Exercise 6.6

Written by
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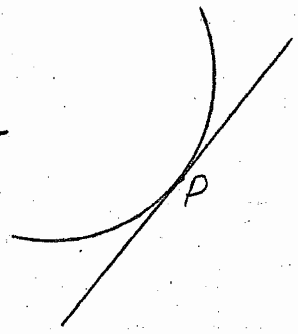
Slope of a Curve.

The slope of a curve at a point on the curve is the slope of the tangent line to the curve at that point.



Theorem.

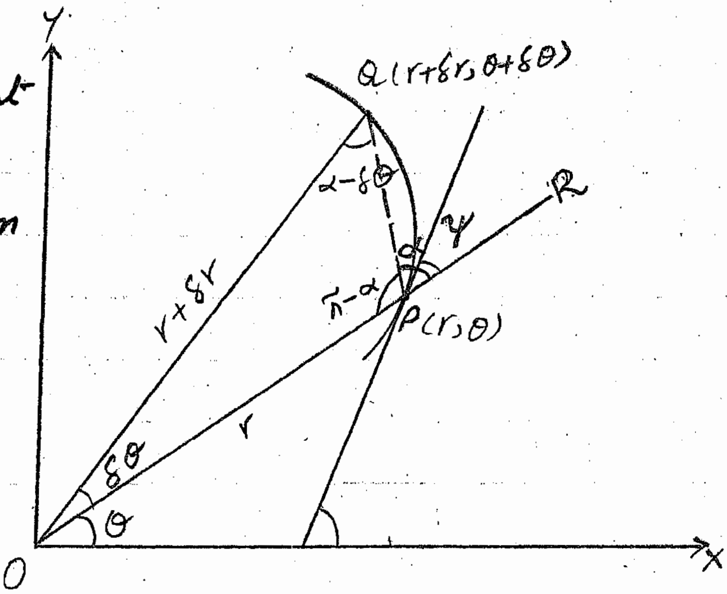
If $P(r,\theta)$, $r \neq 0$ is a point on the curve defined by $r = f(\theta)$. Then the angle between the radius vector and tangent at P in the direction of increasing θ has measured ψ given by $\tan\psi = \dfrac{r}{dr/d\theta}$.

OR Give geometrical significance of derivative in polar coordinates.

OR. Prove that $\tan\psi = \dfrac{r\,d\theta}{dr}$

Proof:- Let $P(r,\theta)$ and $Q(r+\delta r, \theta+\delta\theta)$ be any adjacent points on the curve $r = f(\theta)$.

Let ψ be the angle between the radius vector and the tangent at P.



Further let $Q\hat{P}R = \alpha$

Then $O\hat{P}Q = \pi - \alpha$

and $O\hat{Q}P = \pi - (\pi - \alpha + \delta\theta)$

$O\hat{Q}P = \alpha - \delta\theta$

Apply the law of sine to $\Delta\, OPQ$

$$\frac{r+\delta r}{\sin(\pi-\alpha)} = \frac{r}{\sin(\alpha-\delta\theta)}$$

$$r\sin(\pi-\alpha) = (r+\delta r)\sin(\alpha-\delta\theta)$$

$$r\sin\alpha = (r+\delta r)(\sin\alpha\cos\delta\theta - \cos\alpha\sin\delta\theta)$$

$$r\sin\alpha = r\sin\alpha\cos\delta\theta - r\cos\alpha\sin\delta\theta + \delta r\sin\alpha\cos\delta\theta - \delta r\cos\alpha\sin\delta\theta$$

$$r\sin\alpha - r\sin\alpha\cos\delta\theta = \delta r\sin\alpha\cos\delta\theta - (r+\delta r)\cos\alpha\sin\delta\theta$$

$r\sin\alpha(1-\cos\delta\theta) = \delta r\sin\alpha\cos\delta\theta - (r+\delta r)\cos\alpha\sin\delta\theta.$

Dividing by $\delta\theta$

$$r\sin\alpha\frac{(1-\cos\delta\theta)}{\delta\theta} = \frac{\delta r}{\delta\theta}[\sin\alpha\cos\delta\theta] - \frac{(r+\delta r)}{\delta\theta}\cos\alpha\sin\delta\theta.$$

$$\lim_{\delta\theta\to 0}\frac{r\sin\alpha(1-\cos\delta\theta)}{\delta\theta} = \lim_{\delta\theta\to 0}\frac{\delta r}{\delta\theta}[\sin\alpha\cos\delta\theta] - \lim_{\delta\theta\to 0}\frac{(r+\delta r)}{\delta\theta}\cos\alpha\sin\delta\theta$$

[As point Q tends to P the chord PQ concides with the tangent at P and $\alpha \to \psi$] ($\because \delta\theta\to 0$)

$$\lim_{\delta\theta\to 0}\frac{1-\cos\delta\theta}{\delta\theta} = 0 \quad \text{[By using L'hospital Rule]}$$

So $\Rightarrow$ $$0 = \frac{dr}{d\theta}\sin\psi\cos 0 - (r+0)\cos\psi\cdot 1 \quad \left|\lim_{\delta\theta\to 0}\frac{\sin\delta\theta}{\delta\theta} = 1\right]$$

$$0 = \frac{dr}{d\theta}\sin\psi - r\cos\psi$$

$$\Rightarrow \frac{dr}{d\theta}\sin\psi = r\cos\psi$$

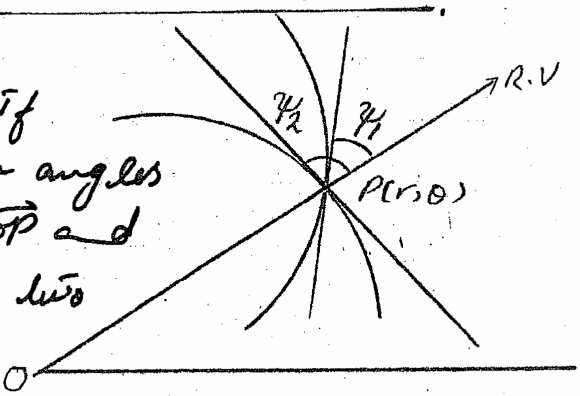
$$\frac{\sin\psi}{\cos\psi} = \frac{r}{\frac{dr}{d\theta}}$$

$$\tan\psi = r\frac{d\theta}{dr}$$

Is as required.

Angle of Intersection of Two Polar Curves.

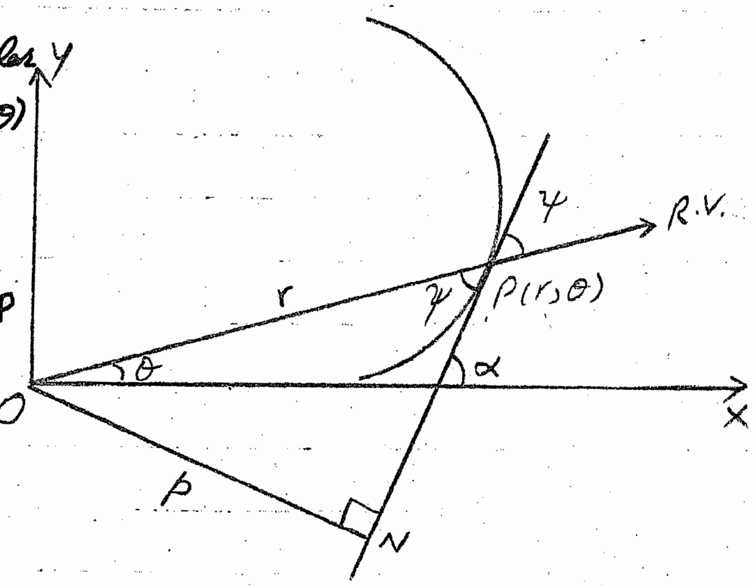
Suppose that two curves intersect at a point $P=(r,\theta)$. If ψ_1 and ψ_2 are measre of the angles between the radius vector $\overrightarrow{OP}$ and the tangents to each of the two curves at P, then measure of their angle of intersection is $|\psi_1 - \psi_2|$.



Length of Perpendicular From the pole to a Tangent.

Let ON be the perpendicular from O to the tangent at $P(r,\theta)$ of the curve defined by

$$r = f(\theta)$$



Let $ON = p$ then from r.t ΔONP

$$\frac{ON}{OP} = \sin\psi$$

$$\frac{p}{r} = \sin\psi$$

$$\Rightarrow \quad p = r\sin\psi$$

Is the required length of perpendicular from pole to tangent.

Now if α is the inclination of tangent.

Then by elementry geometry

$$\alpha = \theta + \psi$$

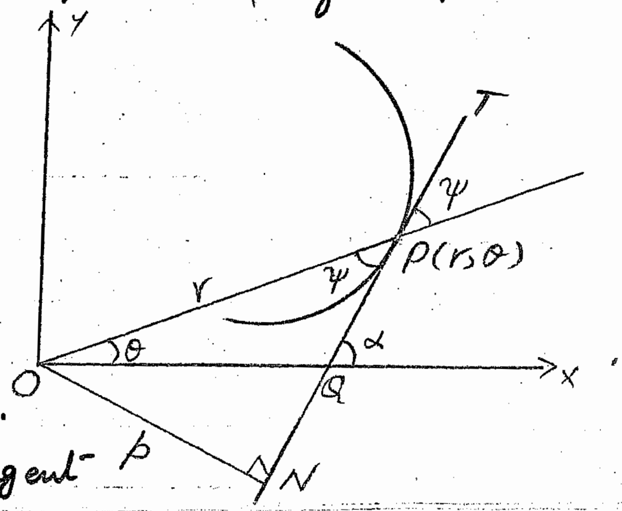
Theorem.

If (r,θ) be the coordinates of a pt. P on the curve and T be the tangent at that point. If α be the inclination of the tangent T and ψ be the angle of between the radius vector and tangent then prove that

I) $$\alpha = \theta + \psi$$

II) $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

where p is the length of perpendicular from pole to the tangent.



Proof: Consider the pt. $P(r,\theta)$ on the curve $r = f(\theta)$ ——— (1)

Then $|OP| = r$

$X\hat{O}P = \theta$.

Let T be the tangent at $P(r,\theta)$.

α is the inclination of this tangent

and ψ is the angle b/w the tangent and radius vector.
p is the length of the $\perp$ar from the pole to the tangent.

I) Consider the ΔOQP, we have

$$\alpha = \theta + \psi \qquad \text{(By elementry geometry.)}$$

II) Consider the ΔONP

$$\sin\psi = \frac{p}{r}$$

$$p = r\sin\psi$$

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\psi}$$

$$= \frac{1}{r^2}\operatorname{Cosec}^2\psi$$

$$= \frac{1}{r^2}(1+\operatorname{Cot}^2\psi)$$

$$= \frac{1}{r^2}\left(1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right) \qquad \operatorname{Tan}\psi = r\frac{d\theta}{dr}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{ ——— (1)}$$

is as required.

we can prove this relation directly by saying $\because p = r\sin\psi$ and proceed as above.

Amendent.

Let $u = \frac{1}{r}$ ——— (2)

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \text{ ——— (3)}$$

Put (2) and (3) in (1)

$$\Rightarrow \quad \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Pedal Eq. of Polar Curves

To obtain the pedal equation of a curve defined by $r = f(\theta)$, we eliminate θ between

$$r = f(\theta) \text{ and } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

Some times it is convenient to obtain pedal Eq. by eliminating θ and ψ from $r = f(\theta)$,

$$\tan\psi = r\frac{d\theta}{dr} \text{ and } p = r\sin\psi.$$

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Find ψ for each of the given curves (1-4)

Q#1. $r = a(1-\cos\theta)$

$$\frac{dr}{d\theta} = a\sin\theta$$

Now $$\tan\psi = \frac{r}{dr/d\theta}$$

$$\tan\psi = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$\tan\psi = \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$\tan\psi = \frac{\sin\theta/2}{\cos\theta/2}$$

$$\tan\psi = \tan\theta/2$$

$$\Rightarrow \psi = \theta/2$$

Q#2. $r = -5\csc\theta$

$$\frac{dr}{d\theta} = 5\csc\theta\cot\theta$$

Now $$\tan\psi = \frac{r}{dr/d\theta}$$

$$\tan\psi = \frac{-5\csc\theta}{5\csc\theta\cot\theta}$$

$$\tan\psi = -\tan\theta = \tan(\pi-\theta)$$

$$\Rightarrow \psi = \pi-\theta$$

Q#3. $\frac{2a}{r} = 1+\sin\theta$

$$r = \frac{2a}{1+\sin\theta}$$

Diff. w.r.t. (θ)

$$\frac{dr}{d\theta} = \frac{-2a\cos\theta}{(1+\sin\theta)^2}$$

Now $$\tan\psi = r/dr/d\theta.$$

$$= \frac{2a}{1+\sin\theta} \cdot \frac{(1+\sin\theta)^2}{-2a\cos\theta}$$

$$\text{Tan}\,\psi = -\frac{1+\sin\theta}{\cos\theta}$$

$$\text{Tan}\,\psi = +\frac{1+2\sin\theta/2\cos\theta/2}{\sin^2\theta/2-\cos^2\theta/2}$$

$$\text{Tan}\,\psi = +\frac{\sin^2\theta/2+\cos^2\theta/2+2\sin\theta/2\cos\theta/2}{\sin^2\theta/2-\cos^2\theta/2}$$

$$\text{Tan}\,\psi = \frac{(\sin\theta/2+\cos\theta/2)^2}{(\sin\theta/2-\cos\theta/2)(\sin\theta/2+\cos\theta/2)}$$

$$\text{Tan}\,\psi = \frac{\sin\theta/2+\cos\theta/2}{\sin\theta/2-\cos\theta/2}$$

$$\text{Tan}\,\psi = \frac{1+\text{Tan}\,\theta/2}{1-\text{Tan}\,\theta/2}$$ By Dividing Num. and Den. by $\sin\theta/2$

$$\text{Tan}\,\psi = -\frac{1-\text{Tan}\,\theta/2}{1+\text{Tan}\,\theta/2}$$

$$\text{Tan}\,\psi = -\text{Tan}\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\text{Tan}\,\psi = \text{Tan}\left(\pi-\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\text{Tan}\,\psi = \text{Tan}\left(\frac{3\pi}{4}-\frac{\theta}{2}\right)$$

$$\Rightarrow \psi = \frac{3\pi}{4}-\frac{\theta}{2}$$

$1+\sin\theta = 1+2\sin\theta/2\cos\theta/2$

$\cos\theta = \cos^2\theta/2-\sin^2\theta/2$

$\text{Tan}\left(\frac{\pi}{4}-\theta\right) = \frac{1-\text{Tan}\,\theta}{1+\text{Tan}\,\theta}$

Q#4.

$$r = \frac{3}{2(1-\cos\theta)} \Rightarrow r = \frac{3}{2-2\cos\theta}$$

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -\frac{3(2\sin\theta)}{(2-2\cos\theta)^2}$$

$$= \frac{-6\sin\theta}{(2-2\cos\theta)^2}$$

Now $\text{Tan}\,\psi = r\dfrac{d\theta}{dr}$

$$= \frac{3}{2(1-\cos\theta)}\cdot\frac{2^2(1-\cos)^2}{-6\sin\theta}$$

$$= \frac{1-\cos\theta}{-\sin\theta}$$

$$= -\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$$

$$= -\frac{\sin\theta/2}{\cos\theta/2} = -\tan\theta/2$$

$$\tan\psi = \tan(\pi - \theta/2)$$

$$\Rightarrow \psi = \pi - \frac{\theta}{2}$$

Find the measure of the angle of intersection of the given curves (5-10)

Q#5. $r=1$ and $r = 2\sin\theta$

Here $r=1$ —— (1) , $r = 2\sin\theta$ —— (2)

Put (1) in (2) $\Rightarrow$ $1 = 2\sin\theta$

$$1/2 = \sin\theta$$

$$\Rightarrow \theta = 30^\circ$$

Diff. (1) w.r.t. 'θ', we have

$$\frac{dr}{d\theta} = 0$$

Now $\tan\psi_1 = r \cdot \frac{d\theta}{dr}$

$$\tan\psi_1 = \frac{1}{0}$$

$$\tan\psi_1 = \infty$$

$$\psi_1 = 90^\circ$$

Diff. (2) w.r.t. 'θ'

$$\frac{dr}{d\theta} = 2\cos\theta$$

Now

$$\tan\psi_2 = \frac{r}{dr/d\theta}$$

$$\tan\psi_2 = \frac{2\sin\theta}{2\cos\theta}$$

$$\tan\psi_2 = \tan\theta$$

$$\psi_2 = \theta$$

$$\psi_2 = 30^\circ \qquad \because \theta = 30^\circ$$

Now $|\psi_1 - \psi_2|$

$$|90^\circ - 30^\circ| = 60^\circ$$

Angle b/w the curves at $\theta = 30^\circ$ is 60°

Q# 6. $r = a\theta$ ——— (i) and $r\theta = a$ ——— (ii)

Put (i) in (ii) =>

$$a\theta \cdot \theta = a$$

$$\theta^2 = 1$$

$$\Rightarrow \theta = \pm 1 \quad \text{——— iii}$$

Diff. (1) w.r.t. 'θ'

$$\frac{dr}{d\theta} = a$$

Now $\quad \tan\psi_1 = \dfrac{r}{dr/d\theta}$

$$\tan\psi_1 = \frac{a\theta}{a}$$

$$\tan\psi_1 = \theta \quad \text{——— iv}$$

at $\theta = 1$

$$\tan\psi_1 = 1$$

$$\psi_1 = 45^\circ$$

Diff. (2) w.r.t. 'θ'

$$\frac{dr}{d\theta}\cdot\theta + r = 0$$

$$\frac{dr}{d\theta}\cdot\theta = -r$$

$$\frac{dr}{d\theta}\cdot\theta = -\frac{a}{\theta}$$

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}$$

Now $\quad \tan\psi_2 = r\cdot\dfrac{d\theta}{dr}$

$$\tan\psi_2 = \frac{a/\theta}{-a/\theta^2}$$

$$\tan\psi_2 = -\frac{a}{\theta}\cdot\frac{\theta^2}{a}$$

$$\tan\psi_2 = -\theta \quad \text{——— v}$$

at $\theta = 1$

$$\tan\psi_2 = -1$$

$$\psi_2 = 135^\circ$$

If the required angle is α then

$$\alpha = |\psi_1 - \psi_2|$$

$$= |45^\circ - 135^\circ|$$

$$= 90^\circ$$

Now if $\theta = -1$

then iv) => $\tan\psi_1 = -1$

$$\psi_1 = 135^\circ$$

v) => $\tan\psi_2 = -(-1)$

$$\psi_2 = 45^\circ$$

angle of intersection $= |\psi_1 - \psi_2|$

$$= |135^\circ - 45^\circ|$$

$$= 90^\circ$$

Q#7. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

$r = \frac{a\theta}{1+\theta}$ —— (I) $\qquad$ $r = \frac{a}{1+\theta^2}$ —— (II)

Diff. w.r.t. θ.

$$\frac{dr}{d\theta} = \frac{(1+\theta)a - a\theta(1)}{(1+\theta)^2}$$

$$= \frac{a(1+\theta-\theta)}{((1+\theta)^2}$$

$$= \frac{a}{(1+\theta)^2}$$

$$\tan\psi_1 = \frac{r}{dr/d\theta}$$

$$= \frac{a\theta}{1+\theta}\cdot\frac{(1+\theta)^2}{a}$$

$$= \theta(1+\theta) \text{ —— III}$$

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -\frac{a\cdot 2\theta}{(1+\theta^2)^2}$$

$$= -\frac{2a\theta}{(1+\theta^2)^2}$$

Now $\tan\psi_2 = \frac{a}{1+\theta^2}\cdot -\frac{(1+\theta^2)^2}{2a\theta}$

$$= -\frac{1+\theta^2}{2\theta} \text{ —— IV}$$

Now (1) and (2)

$$\Rightarrow \frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

$$\theta(1+\theta^2) = 1+\theta$$

$$\theta+\theta^3 = 1+\theta$$

$$\theta^3 = 1$$

$$\theta^3 = 1$$

Put in (3) and (IV)

$\tan\psi_1 = 1(1+1)$

$\tan\psi_1 = 2$

$\psi_1 = 63.43^\circ$

$\tan\psi_2 = -\frac{1+1}{2(1)}$

$= -1$

$\psi_2 = 135^\circ$

Required angle $= |\psi_1 - \psi_2|$

$= |63.43 - 135|$

$= 71.57^\circ$

Q#8 $r = ae^{\theta}$ and $re^{\theta} = b$

$r = ae^{\theta}$

Diff. w.r.t. 'θ'

$\frac{dr}{d\theta} = ae^{\theta}$

So

$\tan\psi_1 = r \cdot \frac{d\theta}{dr}$

$= ae^{\theta} \cdot \frac{1}{ee^{\theta}}$

$\tan\psi_1 = 1$

$\psi_1 = 45^{\circ}$

$r = \frac{b}{e^{\theta}}$

Diff. w.r.t. θ

$\frac{dr}{d\theta} = -\frac{b}{e^{\theta}}$

So

$\tan\psi_2 = \frac{b}{e^{\theta}} \cdot -\frac{e^{\theta}}{b}$

$\tan\psi_2 = -1$

$\psi_2 = 135^{\circ}$

Required angle $= |\psi_1 - \psi_2|$

$= |45^{\circ} - 135^{\circ}|$

$= 90^{\circ}$

Q#9 $r = a(1-\cos\theta)$ and $r = \frac{a}{1-\cos\theta}$

$r = a(1-\cos\theta)$ ——— (I)

Diff. w.r.t. θ.

$\frac{dr}{d\theta} = a\sin\theta$

So $\tan\psi_1 = \frac{r}{dr/d\theta}$

$\tan\psi_1 = \frac{a(1-\cos\theta)}{a\sin\theta}$

$\tan\psi_1 = \frac{1-\cos\theta}{\sin\theta}$

$\tan\psi_1 = \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$

$= \tan\theta/2$

$\Rightarrow \psi_1 = \theta/2$ ——— (III)

$r = a(1-\cos\theta)^{-1}$ ——— (II)

Diff. w.r.t. θ.

$\frac{dr}{d\theta} = -\frac{a\sin\theta}{(1-\cos\theta)^2}$

So $\tan\psi_2 = \frac{r}{dr/d\theta}$

$\tan\psi_2 = \frac{a}{1-\cos\theta} \cdot \frac{(1-\cos\theta)^2}{-a\sin\theta}$

$\tan\psi_2 = -\frac{1-\cos\theta}{\sin\theta}$

$\tan\psi_2 = -\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}$

$\tan\psi_2 = -\tan\theta/2$

$\tan\psi_2 = \tan(\pi - \theta/2)$

$\psi_2 = \pi - \theta/2$ ——— (IV)

(I) and (II) $\Rightarrow$

$a(1-\cos\theta) = \frac{a}{1-\cos\theta}$

$(1-\cos\theta)^2 = 1$

$1-\cos\theta = \pm 1$

$\Rightarrow \quad 1-\cos\theta = 1 \quad$ and $\quad 1-\cos\theta = -1$

$\Rightarrow \quad \cos\theta = 0 \quad$ and $\quad \cos\theta = 2$

$\theta = 90^\circ \quad$ and $\quad \theta = \infty$ which is not accepted.

Now III and IV $\Rightarrow$

at $\theta = 90^\circ \quad \psi_1 = 90^\circ/2 \qquad$ at $\theta = 90^\circ \quad \psi_2 = \pi - 90^\circ/2$

$\Rightarrow \quad \psi_1 = 45^\circ \qquad \psi_2 = \pi - 45^\circ$

So Required angle $= |\psi_1 - \psi_2|$

$= |45^\circ - (\pi - 45^\circ)|$

$= |45^\circ - \pi + 45^\circ|$

$= |90^\circ - \pi|$

$= |-\pi/2|$

$= \pi/2$

Is the required angle of intersection.

#10 $\quad r = \cos 2\theta$ and $r = \sin\theta$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$

Here $r = \cos 2\theta$ —— (I) $\qquad r = \sin\theta$ —— II

Diff. (I) w.r.t. 'θ'

$$\frac{dr}{d\theta} = -2\sin 2\theta$$

So $\tan\psi_1 = \dfrac{r}{dr/d\theta}$

$$\tan\psi_1 = \frac{\cos 2\theta}{-2\sin 2\theta}$$

$$\tan\psi_1 = -\frac{1}{2}\cot 2\theta$$

Now $(\tan\psi_1)_{(\frac{1}{2}, \pi/6)} = -\frac{1}{2}\cot 2(\pi/6)$

$= -\frac{1}{2}\cot\frac{\pi}{3}$

$= -\frac{1}{2}(.577)$

$= -.288$

$\psi_1 = -\tan^{-1}(+.2.888)$

$\psi_1 = -\tan^{-1}(.288)$

$\psi_1 = -16.1^\circ$

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = \cos\theta$$

So $\tan\psi_2 = \dfrac{r}{dr/d\theta}$

$$\tan\psi_2 = \frac{\sin\theta}{\cos\theta}$$

$\tan\psi_2 = \tan\theta$

$\psi_2 = \theta$

$\psi_2 = \pi/6$

$\psi_2 = 30^\circ$

Now

$$\text{Required angle} = |\psi_1 - \psi_2|$$
$$= |-16.1 - 30|$$
$$= |-46.1|$$
$$= 46.1^\circ$$

Find the pedal Equation of the given curves

Q.11.

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{———— I}$$

we know that pedal eq. of any polar curve is

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{———— II}$$

Now

Diff. (I) w.r.t. θ

$$-\frac{l}{r^2}\frac{dr}{d\theta} = -e\sin\theta$$

$$\Rightarrow \frac{1}{r^2}\frac{dr}{d\theta} = \frac{e\sin\theta}{l}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{e^2\sin^2\theta}{l^2} \quad \text{———— III}$$

Put III in II $\Rightarrow$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2\sin^2\theta}{l^2} \quad \text{———— IV}$$

Now I) $\Rightarrow$ $$e\cos\theta = \frac{l}{r} - 1 = \frac{l-r}{r}$$

$$\Rightarrow \cos\theta = \frac{l-r}{er} \quad \text{———— V}$$

Rewriting (4) in the form

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2(1-\cos^2\theta)}{l^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\left(1 - \left(\frac{l-r}{er}\right)^2\right) \quad \text{from V}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\left[1 - \frac{l^2+r^2-2lr}{e^2r^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2}\left[\frac{e^2r^2 - l^2 - r^2 + 2lr}{e^2r^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^4r^2}{l^2e^2r^2} - \frac{e^2l^2}{l^2e^2r^2} - \frac{e^2r^2}{l^2e^2r^2} + \frac{2e^2lr}{l^2e^2r^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{e^2}{l^2} - \frac{1}{r^2} - \frac{1}{l^2} + \frac{2}{lr}$$

$$\frac{1}{p^2} = \frac{e^2}{l^2} - \frac{1}{l^2} + \frac{2}{lr}$$

$$\frac{1}{p^2} = \frac{e^2-1}{l^2} + \frac{2}{lr}$$

Is the required pedal Equation.

Q.12. $\quad r = a\theta \quad \text{———} \quad I$

And $\quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{———} \quad II$

Diff. (I) w.r.t. 'θ' $\Rightarrow$

$$\frac{dr}{d\theta} = a$$

Put in II $\Rightarrow$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\, a^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}$$

which is the required pedal Equation.

Q#13. $\quad r = a\sin m\theta. \quad \text{———} \quad I$

And $\quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{———} \quad II$

Diff. (I) w.r.t. 'θ'

$$\frac{dr}{d\theta} = am\cos m\theta$$

Put in II, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\,(am\cos m\theta)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\, a^2m^2\cos^2 m\theta. \quad \text{———} \quad III$$

But i) $\Rightarrow$ $\quad r = a\sin m\theta$

$$\sin m\theta = \frac{r}{a} \quad \text{———} \quad IV$$

Rewriting III in the form

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\, a^2m^2(1 - \sin^2 m\theta)$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} a^2 m^2 \left(1 - \left(\frac{r}{a}\right)^2\right) \qquad \text{from IV}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[a^2m^2 - \frac{a^2m^2r^2}{a^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[a^2m^2 - m^2r^2\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2m^2}{r^4} - \frac{m^2r^2}{r^4}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2m^2}{r^4} - \frac{m^2}{r^2}$$

$$\frac{1}{p^2} = \frac{r^2 + a^2m^2 - m^2r^2}{r^4}$$

$$r^4 = p^2\left[r^2 + a^2m^2 - m^2r^2\right]$$

$$r^4 = p^2\left[r^2(1-m^2) + a^2m^2\right]$$

Is the required pedal Eq.

Q# 14. $\quad r = a + b\cos\theta$ ——— ①

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -b\sin\theta \qquad \text{——— ②}$$

we know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(-b\sin\theta\right)^2 \qquad \text{from (2)}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} b^2\sin^2\theta$$

$$\frac{1}{p^2} = \frac{r^2 + b^2\sin^2\theta}{r^4}$$

$$\frac{1}{p^2} = \frac{(a + b\cos\theta)^2 + b^2\sin^2\theta}{r^4} \qquad \text{from (1)}$$

$$\frac{1}{p^2} = \frac{1}{r^4}\left[a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta\right]$$

$$\frac{1}{p^2} = \frac{1}{r^4}\left[a^2 + b^2(\cos^2\theta + \sin^2\theta) + 2a(b\cos\theta)\right]$$

$$\frac{1}{p^2} = \frac{1}{r^4}\left[a^2+b^2+2a(r-a)\right]$$

$$\frac{1}{p^2} = \frac{1}{r^4}\left[a^2+b^2-2a^2+2ar\right]$$

$$\frac{1}{p^2} = \frac{1}{r^4}\left[b^2-a^2+2ar\right]$$

$$r^4 = p^2\left[b^2-a^2+2ar\right]$$

which is the required pedal Equation.

Q # 15. $\quad r = a(1-\sin\theta)$ ——— (1)

Diff. w.r.t. 'θ'

$$\frac{dr}{d\theta} = -a\cos\theta \quad \text{——— (2)}$$

We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(-a\cos\theta)^2 \quad \text{from (2)}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}(a^2\cos^2\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\cos^2\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}(1-\sin^2\theta) \quad \text{——— (3)}$$

Now (1) $\Rightarrow$ $\quad \frac{r}{a} = 1-\sin\theta$

$$\sin\theta = 1-\frac{r}{a}$$

$$\sin\theta = \frac{a-r}{a}$$

Put in (3) $\Rightarrow$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\left(1-\left(\frac{a-r}{a}\right)^2\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\left[1-\frac{a^2+r^2-2ar}{a^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\left[\frac{a^2-a^2-r^2+2ar}{a^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{a^2}{r^4}\left[\frac{-r^2+2ar}{a^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[-r^2+2ar\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{2ar}{r^4} - \frac{r^2}{r^4}$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{2a}{r^3} - \frac{1}{r^2}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$r^3 = 2ap^2$$

Is the required Pedal Equation.

Q# 16. Show that the curves $r^m = a^m \cos m\theta$ and $r^m = a^m \sin m\theta$ cut each other orthogonally.

Soln. Here $r^m = a^m \cos m\theta$ ——— ①

$r^m = a^m \sin m\theta$ ——— ②

Taking ln of both sides of ①

① $\Rightarrow$ $m\ln r = m\ln a + \ln\cos m\theta$

Diff. w.r.t. 'θ'

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{1}{\cos m\theta}\cdot -m\sin m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan m\theta$$

Reciprocating

$$r\frac{d\theta}{dr} = -\cot m\theta$$

$$\tan\psi_1 = -\cot m\theta$$

$$\tan\psi_1 = \tan\left(\frac{\pi}{2}+m\theta\right)$$

$$\psi_1 = m\theta \quad \text{———} \quad ③$$

Now Taking ln of both sides of ②

② $\Rightarrow$ $m\ln r = m\ln a + \ln\sin m\theta$

Diff. w.r.t. θ

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{1}{\sin m\theta}\cdot m\cos m\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \text{Cot } m\theta$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = \text{Tan } m\theta$$

$$\Rightarrow \quad \text{Tan } \psi_2 = \text{Tan } m\theta$$

$$\Rightarrow \quad \psi_2 = m\theta$$

Now $\quad |\psi_1 - \psi_2| = \left|\frac{\pi}{2} + m\theta - m\theta\right|$

$$= \frac{\pi}{2}$$

Thus the curves cut each other orthognally.

Q#17. Show that the tangents to the Cardioid

$$r = a(1+\text{Cos}\,\theta)$$

at the point $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are respectively parallel and perpendicular to the initial line.

Soln: Here $\quad r = a(1+\text{Cos}\,\theta)$ ——— (1)

Diff. w.r.t. 'θ', we have

$$\frac{dr}{d\theta} = -a\sin\theta$$

So $\quad \text{Tan}\,\psi = r\frac{d\theta}{dr}$

$$\text{Tan}\,\psi = \frac{a(1+\text{Cos}\,\theta)}{-a\sin\theta} = -\frac{1+\text{Cos}\,\theta}{\sin\theta}$$

$$\text{Tan}\,\psi = -\frac{2\,\text{Cos}^2\theta/2}{2\sin\theta/2\,\text{Cos}\,\theta/2}$$

$$\text{Tan}\,\psi = -\text{Cot}\,\theta/2$$

$$\text{Tan}\,\psi = \text{Tan}(\pi/2 + \theta/2)$$

$$\Rightarrow \quad \psi = \pi/2 + \theta/2 \quad \text{——— (2)}$$

Now

$$\alpha = \theta + \psi$$

$$\psi = \alpha - \theta$$

$$\frac{\pi}{2} + \frac{\theta}{2} = \alpha - \theta$$

$$\Rightarrow \quad \alpha = \frac{\pi}{2} + \frac{3\theta}{2} \quad \text{——— (3)}$$

Note: i) The tangent will ∥ to x-axis if $\alpha = 0°$ or $180°$. (ii) The tangent line will ⊥ar to x-axis if $\alpha = 90°$ or $270°$.

Case (1):

If $\theta = \frac{\pi}{3}$ then 3) $\Rightarrow$

$$\alpha = \frac{\pi}{2} + \frac{3}{2}\cdot\frac{\pi}{3}$$

$$\alpha = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

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$\Rightarrow$ the tangent at $\theta = \pi/3$ is $\parallel$ to the initial line.

Case 2.

If $\theta = \frac{2\pi}{3}$ then 3) $\Rightarrow$

$$\alpha = \frac{3}{2}\cdot\frac{2\pi}{3} + \frac{\pi}{2}$$
$$= \pi + \pi/2$$
$$= 3\pi/2$$

$\Rightarrow$ The tangent is $\perp$ar to the initial line.

Q#18. Show that $\tan\psi = \dfrac{x\,dy/dx - y}{y\,dy/dx + x}$

Soln.

We know that

$$\alpha = \theta + \psi$$
$$\psi = \alpha - \theta$$
$$\Rightarrow \quad \tan\psi = \tan(\alpha - \theta)$$
$$\tan\psi = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

But we know that

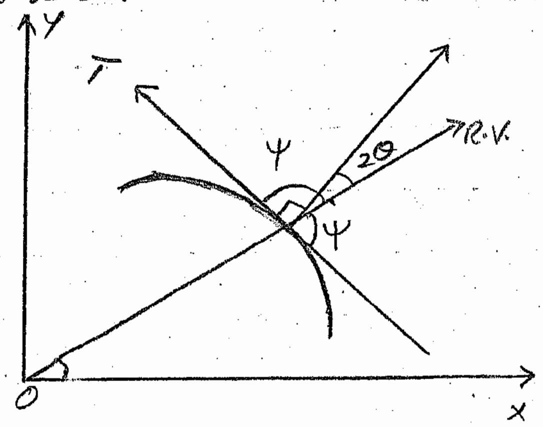
$$\tan\alpha = \frac{dy}{dx} \text{ and } \tan\theta = \frac{y}{x}$$

So

$$\tan\psi = \frac{dy/dx - y/x}{1 + dy/dx \cdot y/x}$$
$$\tan\psi = \frac{1/x\,[x\,dy/dx - y]}{1/x\,[x + y\,dy/dx]}$$
$$\tan\psi = \frac{x\,dy/dx - y}{x + y\,dy/dx}$$
$$\Rightarrow \tan\psi = \frac{x\frac{dy}{dx} - y}{y\frac{dy}{dx} + x}$$

Is as required.

Q.19. Show that at any point of the leminscate
$$r^2 = a^2 \cos 2\theta, \quad 0 \le \theta \le \frac{\pi}{4}$$
the measure of the angle between the radius vector and outward-pointed normal is 2θ.



Soln. Here
$$r^2 = a^2 \cos 2\theta \quad \text{———} (1)$$
$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta$$
$$\frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

Now $\tan\psi = r\dfrac{d\theta}{dr}$
$$= r \cdot 1 \Big/ -\frac{a^2 \sin 2\theta}{r}$$
$$= -\frac{r^2}{a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta}$$
$$= -\cot 2\theta$$
$$\tan\psi = \tan\left(\frac{\pi}{2} + 2\theta\right)$$
$$\Rightarrow \quad \psi = \pi/2 + 2\theta$$

Now if α is the required angle then
$$\alpha = |(\pi/2 + 2\theta) - \pi/2| \qquad \Big|\ |\psi - \pi/2|$$
$$\Rightarrow \quad \alpha = 2\theta$$

Hence the measure of the angle b/w the radius vector and out-word normal is 2θ.

Q#20. Find an equation (in rectangular coordinates) of the tangent to:

i) $r = a \sin 2\theta$ at $(1, \frac{\pi}{4})$. ——(1)

we know that in rectangular coordinates
$$x = r\cos\theta \quad \text{———} (2)$$
$$y = r\sin\theta \quad \text{———} (3)$$

then the point become $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Put (1) into (2) and (3), we have

$$x = \sin 2\theta \cos\theta$$

$$\Rightarrow x = 2\sin\theta\cos\theta \cdot \cos\theta$$

$$\Rightarrow x = 2\sin\theta\cos^2\theta \quad \text{———} (4)$$

and

$$y = \sin 2\theta \sin\theta$$

$$y = 2\sin\theta\cos\theta\sin\theta$$

$$y = 2\sin^2\theta\cos\theta \quad \text{———} (5)$$

Diff. (4) w.r.t. 'θ'

$$\frac{dx}{d\theta} = 2\left[\cos\theta\cos^2\theta + \sin\theta \cdot 2\cos\theta(-\sin\theta)\right]$$

$$= 2\cos^3\theta - 4\sin^2\theta\cos\theta \quad \text{———} (6)$$

Diff. (5) w.r.t. 'θ'

$$\frac{dy}{d\theta} = 2\left[2\sin\theta\cos\theta\cos\theta + \sin^2\theta(-\sin\theta)\right]$$

$$= 4\sin\theta\cos^2\theta - 2\sin^3\theta \quad \text{———} (7)$$

From (6) and (7)

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos^2\theta - 2\sin^3\theta}{2\cos^3\theta - 4\sin^2\theta\cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{(1,\pi/4)} = \frac{4 \cdot 1/\sqrt{2} \cdot 1/2 - 2 \cdot (1/\sqrt{2})^3}{2 \cdot (1/\sqrt{2})^3 - 4 \cdot 1/2 \cdot 1/\sqrt{2}}$$

$$= \frac{\sqrt{2} - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} - \sqrt{2}}$$

$$\left.\frac{dy}{dx}\right|_{(1,\pi/4)} = \frac{2-1}{1-2} = \frac{1}{-1} = -1$$

So equation of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right)$$

$$x + y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$x + y = \frac{1+1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$x + y = \sqrt{2}$$

which is the required eq. of line tangent to

$$r = \sin 2\theta \text{ at } (1, \pi/4)$$

Q#20 $\quad r = 1 + \cos\theta \quad$ at $(1, \pi/2)$

(ii) Soln: $\quad$ ——— (1)

Changing the given eq. in rectangular coordinates

$$x = r\cos\theta \quad \text{———} \quad (2)$$

$$y = r\sin\theta \quad \text{———} \quad (3)$$

the the point become $(0, 1)$

$2) \Rightarrow \quad x = (1+\cos\theta)\cos\theta$

$$x = \cos\theta + \cos^2\theta \quad \text{———} \quad (4)$$

$3) \Rightarrow \quad y = (1+\cos\theta)\sin\theta$

$$y = \sin\theta + \sin\theta\cos\theta \quad \text{———} \quad (5)$$

Diff. (4) w.r.t. 'θ'

$$\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta \quad \text{———} \quad (6)$$

Diff. (5) w.r.t. 'θ'

$$\frac{dy}{d\theta} = \cos\theta - \sin^2\theta + \cos^2\theta \quad \text{———} \quad (7)$$

So by chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{\cos\theta - \sin^2\theta + \cos^2\theta}{-\sin\theta - 2\cos\theta\sin\theta}$$

$$m = \left.\frac{dy}{dx}\right|_{(1, \pi/2)} = \frac{0 - 1 + 0}{-1 - 0} = 1$$

We know that the eq. of the tangent is given by

$$y - y_1 = m(x - x_1)$$

$$y - 01 = 1(x - 0)$$

$$y - 1 = x$$

$$x - y + 1 = 0$$

Is the required eq. of the line tangent to

$$r = 1 + \cos\theta \text{ at } (1, \pi/2)$$