of the circle~~ $= \pi a^2$

~~Area bounded by~~ $r = a$ & $r = a\cos 5\theta$ is

$$\pi a^2 - \frac{a^2\pi}{4} = \frac{3\pi a^2}{4} \quad \text{Ans.}$$

Q18. $r = a(1+\cos\theta)$

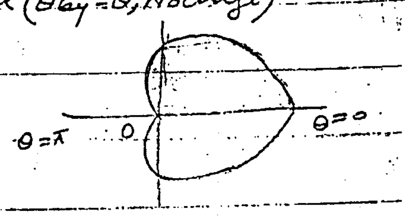
It is symmetrical about initial line (θ by $-\theta$, No change)

Put $r = 0$

$a(1+\cos\theta) = 0$

$1+\cos\theta = 0$

$\cos\theta = -1 \quad \Rightarrow \theta = \pi$



Area of Cardiod $= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta = \frac{2\cdot 1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$

$= a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$ $\quad \because \cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\, d\theta$ $\quad$ Put $\frac{\theta}{2} = u$

$\frac{d\theta}{2} = du$

$= 4a^2\int_0^{\frac{\pi}{2}}\cos^4 u\; 2du$ $\quad d\theta = 2du$

$= 8a^2\int_0^{\frac{\pi}{2}}\cos^4 u\, du$

when $\theta = 0 \quad u = 0$

when $\theta = \pi \quad u = \frac{\pi}{2}$

Using Walli's Formula

$= 8a^2\int_0^{\frac{\pi}{2}}\cos^4 u\,du = 8a^2\left(\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot \cdots \frac{\pi}{2}\right)$

$\int_0^{\frac{\pi}{2}}\cos^{n} x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$ $\quad n \rightarrow$ even

$= 8a^2\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) = \frac{3(3a^2\pi)}{6} = \frac{3a^2\pi}{2}$ Ans

Q19 $\quad r^2 = a^2\cos 2\theta$.

It is symmetrical about initial line (θ by $-\theta$, unchanged)

It is symmetrical about y-axis (θ by $(\pi-\theta)$, unchanged)

Put $r = 0$. $\quad a^2\cos 2\theta = 0$

$\cos 2\theta = 0$

$2\theta = \cos^{-1}(0) = \frac{\pi}{2} \quad \Rightarrow \theta = \frac{\pi}{4}$

$$= \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta \sec^2\theta}{(1+\tan^3\theta)^2}\, d\theta$$

$$= \frac{3a^2}{2}\int_0^{\frac{\pi}{2}} (1+\tan^3\theta)^{-2}\, 3\tan^2\theta \sec^2\theta\, d\theta.$$

$$= \frac{3a^2}{2}\left|\frac{(1+\tan^3\theta)^{-1}}{-1}\right|_0^{\frac{\pi}{2}} = \frac{-3a^2}{2}\left|\frac{1}{1+\tan^3\theta}\right|_0^{\frac{\pi}{2}}$$

$$= -\frac{3a^2}{2}\left[\frac{1}{1+\tan^3(\frac{\pi}{2})} - \frac{1}{1+0}\right]$$

$$= -\frac{3a^2}{2}\left(\frac{1}{\infty} - 1\right) = \frac{-3a^2}{2}\left(0-1\right)$$

$$= \frac{3a^2}{2} \text{ Ans.}$$

x———————x

Imp
Q21 $r = \dfrac{6}{2-\cos\theta}$ ——— ①

$r=0 \Rightarrow \dfrac{6}{2-\cos\theta} = 0$
$6 = 0$ Impossible
$\therefore$ Change in Cartesian form

Change into Cartesian Coord System

$r = \sqrt{x^2+y^2}$ $x = r\cos\theta$

$\therefore$ ① becomes $2r - r\cos\theta = 6.$

$2\sqrt{x^2+y^2} - x = 6.$

$2\sqrt{x^2+y^2} = 6+x$

Squaring $4(x^2+y^2) = x^2 + 36 + 12x.$

Area $= 4\int_0^{\frac{\pi}{4}} \frac{r^2 d\theta}{2} = \frac{4}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta.$

$= 2a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta.$

$= 2a^2 \left| \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{4}}$

$= \frac{2a^2}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right] = a^2\left[\sin\left(\frac{\pi}{2}\right)\right]$

Area $= a^2.$

Q20. $r = \frac{3a \sin\theta \cos\theta}{\cos^3\theta + \sin^3\theta}$ folium $x^3 + y^3 = 3axy$

Put $r = 0$

$3a \sin\theta \cos\theta = 0.$

$\sin\theta \cos\theta = 0.$

$\sin\theta = 0$

$\theta = 0$; $\cos\theta = 0$

$\theta = \frac{\pi}{2}.$

Area $= \frac{1}{2}\int_\alpha^\beta r^2 d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{(3a \sin\theta \cos\theta)^2}{(\cos^3\theta + \sin^3\theta)^2} d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{9a^2 \sin^2\theta \cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2} d\theta$

$= \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{\sin^2\theta \cos^2\theta}{\cos^6\theta}}{\frac{(\cos^3\theta + \sin^3\theta)^2}{\cos^6\theta}} = \frac{9a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\cos^2\theta}{\cos^2\theta} \cdot \frac{1}{\cos^2\theta}}{\left(\frac{\cos^3\theta}{\cos^3\theta} + \frac{\sin^3\theta}{\cos^3\theta}\right)^2}$

$4x^2+4y^2-12x-x^2=36$

$3x^2-12x+4y^2=36$

$3(x^2-4x)+4y^2=36$

$3(x^2-4x+4-4)+4y^2=36$ Completing Square

$3((x-2)^2-4)+4y^2=36$

$3(x-2)^2-12+4y^2=36$

$3(x-2)^2+4y^2=48$

$\frac{3(x-2)^2}{48}+\frac{4y^2}{48}=1$

$\frac{(x-2)^2}{16}+\frac{y^2}{12}=1$

$\frac{(x-2)^2}{(4)^2}+\frac{y^2}{(\sqrt{12})^2}=1$

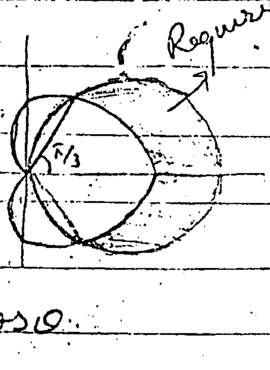
Area of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $ab\pi$.

$a=4 \quad b=\sqrt{12}$

So Area of the curve is $4(\sqrt{12})\pi$

$=4(2)\sqrt{3}\,\pi = 8\sqrt{3}\,\pi$

Q22. Area of the region that lies outside the cardiod $r=1+\cos\theta$ and inside the circle $r=3\cos\theta$.



$r=1+\cos\theta$

$r=3\cos\theta$ $\therefore$ $1+\cos\theta=3\cos\theta$

$1=2\cos\theta$

$\frac{1}{2}=\cos\theta$ $\quad\theta=\pm\frac{\pi}{3}$

Cartesian Polar

$$\text{Area} = \int_a^b f(x) - g(x)\,dx = \int_\alpha^\beta \frac{(r_1^2 - r_2^2)}{2}\,d\theta$$

$$\text{Required Area} = 2\int_0^{\frac{\pi}{3}} \frac{1}{2}\left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$

$$= \frac{2}{2}\int_0^{\pi/3} (4\cos^2\theta - 1 - \cos^2\theta - 2\cos\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} (8\cos^2\theta - 2\cos\theta - 1)\,d\theta$$

$2\cos^2\theta - 1 = \cos 2\theta$

$$= \int_0^{\frac{\pi}{3}} \left[8\left(\frac{1+\cos 2\theta}{2}\right) - 2\cos\theta - 1\right] d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4 + 4\cos 2\theta - 2\cos\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4\cos 2\theta - 2\cos\theta + 3)\,d\theta$$

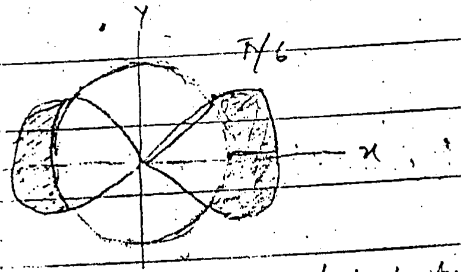
$$= \int_0^{\pi/3} 4\cos 2\theta\,d\theta - \int_0^{\pi/3} 2\cos\theta\,d\theta + 3\int_0^{\pi/3} d\theta$$

$$= \left|4\frac{\sin 2\theta}{2}\right|_0^{\pi/3} - \left|2\sin\theta\right|_0^{\pi/3} + 3\left|\theta\right|_0^{\pi/3}$$

$$= \left(2\sin 2\left(\frac{\pi}{3}\right)\right) - 2\sin\left(\frac{\pi}{3}\right) + \frac{3\pi}{3}$$

$$= 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} + \pi \qquad = \pi \text{ Ans}$$

Q 23 Inside $r^2 = 2a^2\cos 2\theta$ and outside $r = a$.



$r^2 = 2a^2\cos 2\theta$

$r = a$

$\therefore r^2 = a^2$

$a^2 = 2a^2\cos 2\theta$

$\dfrac{a^2}{2a^2} = \cos 2\theta \Rightarrow \cos 2\theta = \dfrac{1}{2}$

Symmetric about ~~initial line~~

Symmetric about y-axis

$2\theta = \dfrac{\pi}{3}, -\dfrac{\pi}{3}$

$\theta = \dfrac{\pi}{6}, -\dfrac{\pi}{6}$

$$\text{Required Area} = 4\int_{\alpha}^{\beta} \frac{(r_1^2 - r_2^2)}{2}\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/6} (2a^2\cos 2\theta - a^2)\,d\theta$$

$$= 2\int_0^{\pi/6} 2a^2\cos 2\theta\,d\theta - \int_0^{\pi/6} 2a^2\,d\theta$$

$$= 4a^2\left|\frac{\sin 2\theta}{2}\right|_0^{\pi/6} - 2\left|a^2\theta\right|_0^{\pi/6}$$

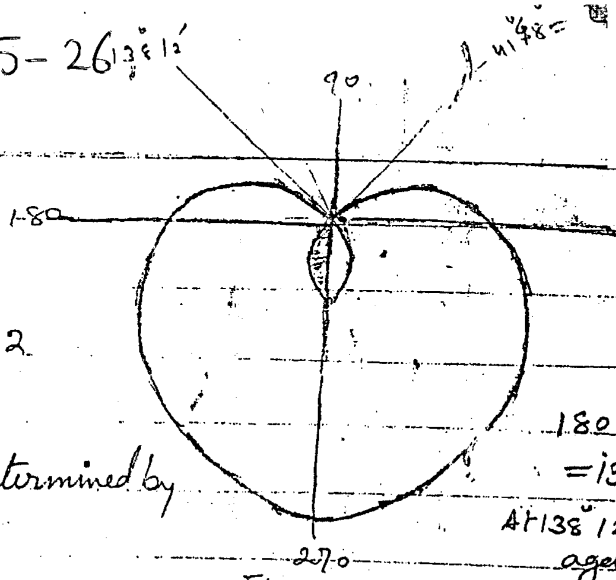
$$= 2a^2\left[\sin 2\left(\frac{\pi}{6}\right) - 0\right] - 2a^2\frac{\pi}{6}$$

$$= 2a^2\left[\sin\frac{\pi}{3}\right] - a^2\frac{\pi}{3}$$

$$= 2a^2\left(\frac{\sqrt{3}}{2}\right) - a^2\frac{\pi}{3}$$

$$= a^2\sqrt{3} - a^2\pi/3 = \frac{a^2}{3}\left(3\sqrt{3} - \pi\right) \text{ Ans.}$$

7.5-26 $138^\circ 12'$



Q24 $r = 2 - 3\sin\theta$

Put $r = 0$

$0 = 2 - 3\sin\theta \Rightarrow 3\sin\theta = 2$

$\therefore \theta = \sin^{-1}\frac{2}{3} = 41^\circ 48'$

$180 - 41^\circ 48' = 138^\circ 12'$

At $138^\circ 12'$ $r = 0$ again.

The left half of smaller loop is determined by $\theta = \sin^{-1}\frac{2}{3}$ and $\theta = \frac{\pi}{2}$

Area of smaller loop $= 2$ (shaded Area) $= 2\int_{\sin^{-1}(\frac{2}{3})}^{\pi/2} \frac{r^2\,d\theta}{2}$

symmetric about $\pi/2$ ∵ Replace θ by $(\pi - \theta)$ & ready.

$$= \int_{\sin^{-1}(\frac{2}{3})}^{\pi/2} (2 - 3\sin\theta)^2\,d\theta = \int_{\sin^{-1}(\frac{2}{3})}^{\pi/2} (4 + 9\sin^2\theta - 12\sin\theta)\,d\theta$$

$$= \int_{\sin^{-1}(\frac{2}{3})}^{\pi/2} \left(4 + 9\left(\frac{1-\cos 2\theta}{2}\right) - 12\sin\theta\right)d\theta \quad = \int_{\sin^{-1}\frac{2}{3}}^{\pi/2} \left(\frac{17}{2} - \frac{9}{2}\cos 2\theta - 12\sin\theta\right)d\theta$$

$$= \left|\frac{17}{2}\theta - \frac{9}{2}\frac{\sin 2\theta}{2} - 12(-\cos\theta)\right|_{\sin^{-1}(\frac{2}{3})}^{\pi/2} = \left|\frac{17}{2}\theta - \frac{9}{4}\sin 2\theta + 12\cos\theta\right|_{\sin^{-1}(\frac{2}{3})}^{\pi/2} =$$

$$= \left|\frac{17}{2}\theta - \frac{9}{\cancel{4}_2}\cancel{2}\sin\theta\cos\theta + 12\cos\theta\right|_{\sin^{-1}(2/3)}^{\pi/2}$$

We know $\theta = \sin^{-1}\frac{2}{3} \Rightarrow \sin\theta = \frac{2}{3}$

Also $\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

$= 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2} + \pi$ $\quad = \pi$ Ans.

7.5 - 27

$$= \cancel{2}\int_0^{\cos^{-1}(-2/3)} \frac{(2+3\cos\theta)^2}{\cancel{2}}\,d\theta \quad \cancel{2}\int_{\cos^{-1}(-2/3)}^{\pi} \frac{(2+3\cos\theta)^2}{\cancel{2}}\,d\theta$$

$$= \int_0^{\cos^{-1}(-2/3)} (4+9\cos^2\theta+12\cos\theta)\,d\theta + \int_{\cos^{-1}(-\frac{2}{3})}^{\pi} (4+9\cos^2\theta+12\cos\theta)\,d\theta$$

$$= \int_0^{\cos^{-1}(-2/3)} \left(4+9\left(\frac{1+\cos 2\theta}{2}\right)+12\cos\theta\right)d\theta - \int_{\cos^{-1}(-\frac{2}{3})}^{\pi} \left(4+9\left(\frac{1+\cos 2\theta}{2}\right)+12\cos\theta\right)d\theta$$

$$= \int_0^{\cos^{-1}(-2/3)} \left(4+\frac{9}{2}+\frac{9}{2}\cos 2\theta+12\cos\theta\right)d\theta - \int_{\cos^{-1}(-\frac{2}{3})}^{\pi} \left(4+\frac{9}{2}+\frac{9}{2}\cos 2\theta+12\cos\theta\right)d\theta$$

$$= \int_0^{\cos^{-1}(-\frac{2}{3})} \left(\frac{17}{2}+\frac{9}{2}\cos 2\theta+12\cos\theta\right)d\theta - \int_{\cos^{-1}(-\frac{2}{3})}^{\pi} \left(\frac{17}{2}+\frac{9}{2}\cos 2\theta+12\cos\theta\right)d\theta$$

$$= \left|\frac{17\theta}{2}+\frac{9}{4}\sin 2\theta+12\sin\theta\right|_0^{\cos^{-1}(-\frac{2}{3})} - \left|\frac{17}{2}\theta+\frac{9}{4}\sin 2\theta+12\sin\theta\right|_{\cos^{-1}(-\frac{2}{3})}^{\pi}$$

$$= \left|\frac{17\theta}{2}+\frac{9}{4}2\sin\theta\cos\theta+12\sin\theta\right|_0^{\cos^{-1}(-\frac{2}{3})} - \left|\frac{17\theta}{2}+\frac{9}{4}2\sin\theta\cos\theta+12\sin\theta\right|_{\cos^{-1}(-\frac{2}{3})}^{\pi} -$$

we know $2+3\cos\theta = 0 \Rightarrow \cos\theta = -\frac{2}{3}$

=

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$$

i.e. Area of smaller loop $= \left\{\frac{17}{2}\left(\frac{\pi}{2}\right) - 0 + 0\right\} - \left\{\frac{17}{2}\sin^{-1}\left(\frac{2}{3}\right) - \frac{9}{2}\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) + \frac{12\sqrt{5}}{3}\right\}$

$$= \frac{17\pi}{4} - \frac{17}{2}\sin^{-1}\frac{2}{3} + \sqrt{5} - 4\sqrt{5}$$

$$= \frac{17\pi}{4} - \frac{17}{2}\sin^{-1}\frac{2}{3} - 3\sqrt{5}$$

131

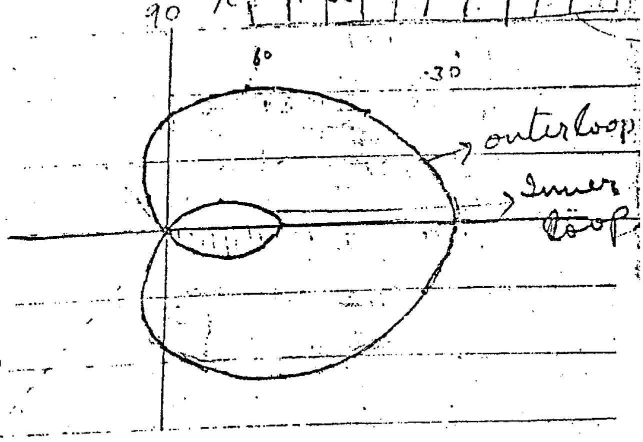
(25) $r = 2 + 3\cos\theta$

Put $r = 0$.

$$\frac{-2}{3} = \cos\theta$$

$$\theta = \cos^{-1}\left(-\frac{2}{3}\right) = 131^\circ 48'$$

θ	0	30	60	90	120	150	180	210	240	270
r	5	4.6	3.5	2	.5	-.6	-1	-.6	.5	2



The upper half of the outer loop is determined by

$\theta = 0$ to $\theta = \cos^{-1}\left(-\frac{2}{3}\right)$ $228^\circ 12'$

and lower half of the smaller loop is traced by

$\theta = \cos^{-1}\left(-\frac{2}{3}\right)$ to $\theta = \pi$

Area between the outer loop & inner, i.e. shaded area

$$= \text{Area of outer loop} - \text{Area of inner smaller loop}$$

$$= 2\int_0^{\cos^{-1}(-2/3)} \frac{r^2\,d\theta}{2} - 2\int_{\cos^{-1}(-\frac{2}{3})}^{\pi} \frac{r^2\,d\theta}{2}$$

∴ Area between the outer loop & inner loop =

$$\left[\frac{17}{2}\cos^{-1}\left(-\frac{2}{3}\right)+\frac{9}{2}\frac{\sqrt{5}}{3}\left(-\frac{2}{3}\right)+12\frac{\sqrt{5}}{3}-0\right]-\left[\frac{17\pi}{2}+0+0-\frac{17}{2}\cos^{-1}\left(-\frac{2}{3}\right)-\frac{\cancel{9}}{\cancel{2}}\frac{\sqrt{5}}{\cancel{3}}\left(\frac{-\cancel{2}}{\cancel{3}}\right)-12\frac{\sqrt{5}}{3}\right]$$

$$=\frac{17}{2}\cos^{-1}\left(\frac{-2}{3}\right)-\sqrt{5}+4\sqrt{5}-\frac{17\pi}{2}+\frac{17}{2}\cos^{-1}\left(-\frac{2}{3}\right)-\sqrt{5}+4\sqrt{5}$$

$$\Rightarrow\ =17\cos^{-1}\left(-\frac{2}{3}\right)+3\sqrt{5}-\frac{17\pi}{2}+3\sqrt{5}$$

$$=17\cos^{-1}\left(-\frac{2}{3}\right)-\frac{17\pi}{2}+6\sqrt{5}$$ Ans.

x————————x