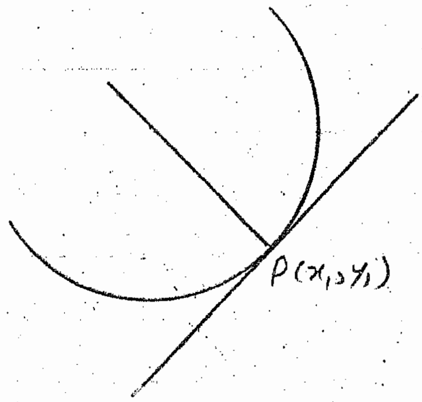
Tangent and Normal



Let $y = f(x)$ be the curve and $P(x_1, y_1)$ be any point on it.

$\because \; y = f(x)$

$\therefore \; \dfrac{dy}{dx} = f'(x)$

$\left(dy/dx\right)_{P(x_1, y_1)} = f'(x_1)$

Tangent. If a line touches the curve at the pt. $P(x_1, y_1)$. Then this line is called Tangent to the curve at the pt. $P(x_1, y_1)$. The equation of the Tangent at this point $P(x_1, y_1)$ to the curve $y = f(x)$ is given by

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = f'(x_1)(x - x_1)$$

Generally $\quad y - y_1 = f'(x)\Big|_P (x - x_1)$

Normal. A line passing through the pt. P and $\perp$ to the tangent at the pt. P to the curve is called the Normal at that pt. P.

The equation of the normal at the pt. $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1)$$

Generally

$$y - y_1 = -\frac{1}{f'(x)}\Big|_P (x - x_1).$$

Example # 7

Find the eq. of the normal to the parabola $y^2 = 4ax$ in the form $y = mx - 2am - am^3$

Soln. $\quad y^2 = 4ax$

$2y\dfrac{dy}{dx} = 4a \implies \dfrac{dy}{dx} = \dfrac{2a}{y}$

$\dfrac{dy}{dx}\Big|_{P(x_1, y_1)} = \dfrac{2a}{y_1}$

Slope of the normal $= -\dfrac{y_1}{2a} = m \quad (\text{say})$

$$-y_1 = 2am \quad \Rightarrow \quad y_1 = -2am \quad \text{———} (2)$$

$\because$ (x_1, y_1) lies on $y^2 = 4ax$

$$\therefore \quad y_1^2 = 4ax_1 \quad \text{———} (3)$$

Put $y_1 = -2am$ in (3)

$$4a^2m^2 = 4ax_1$$

$$\Rightarrow \quad x_1 = am^2 \quad \text{———} (4)$$

$$P(x_1, y_1) = P(am^2, -2am)$$

eq. of the normal

$$y - (-2am) = m(x - am^2)$$

$$y + 2am = mx - am^3$$

$$y = mx - 2am - am^3$$

is as required.

Parametric form of Parabola:-

$$y^2 = 4ax$$

$$x = at^2, \; y = 2at$$

$x = at^2$, $y = 2at$ are the equations representing the parabola in parametric form. where 't' is called the parameter.

Parametric form of Ellipse:-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a\cos\theta$, $y = b\sin\theta$ are the parametric equations of the ellipse.

Parametric form of Hyperbola:-

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x = a\sec\theta; \; y = b\tan\theta$$

or $x = a\cosh\theta$; $y = b\sinh\theta$

are the parametric equations of the hyperbola.

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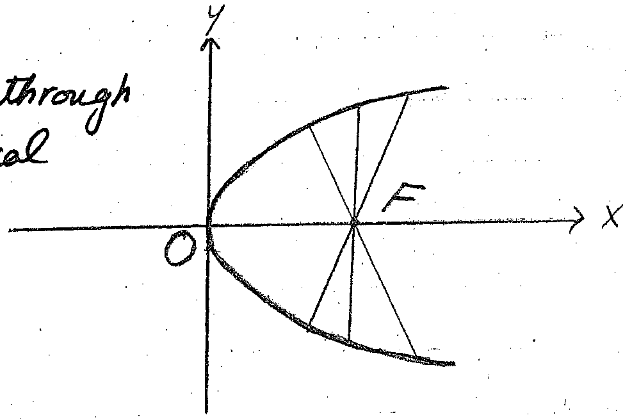
Focal Chord

The chord passes through the focus is called focal chord.



Example # 8

Show that the points $(at, 2at)$ always lies on the parabola $y^2 = 4ax$. Find the condition that the chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ may be a focal chord. Find an equation of the Tangent to the parabola at $(at^2, 2at)$.

Soln. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$

Using two points formula.

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

Putting $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$

$$\frac{y-2at_1}{2at_2-2at_1} = \frac{x-at_1^2}{at_2^2-at_1^2}$$

$$\frac{y-2at_1}{2\cancel{a}(t_2-t_1)} = \frac{x-at_1^2}{\cancel{a}(t_2^2-t_1^2)}$$

$$\frac{y-2at_1}{2(t_2-t_1)} = \frac{x-at_1^2}{(t_2-t_1)(t_2+t_1)}$$

$$\frac{y-2at_1}{2} = \frac{x-at_1^2}{t_2+t_1} \qquad \text{------} \ (1)$$

Focus: $(a, 0)$, put $x = a$, $y = 0$ in (1)

$$\frac{0-2at_1}{2} = \frac{a-at_1^2}{t_2+t_1}$$

$$-at_1 = \frac{a(1-t_1^2)}{t_2+t_1}$$

$$-t_1 = \frac{1-t_1^2}{t_1+t_2}$$

$$-t_1^2 - t_1t_2 = 1-t_1^2$$

$$-t_1t_2 = 1$$

$$t_1t_2 = -1$$

Is the required condition for the chord PQ to be focal chord.

Now equation of the Tangent at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y - y_1 = m(x - x_1) \qquad \text{———— (2)}$$

m = slope of tangent at $P(at^2, 2at)$ is

$$m = \frac{dy}{dx} = \frac{d(2at)}{d(at^2)} = \frac{2a}{2at} = \frac{1}{t}$$

$\therefore$ we have to find the equation of Tangent at $P(at^2, 2at)$.

$\therefore$ Put $x_1 = at^2$, $y_1 = 2at$ and $m = \frac{1}{t}$ in (2) we have

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

$$yt = x - at^2 + 2at^2$$

i.e $\quad yt = x + at^2$

<u>Note</u>. $x = at^2$, $y = 2at$ are called the parametric equations of the parabola $y^2 = 4ax$. The point $(at^2, 2at)$ is also referred to as point "t" on the parabola.

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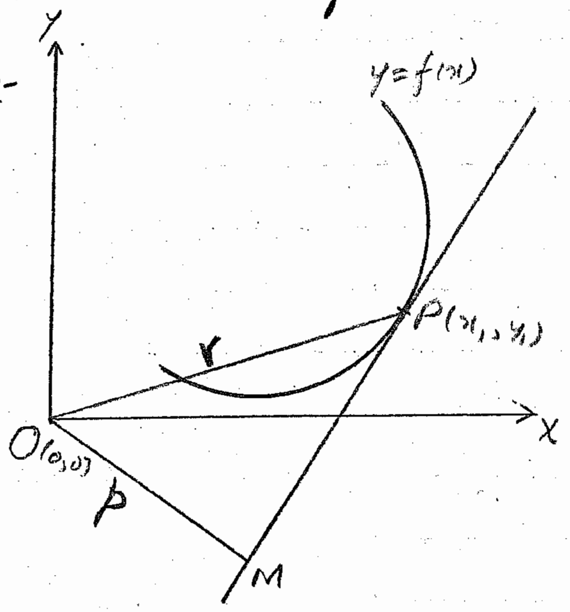
Pedal Equation:-

The pedal equation is an equation in 'p' and 'r' where 'r' is the distance of any point 'P' on the curve from O and p is the is the ⊥ distance of O from the tangent at P.



Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$.

Then $r = |OP| = \sqrt{x_1^2 + y_1^2}$ by distance formula.

$$r^2 = x_1^2 + y_1^2 \qquad \text{——— (1)}$$

∵ P lies on the curve

$$\therefore \quad y_1 = f(x_1) \qquad \text{——— (2)}$$

equation of the Tangent at $P(x_1, y_1)$.

$$(y - y_1) = f'(x_1)(x - x_1)$$

$$(y - y_1) = x f'(x_1) - x_1 f'(x_1)$$

$$x f'(x_1) - y + y_1 = x_1 f'(x_1)$$

Now p = Distance of $O(0,0)$ from tangent line.

$$p = \frac{|f'(x_1)(0) - 0 + y_1 - x_1 f'(x_1)|}{\sqrt{[f'(x_1)]^2 + 1^2}}$$

$$p = \frac{|y_1 - x_1 f'(x_1)|}{\sqrt{1 + f'^2(x_1)}} \qquad \text{——— (3)}$$

The distance of Point $P(x_1, y_1)$ from line $ax + by + c = 0$ is given by
$$p = d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

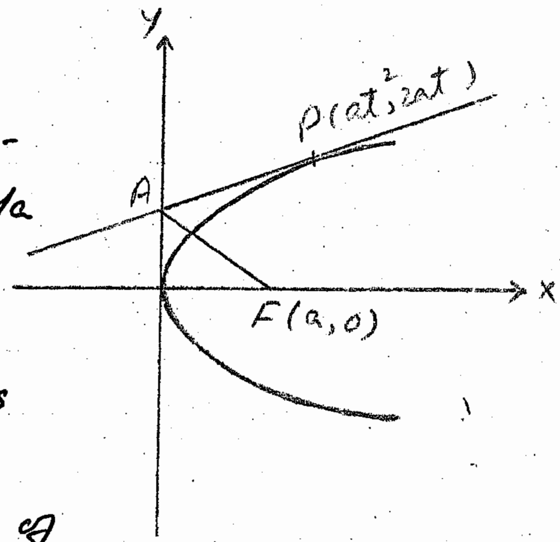
Elimination of x_1 and y_1 from eq (1) to (3) will gives us the equation in p and r called the pedal equation.

Example #9.

If the tangent at any point of the parabola meets y-axis at A, then prove that

$$P\hat{A}F = 90^\circ$$

Where P is any point on the parabola, A is point on y-axis and F is Focus.



Proof.

Consider that the tangent at pt. $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ meets y-axis at the pt. A.

$F(a,0)$ is the focus of this parabola.

We know that the equation of the tangent at the pt. $(at^2, 2at)$ is

$$x - ty + at^2 = 0$$

for the coordinate of A put $x = 0$

then
$$0 - ty + at^2 = 0$$
$$at^2 = ty$$
$$y = at$$

the coordinate of $A(0, at)$

$$m_1 = \text{Slope of } (\overline{PA}) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{at - 2at}{0 - at^2} = \frac{-at}{-at^2} = \frac{1}{t}$$

by

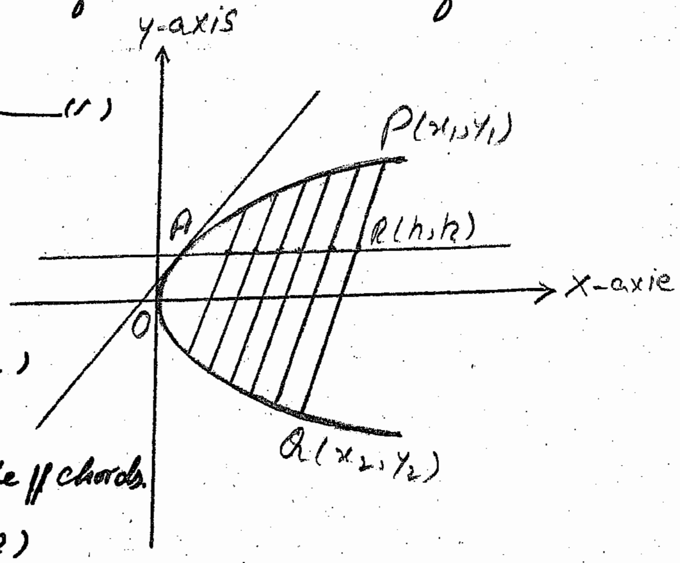
$$m_2 = \text{Slope of } (\overline{FA}) = \frac{0 - at}{a - 0} = -\frac{at}{a} = -t$$

Now
$$m_1 m_2 = \frac{1}{t} \cdot -t$$
$$= -1$$

$$\Rightarrow \overline{PA} \perp \overline{FA}$$
$$\Rightarrow P\hat{A}F = 90^\circ$$

Example #10

Find the locus of the middle points of a sysstem of parallel chords of the parabola of the parabola $y^2 = 4ax$.



Soln. given $y^2 = 4ax$ ——— (1)

Consider a system of // chords of the parabola.

Let m be their slopes and $y = mx + c$ ——— (2)

be the equation of the chord PQ representative of the // chords.

Further let $R = (h,k)$ be the middle point of the chord PQ ~~~~ of the parabola.

Now put (2) in (1)

$$(mx+c)^2 = 4ax$$

$$m^2x^2 + 2mcx + c^2 = 4ax$$

$$m^2x^2 + 2(mc - 2a)x + c^2 = 0 \qquad \text{——— (3)}$$

$\because$ It is quadratic in x

$\therefore$ The values of x obtained from (3) will serve as the x-cordinates of P and Q.

$\because$ R is the mid pt. of PQ

$$\therefore \quad h = \frac{x_1 + x_2}{2}$$

$$h = \frac{\frac{-2(mc-2a)}{m^2}}{2} \qquad x_1 + x_2 = -\frac{b}{a}$$

$$h = \frac{-(mc-2a)}{m^2} \qquad \text{——— (4)}$$

$\because$ $R = (h,k)$ lies on (2)

$\therefore$ $k = mh + c$

$\Rightarrow$ $c = k - mh$ ——— (5)

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(5) in (4) $\Rightarrow \quad h = \dfrac{-(m(k-mh)-2a)}{m^2}$

$$hm^2 = -(mk-m^2h)-2a)$$
$$m^2h = -mk+m^2h+2a$$
$$m^2h-m^2h = -mk+2a$$
$$0 = -mk+2a$$
$$mk = 2a$$
$$k = \frac{2a}{m}$$

i.e the pt. $R(h,k)$ lies on the locus $y=\dfrac{2a}{m}$

x^n of $y=\dfrac{2a}{m}$ and $y^2=4ax$

Put $y=\dfrac{2a}{m}$ in $y^2=4ax$

$$\frac{4a^2}{m^2} = 4ax$$
$$\frac{a}{m^2} = x$$
$$x = \frac{a}{m^2}$$

Point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$ is the Locus.

Note. The locus of the middle points of parallel chords of a parabola is called a diameter of the parabola.

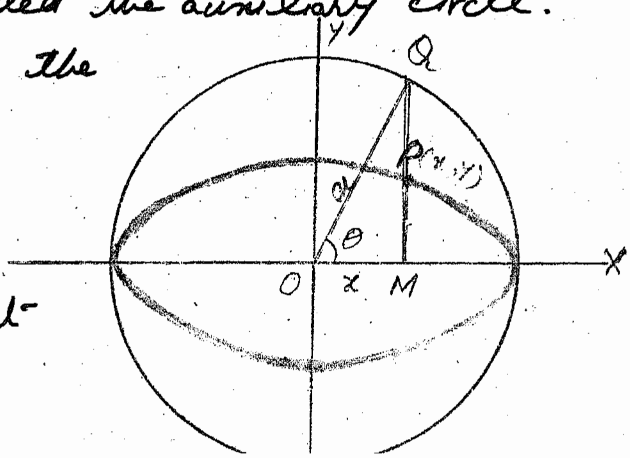
Auxiliary Circle.

Def. "The circle constructed on the Major axis of the ellipse as a diameter is called the auxiliary circle".

Let $p(x,y)$ be any point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

Draw $pM \perp$ to x-axis.

Produce MP so that it meets the circle at Q. Join Q to O.



Let $Q\hat{O}M = \theta$

Then we call θ as the eccentric angle of P.

from r.t Δ OMQ

$$\frac{x}{a} = \cos\theta$$

$$x = a\cos\theta$$

Put $x = a\cos\theta$ in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{a^2\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1$$

$$\cos^2\theta + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \cos^2\theta$$

$$\frac{y^2}{b^2} = \sin^2\theta$$

$$y^2 = b^2\sin^2\theta$$

$$y = \pm b\sin\theta$$

$$y = b\sin\theta$$

i.e $\quad P(x, y) = (a\cos\theta, b\sin\theta)$

Theorem.

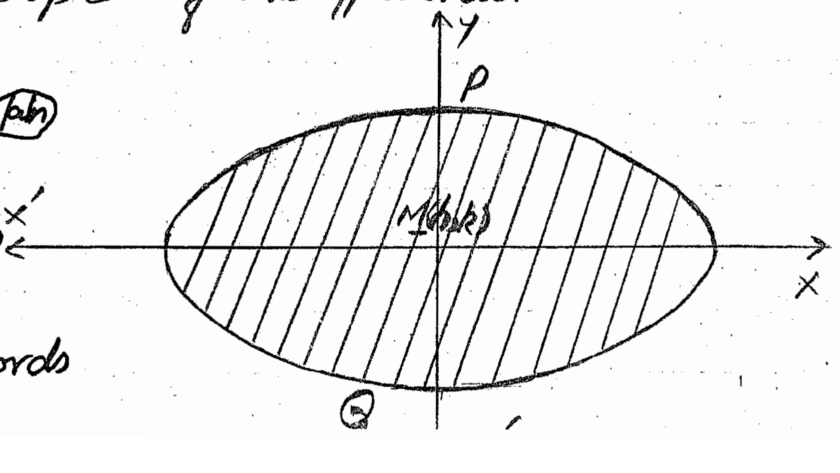
Show that the locus of the middle points of a system of $\parallel$ chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = -\frac{b^2}{a^2 m}x$$

where m is the slope of the $\parallel$ chords.

Proof: The equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{---(1)}$$



* The parallel chords

have slope m so that the equation to any one of them, say PQ is

$$y = mx + c \qquad \text{———— (1)}$$

The straight line (1) meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at points whose abscissae are given by

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$\Rightarrow \quad b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2$$

$$\Rightarrow \quad b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2$$

$$\Rightarrow \quad x^2(b^2 + a^2m^2) + 2a^2mcx + a^2c^2 - a^2b^2 = 0$$

i.e $\quad x^2(a^2m^2 + b^2) + 2a^2mcx + a^2(c^2 - b^2) = 0$

Let the roots of this equation be x_1, x_2. Then x_1, x_2 are the abscissae of P and Q. Let $M(h,k)$ be the middle point of PQ. Then by using sum of the roots we have

$$h = \frac{x_1 + x_2}{2} = \frac{-\dfrac{2a^2mc}{a^2m^2+b^2}}{2} = -\frac{a^2mc}{a^2m^2+b^2} \qquad \text{———— (2)}$$

$\because \quad M(h,k)$ lies on (1)

$\therefore \quad k = mh + c$

$$c = k - mh \qquad \text{———— (3)}$$

$ax^2 + bx + c = 0$

$x_1 + x_2 = -\dfrac{b}{a}$

(3) in (2)

$$\Rightarrow \quad h = -\frac{a^2m(k - mh)}{a^2m^2 + b^2}$$

$$a^2m^2h + b^2h = -a^2mk + a^2m^2h$$

$$b^2h = -a^2mk$$

$$k = -\frac{b^2h}{a^2m}$$

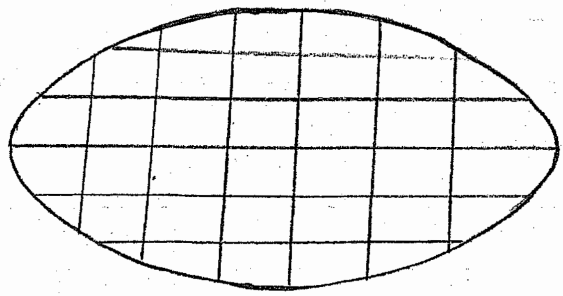
i.e the pt. $M(h,k)$ lies on the locus, $y = -\dfrac{b^2x}{a^2m}$.

Diameter of an Ellipse

Def. The locus of the middle pts. of a system of // chords of an ellipse is called a diameter of an ellipse.

Conjugate Diameter:-

Def. Two diameters of an ellipse are called conjugate if each bisects chord // to other.



Result for Conjugate Diameters.

For Conjugate diameters the product of their slopes $= -\dfrac{b^2}{a^2}$

Theorem.

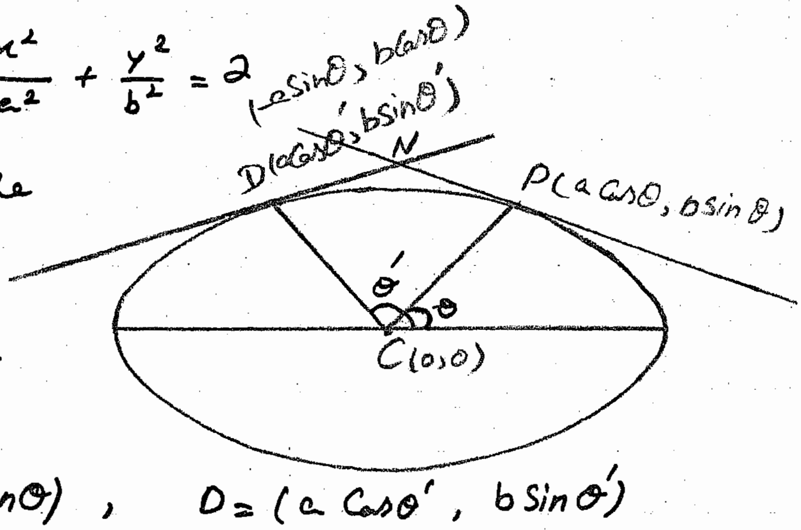
If CP and CD are semiconjugate diameters of an ellipse with centre C, show that

i) The ecentric angles of P and D differ by a right angle.

ii) $CP^2 + CD^2 = a^2+b^2$ is a constant.

iii) The locus of the point of intersection of tangents at P and D is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$



Proof: Let θ and θ' be the ecentric angles of P and D where CP and CD are the semi conjugate Diameters.

Then $P = (a\cos\theta, b\sin\theta)$, $D = (a\cos\theta', b\sin\theta')$

Slope of CP $= \dfrac{b\sin\theta - 0}{a\cos\theta - 0}$

$= \dfrac{b\sin\theta}{a\cos\theta}$

We use here the slope formula i.e. when the points are given i.e $\dfrac{y_2 - y_1}{x_2 - x_1}$

Similarly slope of CD $= \dfrac{b\sin\theta'}{a\cos\theta'}$

$\because$ CP and CD are semi conjugate diameters

$$\therefore \quad \frac{b\sin\theta}{a\cos\theta} \cdot \frac{b\sin\theta'}{a\cos\theta'} = -\frac{b^2}{a^2}$$

$$\Rightarrow \quad \frac{\sin\theta\sin\theta'}{\cos\theta\cos\theta'} = -1$$

$$\Rightarrow \quad \sin\theta\sin\theta' + \cos\theta\cos\theta' = 0$$

$$\cos(\theta' - \theta) = 0$$

$$\theta' - \theta = 90^\circ$$

$$\text{i.e} \quad \theta' - \theta = \pi/2$$

i.e the ecentric angles θ' and θ of D and P differ by right angle.

<u>Deduction</u>

$$\because \quad \theta' = \pi/2 + \theta$$

$$\therefore \quad D = \left(a\cos(\pi/2 + \theta),\ b\sin(\pi/2 + \theta) \right)$$

$$= (-a\sin\theta,\ b\cos\theta)$$

ii)

Target: $CP^2 + CD^2 = a^2 + b^2$

$$|CP|^2 = (a\cos\theta - 0)^2 + (b\sin\theta - 0)^2$$

$$= a^2\cos^2\theta + b^2\sin^2\theta$$

Now $|CD|^2 = (-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2$

$$= a^2\sin^2\theta + b^2\cos^2\theta$$

Now

$$|CP|^2 + |CD|^2 = a^2\cos^2\theta + b^2\sin^2\theta + a^2\sin^2\theta + b^2\cos^2\theta$$

$$CP^2 + CD^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2(\cos^2\theta + \sin^2\theta)$$

$$CP^2 + CD^2 = a^2 + b^2$$

Proved.

iii) We know that the Eq. of Ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It can be written as

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$$

Now Tangent at P is

$$\frac{x \cdot a\cos\theta}{a^2} + \frac{y \cdot b\sin\theta}{b^2} = 1$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \quad \text{———— (i)}$$

Tangent at D is

$$\frac{x(-a\sin\theta)}{a^2} + \frac{y}{b^2}(b\cos\theta) = 1$$

$$-\frac{x\sin\theta}{a} + \frac{y\cos\theta}{b} = 1 \quad \text{———— (ii)}$$

Squaring and adding (i) and (ii)

$$\frac{x^2\cos^2\theta}{a^2} + \frac{y^2\sin^2\theta}{b^2} + \frac{2xy\sin\theta\cos\theta}{ab} + \frac{x^2\sin^2\theta}{a^2} + \frac{y^2\cos^2\theta}{b^2} - 2\frac{xy\sin\theta\cos\theta}{ab} = 1+1$$

$$\frac{x^2}{a^2}\left[\cos^2\theta + \sin^2\theta\right] + \frac{y^2}{b^2}\left[\cos^2\theta + \sin^2\theta\right] = 2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

Proved.