Exercise 6.3

Written by
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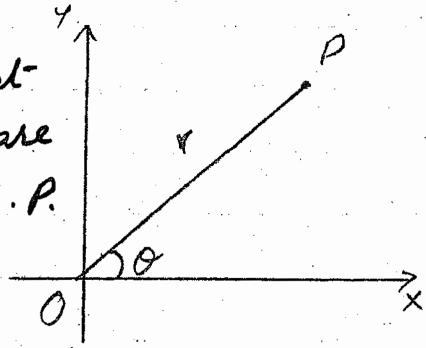
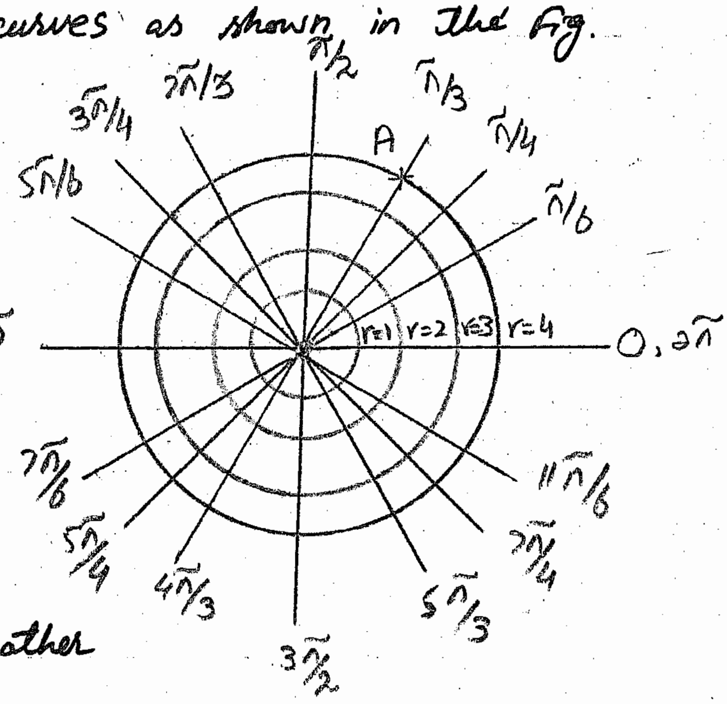
Polar Coordinates.

If P is a point, such that $|OP| = r$ and $P\hat{O}X = \theta$ then (r, θ) are called Polar coordinates of the pt. P. The curves in (r, θ) are called the Polar curves.



To draw a graph of polar curves, we select a pt. O of the plane, which is called the called the pole then we draw the family of curves as shown in the fig. and also a family of straight lines through the pole as in the Fig.



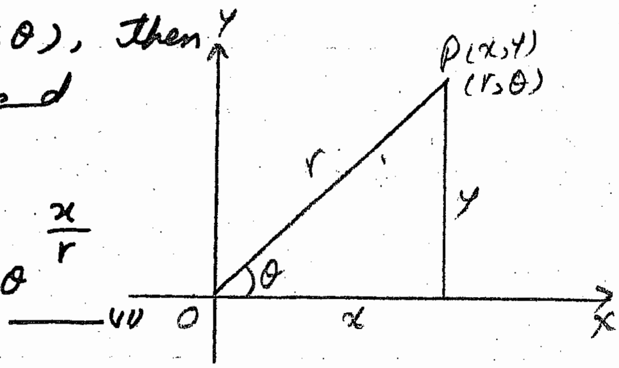
Now if a pt. A has the polar coordinates, $(4, \pi/3)$, then the position of this pt. A is shown in the Fig. Similarly all other pts. can be drawn.

Relation between Polar and Cartesian Coordinates:-

Consider the pt. P whose cartesian coordinates are (x, y) and polar coordinates are (r, θ), then

$|OA| = x$; $|AP| = y$; $|OP| = r$ and

$A\hat{O}P = \theta$



Then $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$

$\Rightarrow \quad x = r\cos\theta$ and $y = r\sin\theta$

——— (i) ——— (ii)

$(i)^2 + (ii)^2 \Rightarrow$

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2} \quad \text{——— (iii)}$$

$(ii) \div (i) \Rightarrow$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\Rightarrow \quad \theta = \tan^{-1}\frac{y}{x} \quad \text{——— (iv)}$$

(i), (ii) gives cartesian in form of polar and (iii), (iv) polar in form of cartesian.

Note:

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$$

and

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

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In Problems 1-8, express the given equations in rectangular coordinates.

Question 1.

Soln:

$$r^2 = a^2 \sin 2\theta \quad \text{------} (1)$$

$$r^2 = 2a^2 \sin\theta \cos\theta \quad \text{------} (2)$$

$\because$ $r^2 = x^2+y^2$, $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$

$\therefore$ (2) becomes

$$r^2 = 2a^2 \frac{y}{r} \cdot \frac{x}{r}$$

$$r^4 = 2a^2 yx$$

$$(x^2+y^2)^2 = 2a^2 yx$$

Question 2.

Soln.

$$r^4 \sin 4\theta = a^4$$

$$r^4 = \frac{a^4}{\sin 4\theta}$$

$$r^4 = \frac{a^4}{2\sin 2\theta \cos 2\theta}$$

$$r^4 = \frac{a^4}{4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)} \quad \text{------} (1)$$

$\because$ $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$

$\therefore$ 1) $\Rightarrow$

$$= \frac{a^4}{4\left(\frac{y}{r}\right)\left(\frac{x}{r}\right)\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right)}$$

$$= \frac{a^4}{\frac{4xy}{r^2}\left(\frac{x^2-y^2}{r^2}\right)}$$

$$r^4 = \frac{a^4 r^4}{4xy(x^2-y^2)}$$

$$\Rightarrow \quad 4xy(x^2-y^2) = a^4$$

Question 3. $r^2 = a^2 \cos 2\theta$

Soln:

$$r^2 = a^2(\cos^2\theta - \sin^2\theta)$$

$$r^2 = a^2\left(\frac{x^2}{r^2} - \frac{y^2}{r^2}\right) \qquad \because \frac{x}{r} = \cos\theta, \frac{y}{r} \sin\theta$$

$$r^2 = a^2\left(\frac{x^2 - y^2}{r^2}\right)$$

$$r^4 = a^2(x^2 - y^2)$$

$$(x^2+y^2)^2 = a^2(x^2-y^2) \qquad \because r^2 = x^2+y^2$$

Question 4. $r = 2a \sin\theta \tan\theta$

Soln.

$$r = 2a \frac{y}{r} \cdot \frac{y}{x} \qquad \because \sin\theta = \frac{y}{r}, \tan\theta = \frac{y}{x}$$

$$r^2 = \frac{2ay^2}{x}$$

$$x(x^2+y^2) = 2ay^2$$

Question 5. $r = 1 - \cos\theta$

Soln.

$$r = 1 - \frac{x}{r} \qquad \because \cos\theta = \frac{x}{r}$$

$$r = \frac{r - x}{r}$$

$$r^2 = r - x$$

$$\Rightarrow \quad x^2+y^2 = \sqrt{x^2+y^2} - x \qquad \because r^2 = x^2+y^2$$

$$x^2+y^2+x = \sqrt{x^2+y^2} \qquad \Rightarrow r = \sqrt{x^2+y^2}$$

$$(x^2+y^2+x)^2 = x^2+y^2$$

Question 6. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$

Soln.

$$r^2\left(4\frac{y^2}{r^2} - 9\frac{x^2}{r^2}\right) = 36$$

$$r^2\frac{(4y^2 - 9x^2)}{r^2} = 36$$

$$4y^2 - 9x^2 = 36$$

$$\frac{4y^2}{36} - \frac{9x^2}{36} = 1$$

$$\Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$$

Question 7. $r = \dfrac{8}{2 - \cos\theta}$

Soln.

$$r = \frac{8}{2 - \frac{x}{r}} = \frac{8}{\frac{2r - x}{r}}$$

$$r = \frac{8r}{2r - x} \Rightarrow 1 = \frac{8}{2r - x}$$

$$\Rightarrow 2r - x = 8$$

$$\Rightarrow \quad 2r = x + 8$$

$$2\sqrt{x^2+y^2} = x + 8 \qquad \because r = \sqrt{x^2+y^2}$$

$$(2)^2(x^2+y^2) = x^2 + 64 + 16x$$

$$4x^2 + 4y^2 = x^2 + 64 + 16x$$

$$\Rightarrow \quad 3x^2 + 4y^2 - 16x - 64 = 0$$

Question 8.

$$r = 2\sin\theta + 3\cos\theta$$

Soln.

$$r = 2\frac{y}{r} + 3\frac{x}{r} \qquad \because \sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r}$$

$$\Rightarrow \quad r^2 = 2y + 3x$$

$$\Rightarrow \quad x^2 + y^2 = 2y + 3x \qquad \because r^2 = x^2 + y^2$$

Transform the given equations in polar coordinates. (9-15)

Question 9.

$$xy = a \qquad \text{--- (1)}$$

Soln.

$$\because \quad x = r\cos\theta \text{ and } y = r\sin\theta$$

$$\therefore (1) \Rightarrow r\cos\theta \cdot r\sin\theta = a$$

$$r^2\sin\theta\cos\theta = a$$

$$\Rightarrow \quad r^2 = \frac{a}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad r^2 = a\csc\theta\sec\theta.$$

Question 10

$$y^2 = 4x$$

Soln.

$$r^2\sin^2\theta = 4r\cos\theta \qquad \because y = r\sin\theta,\ x = r\cos\theta$$

$$r(r\sin^2\theta - 4\cos\theta) = 0$$

$$r\sin^2\theta - 4\cos\theta = 0$$

Question 11

$$y = \frac{x}{x+1}$$

$$r\sin\theta = \frac{r\cos\theta}{r\cos\theta + 1}$$

$$r\sin\theta(r\cos\theta + 1) = r\cos\theta$$

$$\sin\theta(r\cos\theta + 1) = \cos\theta$$

$$r\sin\theta\cos\theta + \sin\theta = \cos\theta$$

$$r\sin\theta\cos\theta = \cos\theta - \sin\theta$$

$$r = \frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta}$$

$$\Rightarrow \quad r = \frac{1}{\sin\theta} - \frac{1}{\cos\theta}$$

$$r = \operatorname{cosec}\theta - \sec\theta.$$

Question 12. $x^2 + y^2 - 8x + 6y + 7 = 0 \quad \text{—— (1)}$

Soln. $\because \; r^2 = x^2 + y^2, \; x = r\cos\theta, \; y = r\sin\theta$

$$\therefore (1) \Rightarrow r^2 - 8r\cos\theta + 6r\sin\theta + 7 = 0$$

$$r^2 - 2r(4\cos\theta + 3\sin\theta) + 7 = 0$$

Question 13. $(x^2 + y^2)\,y^2 = a^2 x^2$

Soln. $r^2 \cdot r^2\sin^2\theta = a^2 r^2\cos^2\theta \qquad \because \; x^2 + y^2 = r^2, \; y = r\sin\theta, \; x = r\cos\theta.$

$$r^2\sin^2\theta = a^2\cos^2\theta$$

$$r^2 = \frac{a^2\cos^2\theta}{\sin^2\theta}$$

$$\Rightarrow r^2 = a^2\cot^2\theta$$

$$\Rightarrow r = a\cot\theta$$

Question 14. $x^3 + 4x^2 + xy^2 - 4y^2 = 0$

Soln. $r^3\cos^3\theta + 4r^2\cos^2\theta + r\cos\theta \cdot r^2\sin^2\theta - 4r^2\sin^2\theta = 0$

$$r^2(r\cos^3\theta + r\cos\theta\sin^2\theta + 4\cos^2\theta - 4\sin^2\theta) = 0$$

$$r\cos^3\theta + 4(\cos^2\theta - \sin^2\theta) + r\sin^2\theta\cos\theta = 0$$

$$r\cos^3\theta + 4\cos 2\theta + r\sin^2\theta\cos\theta = 0$$

Question 15 $x^4 + 2x^2y^2 + y^4 - 6x^2y - 2y^3 = 0$

Soln. $(x^2 + y^2)^2 - 2y(3x^2 + y^2) = 0$

$$r^4 - 2r\sin\theta(3r^2\cos^2\theta + r^2\sin^2\theta) = 0$$

$$r^4 - 2r^3\sin\theta(3\cos^2\theta + \sin^2\theta) = 0$$

$$r - 2\sin\theta(3\cos^2\theta + \sin^2\theta) = 0$$