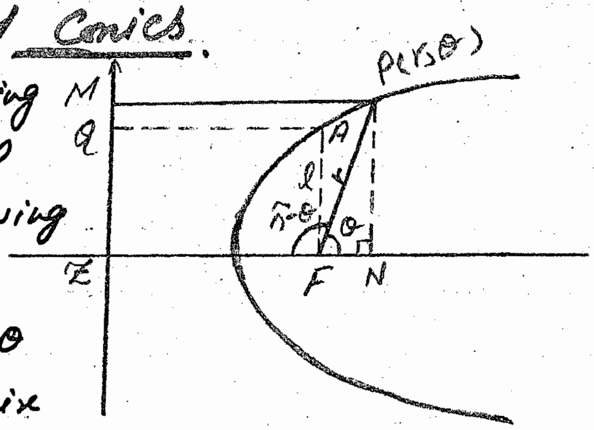
Polar Equations of Conics.



Consider the conic having the focus at the pole. let P be a pt. on the conic having the coordinates (r,θ)

$\Rightarrow |FP| = r$ and $P\hat{F}N = \theta$

And ZM be the Directrix of the conic and $|FA| = l$ length of semi latus rectum. and $|PM|$ is the length of the perpendicular from P on the directrix.

By definition of a conic

$$\frac{|PF|}{|PM|} = e$$

$$|PF| = e|PM|$$

$$r = e|ZN|$$

$$= e(|ZF| + |FN|)$$

$$= e|ZF| + e|FN| \quad \text{———} \quad (1)$$

Also we have

$$\frac{|AF|}{|AQ|} = e$$

$$\Rightarrow |AF| = e|AQ|$$

$$|AF| = e|ZF| \quad \text{———} \quad (2)$$

$\because |AF| = l$

$$l = e|ZF| \quad \text{———} \quad (3)$$

(3) in (1) $\Rightarrow$

$$r = l + e|FN| \quad \text{———} \quad (4)$$

Now from r.t. Δ PNF

$$\frac{|FN|}{r} = \cos\theta$$

$$|FN| = r\cos\theta \quad \text{———} \quad (5)$$

(5) in (4) $\Rightarrow$

$$r = l + re\cos\theta$$

$$r - re\cos\theta = l$$

$$r(1 - e\cos\theta) = l$$

$$r = \frac{l}{1 - e\cos\theta}$$

or $\quad 1-e\cos\theta = \frac{l}{r}$

which is called polar equations of Conics.

This represents parabola, ellipse or hyperbola according as $e=1$, $e<1$ or $e>1$

Deduction

In the equation $\frac{l}{r} = 1-e\cos\theta$ we have taken $\vec{FN}$ to be the positive direction of the initial line. If we regard $\vec{FZ}$ to be the positive direction of the initial line, then eq. of the Conic is given by

$$\frac{l}{r} = 1-e\cos(\pi-\theta)$$

$\because \vec{FP}$ makes angle $\pi-\theta$ with $\vec{ZF}$ $\qquad \because \cos(\pi-\theta) = -\cos\theta$

$$\Rightarrow \frac{l}{r} = 1+e\cos\theta$$

Some useful results to recognize the Conic:-

The Conic is a parabola if $e=1$

The Conic is an ellipse if $e<1$

The Conic is a hyperbola if $e>1$

Exercise 6.4

In Problems (1-6), identify and graph the given polar equations:

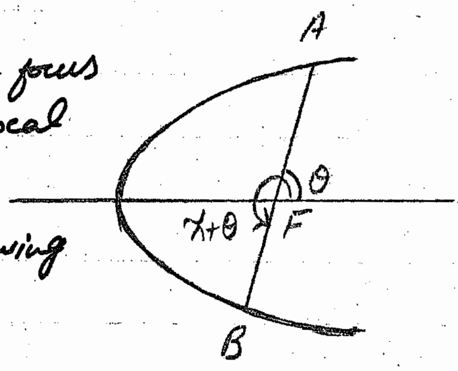
For Questions 1-6 Please see Graph Book.

Question 7

Show that in any Conic the sum of the reciprocals of the segments of any focal chord is constant.

Soln.

Consider a Conic having the focus at $O(0,0)$. Let AFB be the focal chord. Then AF and FB be the segments of focal chord having the angles θ and $\pi+\theta$.



We know that

$$r = \frac{l}{1-e\cos\theta}$$

$$\Rightarrow |AF| = \frac{l}{1-e\cos\theta} \qquad \text{——— (1)}$$

and

$$|FB| = \frac{l}{1-e\cos(\pi+\theta)}$$

$$|FB| = \frac{l}{1+e\cos\theta} \qquad \text{——— (2)}$$

Now sum of the reciprocals of the segments

$$\frac{1}{|AF|} + \frac{1}{|FB|} = \frac{1-e\cos\theta}{l} + \frac{1+e\cos\theta}{l}$$

$$= \frac{1-e\cos\theta+1+e\cos\theta}{l}$$

$$= \frac{2}{l} = \text{Constant.}$$

Note:

$$\frac{|AF|+|BF|}{|AF||BF|} = \frac{2}{l}$$

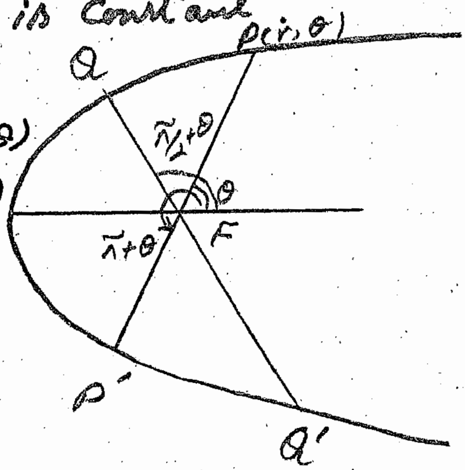
$$l = \frac{2|AF||BF|}{|AF|+|BF|}$$

$$= \frac{2ab}{a+b}$$

we also observe that l is the semi latus rectum is the harmonic mean between the two segments of the focal chord.

<u>Question 8.</u> If PFP' and QFQ' are two perpendiculars focal chords of a conic, prove that

$$\frac{1}{|PF|\cdot|FP'|} + \frac{1}{|QF|\cdot|FQ'|} \text{ is constant}$$



<u>Soln.</u> Let the coordinates of P be (r, θ)

Then the coordinate of P' be $(r', \pi+\theta)$

The coordinates of Q, $(r'', \frac{\pi}{2}+\theta)$

The coordinates of Q' $(r''', 3\pi/2+\theta)$

Now we know that

$$\frac{1}{r} = \frac{1-e\cos\theta}{l}$$

$$\frac{1}{|PF|} = \frac{1-e\cos\theta}{l}$$

and $$\frac{1}{|FP'|} = \frac{1-e\cos(\pi+\theta)}{l} = \frac{1+e\cos\theta}{l}$$

now $$\frac{1}{|PF|\cdot|FP'|} = \frac{1-e\cos\theta}{l}\cdot\frac{1+e\cos\theta}{l}$$

$$= \frac{1-e^2\cos^2\theta}{l^2} \quad \text{———} (1)$$

Also

$$\frac{1}{|QF|} = \frac{1-e\cos(\pi/2+\theta)}{l}$$

$$= \frac{1+e\sin\theta}{l}$$

and

$$\frac{1}{|FQ'|} = \frac{1-e\cos(3\pi/2+\theta)}{l}$$

$$= \frac{1-e\sin\theta}{l}$$

Now $$\frac{1}{|QF|\cdot|FQ'|} = \frac{1+e\sin\theta}{l}\cdot\frac{1-e\sin\theta}{l}$$
$$= \frac{1-e^2\sin^2\theta}{l^2} \quad \text{———— (2)}$$

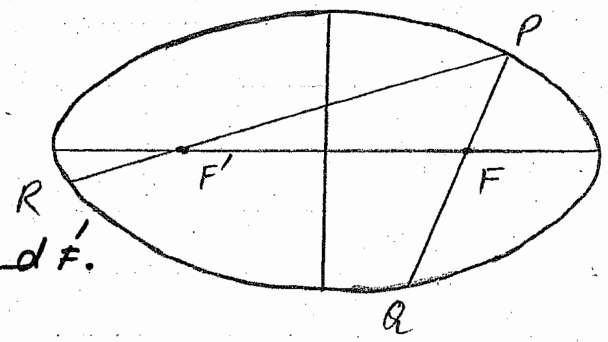
(1) + (2) $\Rightarrow$
$$\frac{1}{|PF|\cdot|FP'|} + \frac{1}{|QF|\cdot|FQ'|} = \frac{1-e^2\cos^2\theta}{l^2} + \frac{1-e^2\sin^2\theta}{l^2}$$
$$= \frac{1-e^2\cos^2\theta + 1 - e^2\sin^2\theta}{l^2}$$
$$= \frac{2-e^2(\cos^2\theta+\sin^2\theta)}{l^2}$$
$$= \frac{2-e^2}{l^2} = \text{Constant.}$$

Question 9.

If PFQ, $PF'R$ be two chords of an ellipse through the foci F, F', show that
$$\frac{|PF|}{|FQ|} + \frac{|PF'|}{|F'R|} \text{ is independent of}$$
the position of P.

Soln.

Consider that PFQ and $PF'R$ are the two focal chords of the ellipse having the foci at F and F'.



$\because$ PFQ is a focal chord

$\therefore$ PF and FQ are its two segments

$\therefore$ l is the H.M. between PF and FQ where l is the semi latus rectum

$\therefore$ $$\frac{1}{|PF|} + \frac{1}{|FQ|} = \frac{2}{l}$$

$H.M = \dfrac{2ab}{a+b}$

$\dfrac{1}{a}+\dfrac{1}{b} = \dfrac{2}{H}$

$\times$ by $|PF|$

$\Rightarrow$ $$1 + \frac{|PF|}{|FQ|} = \frac{2|PF|}{l} \quad \text{———— (1)}$$

Also $PF'R$ is a focal chord

therefore $|PF'|$ and $|F'R|$ are its two segments

Therefor proceeding as above

$$1+\frac{|PF'|}{|F'R|} = \frac{2\,|PF'|}{l} \qquad \text{——— (2)}$$

$(1)+(2) \Rightarrow$

$$2+\frac{|PF|}{|FQ|}+\frac{|PF'|}{|F'R|} = \frac{2}{l}\left(|PF|+|PF'|\right)$$

$$\frac{|PF|}{|FQ|}+\frac{|PF'|}{|F'R|} = \frac{2}{l}(2a)-2$$

$$\frac{|PF|}{|FQ|}+\frac{|PF'|}{|F'R|} = \frac{4a}{l}-2 = \text{Constant.}$$

In Ellipse if we take a point on it. If we sum up the distance of this point from F and F' i.e $PF+PF'$ then it is equal to $2a$.

Express each of the given equations in polar form and find the eccentricity and equation of the directrix.

Question 10 $\quad y^2 = 4-4x$

$$r^2\sin^2\theta = 4-4r\cos\theta$$

put $x = r\cos\theta$
$y = r\sin\theta$

$$\sin^2\theta\, r^2 + 4\cos\theta\, r - 4 = 0$$

which is quadratic in r

So
$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta - 4\sin^2\theta(-4)}}{2\sin^2\theta}$$

$$r = \frac{4\left(-\cos\theta \pm \sqrt{\cos^2\theta+\sin^2\theta}\right)}{2\sin^2\theta}$$

$$r = \frac{4(-\cos\theta \pm 1)}{2\sin^2\theta}$$

Neglecting –ve sign

$$r = \frac{2(-\cos\theta+1)}{1-\cos^2\theta}$$

$$r = \frac{2(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}$$

$$r = \frac{2}{1+\cos\theta}$$

$$\Rightarrow \quad e = 1$$

This is a parabola with eccentricity, $e = 1$

eq. of Directrix

~~We prove that eq. of the directrix of parabola is $x+a=0$ ——— (i)~~

$\because\ l = 2a$

latus rectum $= 4a$
semi latus rectum $= 2a$

$\Rightarrow \quad 2 = 2a$

$a = 1$

$\because$ The distance of the directrix line from the focus is equal to $2a$. And the form of the conic focus is at the origin.

$\therefore$ The directrix line is at a distance of $2a$ from the focus

i.e $\quad x = 2a$

$\Rightarrow \quad x = 2$ $\qquad \Big| \; x = \frac{l}{e}$

$r\cos\theta = 2$

$r = 2\sec\theta$

Question 11.

$3y^2 - 16y - x^2 + 16 = 0$

$3r^2\sin^2\theta - 16r\sin\theta - r^2\cos^2\theta + 16 = 0$ $\qquad$ Put $x = r\cos\theta$

$(3\sin^2\theta - \cos^2\theta)r^2 - 16\sin\theta\, r + 16 = 0$ $\qquad y = r\sin\theta$

$$r = \frac{16\sin\theta \pm \sqrt{256\sin^2\theta - 4(3\sin^2\theta - \cos^2\theta)(16)}}{2(3\sin^2\theta - \cos^2\theta)}$$

$$r = \frac{16\sin\theta \pm \sqrt{256\sin^2\theta - 192\sin^2\theta + 64\cos^2\theta}}{2(3\sin^2\theta - \cos^2\theta)}$$

$$r = \frac{16\sin\theta \pm \sqrt{64\sin^2\theta + 64\cos^2\theta}}{2(3\sin^2\theta - \cos^2\theta)}$$

$$r = \frac{16\sin\theta \pm 8\sqrt{\sin^2 + \cos^2\theta}}{2(3\sin^2\theta - (1-\sin^2\theta))}$$

$$r = \frac{16\sin\theta \pm 8}{2(3\sin^2\theta - 1 + \sin^2\theta)}$$

$$r = \frac{8(2\sin\theta \pm 1)}{2(4\sin^2\theta - 1)}$$

$$r = \frac{4(2\sin\theta + 1)}{4\sin^2\theta - 1}$$ Neglecting -ve sign.

$$r = \frac{4(2\sin\theta + 1)}{(2\sin\theta - 1)(2\sin\theta + 1)}$$

$$r = \frac{4}{2\sin\theta - 1}$$

$$r = \frac{4}{-(1-2\sin\theta)}$$

$$r = \frac{-4}{1-2\cos(\pi/2-\theta)}$$

$$\Rightarrow \quad e = 2, >1$$

Thus this is a hyperbola with eccentricity, $e = 2$

Equation of a directrix is $x=2$

$$r\sin\theta = 2$$

$$r = 2\operatorname{cosec}\theta$$

Eq. of Directrix

$x + a = 0$

$\because l = -4 \Rightarrow 4 = 2a \quad \because l = 2a$

$\Rightarrow a = -2$

$\Rightarrow x - 2 = 0$

$\Rightarrow x = 2$

Question 22. $8x^2 + 9y^2 + 4x - 4 = 0$

The equation in polar form is

$$8r^2\cos^2\theta + 9r^2\sin^2\theta + 4r\cos\theta - 4 = 0$$

$$(8\cos^2\theta + 9\sin^2\theta)r^2 + 4\cos\theta - 4 = 0$$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta - 4(8\cos^2\theta + 9\sin^2\theta)(-4)}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta + 128\cos^2\theta + 144\sin^2\theta}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$r = \frac{-4\cos\theta \pm \sqrt{144\cos^2\theta + 144\sin^2\theta}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$r = \frac{-4\cos\theta \pm \sqrt{144}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$r = \frac{-4\cos\theta \pm 12}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$r = \frac{-2\cos\theta \pm 6}{8\cos^2\theta + 9\sin^2\theta}$$

$$r = \frac{6 - 2\cos\theta}{8\cos^2\theta + 9(1-\cos^2\theta)}$$

$$r = \frac{6 - 2\cos\theta}{8\cos^2\theta + 9 - 9\cos^2\theta}$$

$$r = \frac{6 - 2\cos\theta}{9 - \cos^2\theta}$$

$$r = \frac{2(3 - \cos\theta)}{(3+\cos\theta)(3-\cos\theta)}$$

$\Rightarrow \qquad r = \dfrac{2}{3+\cos\theta}$

$$r = \frac{2}{3\left(1+\frac{1}{3}\cos\theta\right)}$$

$$r = \frac{2/3}{1+\frac{1}{3}\cos\theta}$$

$\Rightarrow \qquad e = \frac{1}{3}, < 1$

Thus the conic is ellipse

$$\frac{\frac{2}{3}}{\frac{1}{3}}$$

Eq. of the directrix is $x = 2$

$\Rightarrow \quad r\cos\theta = 2$

$r = 2\sec\theta$.

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