Submitted by
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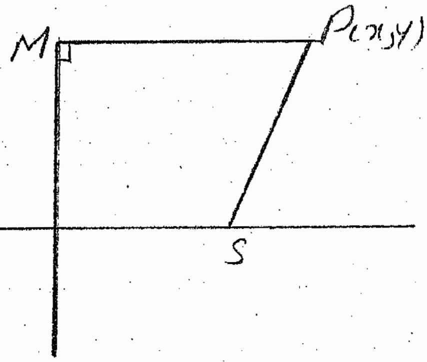
Chapter 6.

Plane Curves I

Topics of Exercise 6.1

Conic.

Def. The Conic is the set of points in a plane, the distance of each of which from a given point bears a constant ratio to its distance from a given straight line in the plane.



Focus.

The fixed point is called the Focus.

Directrix.

The fixed line is said to be the Directrix.

Eccentericity.

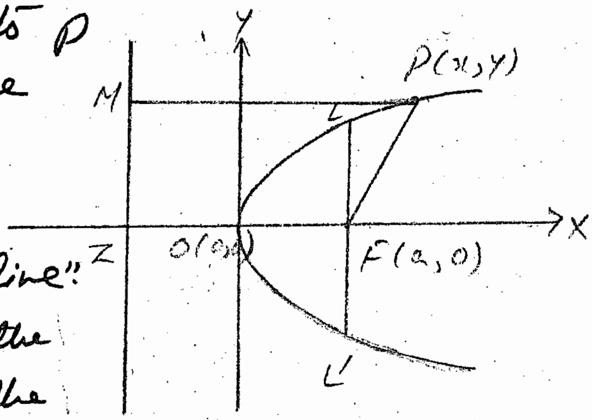
The constant ratio which is denoted by e, is known as the eccentericity of the Conic.

$$\text{i.e } \frac{|PS|}{|PM|} = e$$

If $e = 1$ The Conic is a parabola.
If $e < 1$ The Conic is an ellipse.
If $e > 1$ The Conic is a hyperbola.

The Parabola.

Def. "A Parabola is set of all points P in the plane that are equidistance from a fixed line and a fixed point in the plane. The fixed point does not lie on the fixed line."



The fixed line is called the Directrix of the parabola. and the fixed point is called it's focus. The straight line through the focus and perpendicular to the directrix is called axis of the parabola. The point where the parabola meets it's axis is called the vertex of the parabola.

The equation of a parabola with focus $F(a,0), a>0$ and vertex at the origin is quite simple. In this case the equation of the directrix ZM is $x=-a$. If $P(x,y)$ is a point on the parabola, then By definition, P is equidistance from F and ZM

Hence $|PF| = |PM|$

i.e $(x-a)^2 + y^2 = (x+a)^2$

$$y^2 = 4ax$$

Which is standard equation of parabola.

Let the line through $F(a,0)$ and perpendicular to the axis of the parabola meet the parabola at L and L'. If y' is ordinate of L, then $L(a,y')$ lies on $y^2 = 4ax$

$$\therefore \quad y'^2 = 4a^2$$

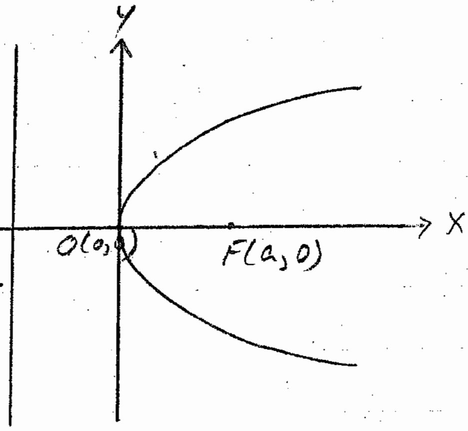
$$y' = \pm 2a$$

The length $LFL' = 4a$ and is called latus rectum of the parabola.

Different Cases of Parabola.

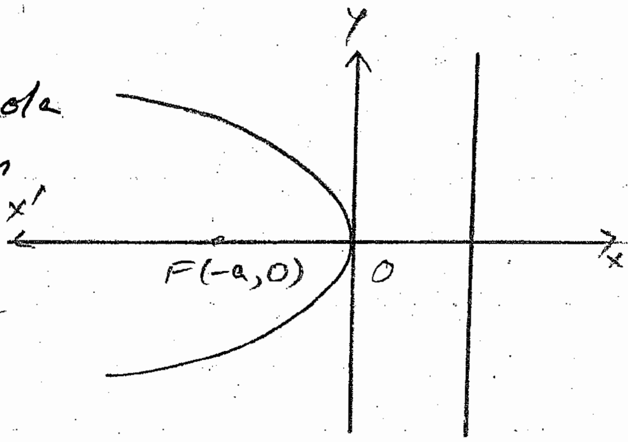
Case I)

If the equation of the Parabola is $y^2 = 4ax$, $a>0$ then the shape of the parabola is shown below.



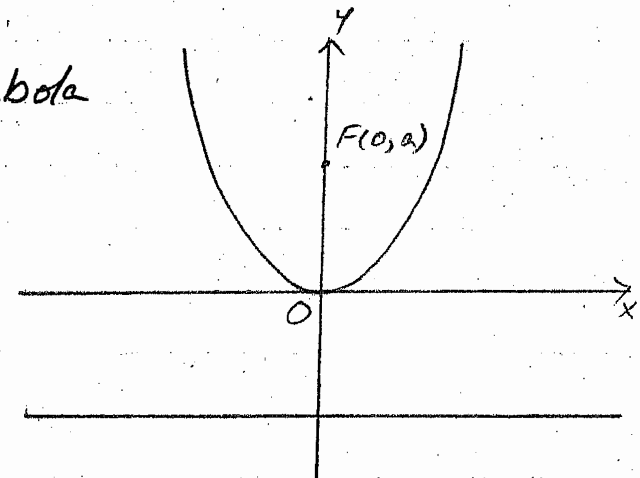
Case 2.

If the equation of the parabola is $y^2 = -4ax$ where $a > 0$, then shape of this parabola is shown.



Case 3.

If the equation of the parabola is $x^2 = 4ay$ where, $a > 0$ then the shape of the hyperbola is as shown.

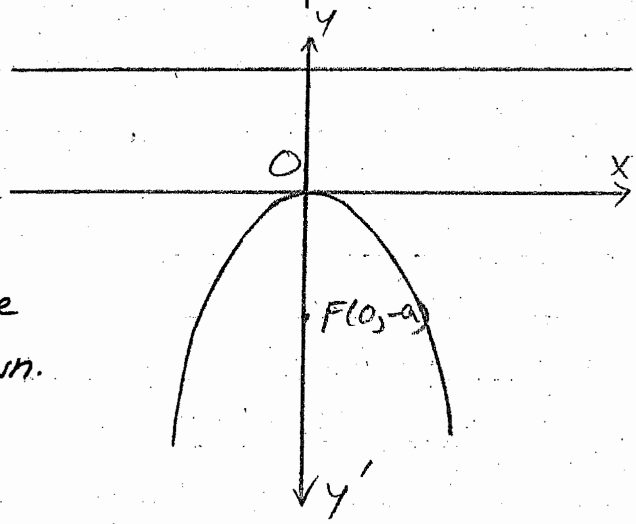


Case 4.

If the equation of the parabola is

$$x^2 = -4ay$$

where $a > 0$, then the shape of the parabola is as shown.



Case 5.

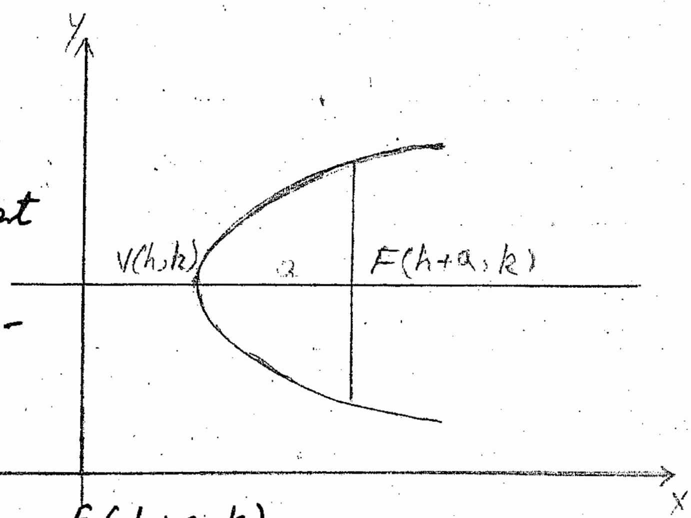
If the equation of the parabola is $(y-k)^2 = 4a(x-h)$. Then

i) The vertex of parabola is at (h,k)

ii) Parabola will open in right side.

iii) The shape is shown.

iv) Focus of the parabola is $F(h+a,k)$.



Case 6.

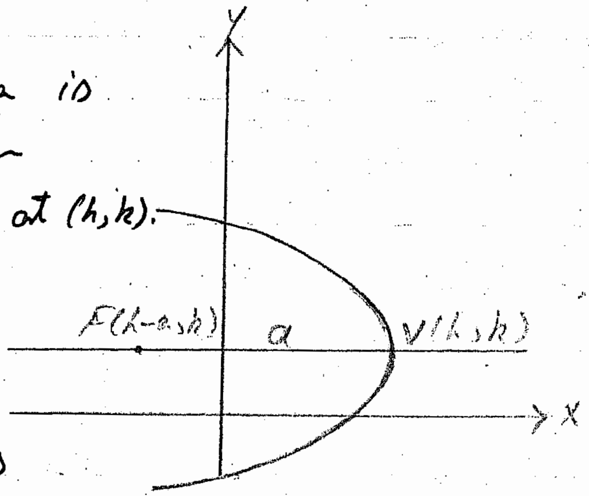
If the equation of the parabola is $(y-k)^2 = -4a(x-h)$; $a>0$ Then

i) The vertex of the parabola is at (h,k).

ii) Parabola will open in left side.

iii) The shape is shown.

iv) The focus of the parabola is $(h-a, k)$.



Case 7.

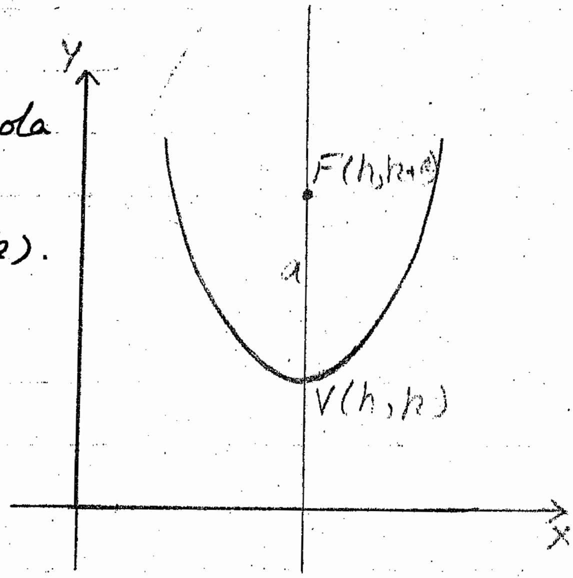
If the equation of the parabola is $(x-h)^2 = 4a(y-k)$ $a>0$, then

i) The vertex of the parabola is (h,k).

ii) Parabola will open in upward direction.

iii) The focus will be $(h, k+a)$

iv) The shape is as shown.



Case 8.

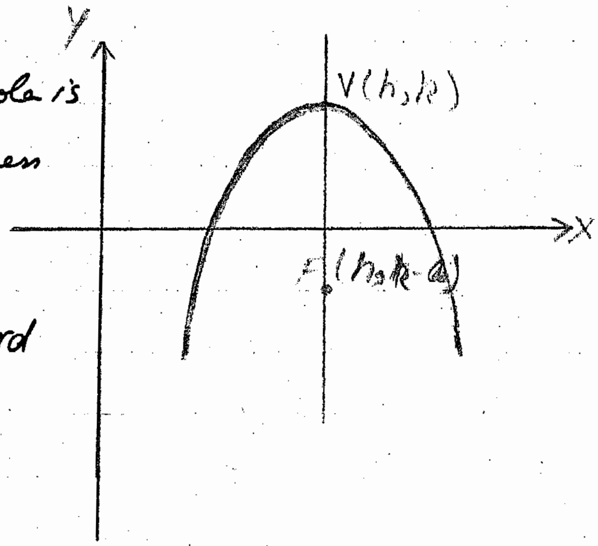
If the equation of the parabola is $(x-h)^2 = -4a(y-k)$; $a>0$ then

i) The vertex of the parabola is $V(h,k)$.

ii) Parabola will open in downword direction.

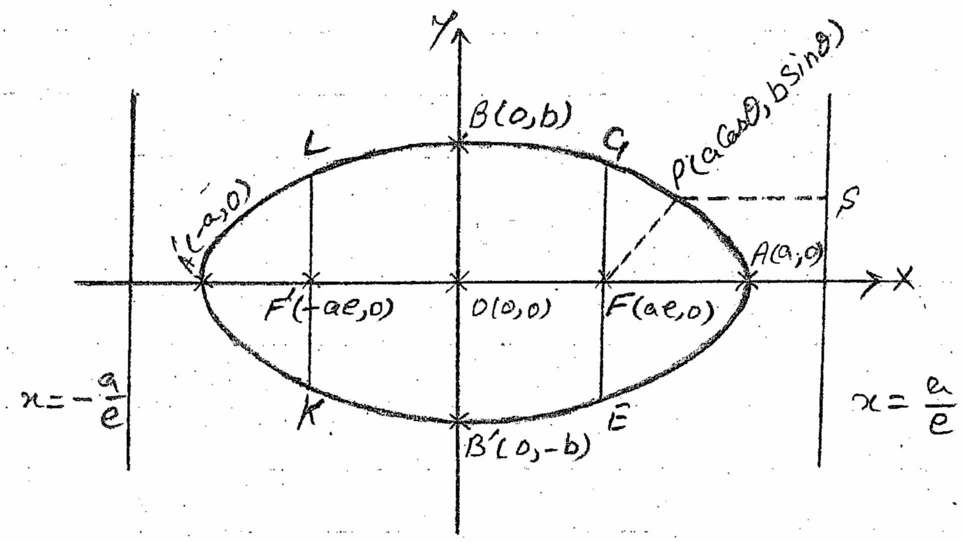
iii) The focus of the parabola is $(h, h-a)$.

iv) The shape is shown.



The Ellipse

1) The equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $a^2>b^2$ is called the equation of an ellipse.
2) The shape of ellipse is shown.



3) The pt. $O(0,0)$ is called centre of the ellipse.
4) The pts. $F(ae,0)$ and $F'(-ae,0)$ are called Foci of the ellipse.
5) The pt $P(a\cos\theta, b\sin\theta)$ lies on the ellipse.
6) $\frac{|PS|}{|PF|}$ is called eccentricity and is denoted by e i.e $\frac{|PS|}{|PF|}=e$. For an ellipse $e<1$.
7) $x=a\cos\theta$, $y=b\sin\theta$ are called parameter equation of the ellipse.
8) $A(a,0)$ and $A'(-a,0)$ are called vertices or end pts. of the major axis.
9) The segment $\overline{AA'}$ is called major axis of the ellipse. The length of the major axis of the ellipse is $2a$ i.e $|\overline{AA'}|=2a$.
10) The line segment $\overline{OA}$ and $\overline{OA'}$ are called semi major axis. The length of the semi major axis is 'a' i.e $|OA|=|OA'|=a$.
11) The pt. $B(0,b)$ and $B'(0,-b)$ are called end pt. of the minor axis.
12) The equation $x=\frac{a}{e}$ and $x=-\frac{a}{e}$ are called directrices.
13) The lines $\overline{EG}$ and $\overline{KL}$ are called latus recta of the ellipse.

14) The lines segment $\overline{BB'}$ is called minor axis of the ellipse. The length of the minor axis is $2b$ i.e $\overline{BB'} = 2b$

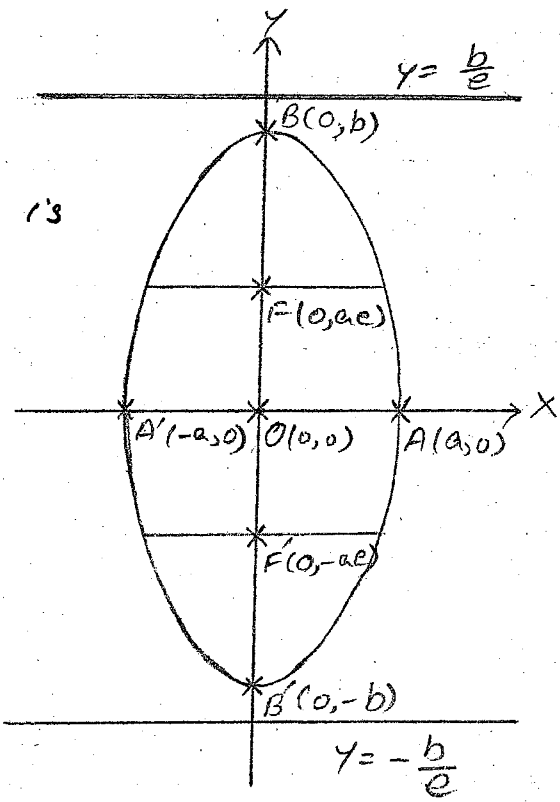
15) The line segment $\overline{OB}$ and $\overline{OB'}$ are called semi minor axis. The length of the semi minor axis is b i.e $|\overline{OB}| = |\overline{OB'}| = b$

16) Ellipse obeys the relation

$$a^2 = b^2 + a^2e^2$$

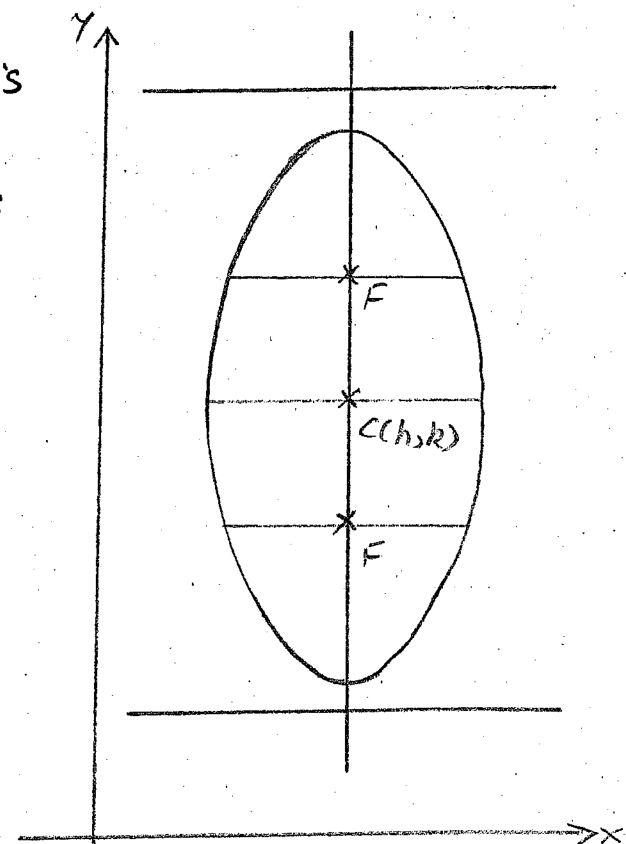
<u>Case 2.</u>

If the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; $a^2 < b^2$ then the shape of the ellipse is shown.



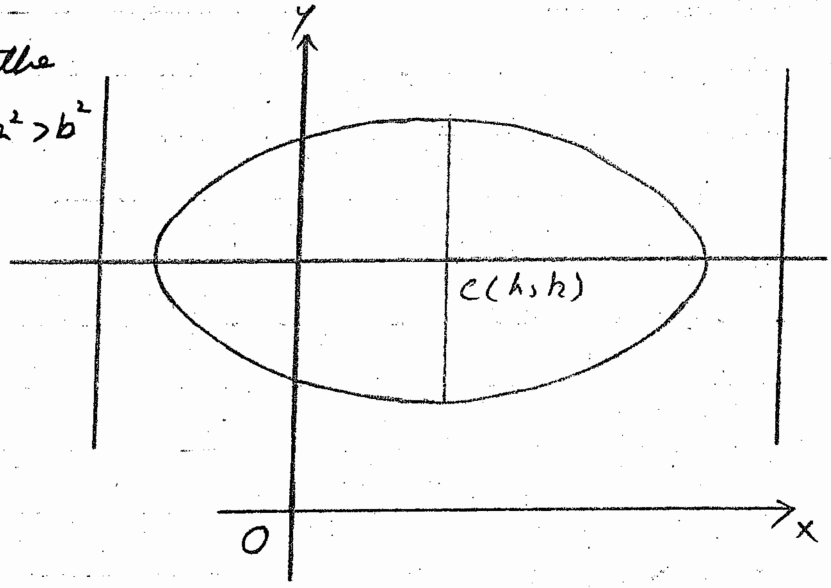
<u>Case 3.</u>

If the equation of the ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $a^2 < b^2$ then centre of the ellipse is $C(h,k)$ and the shape is shown.



Case 4.

If the equation of the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$; $a^2>b^2$ Then the centre of the ellipse is $C(h,k)$ and the shape is shown.



The Hyperbola.

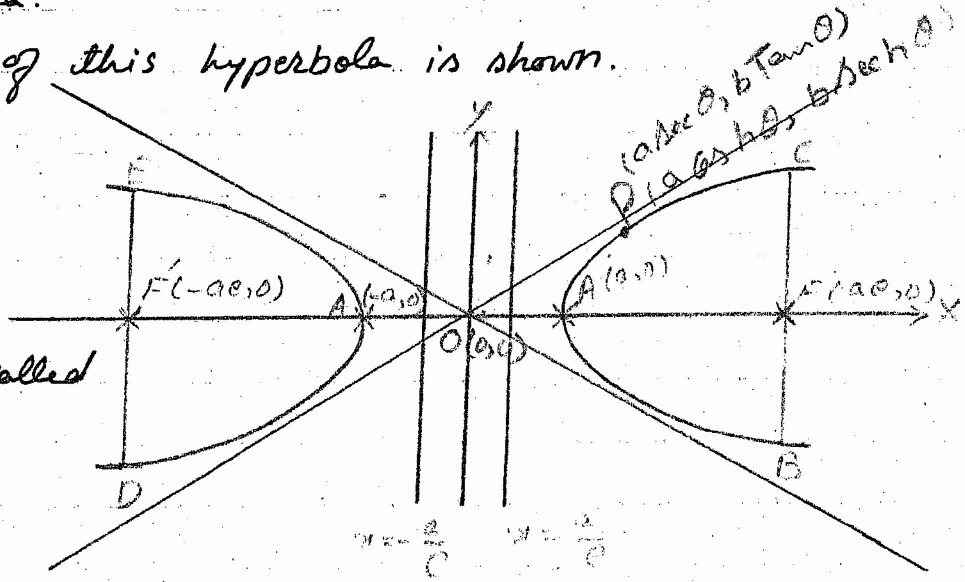
1) The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is called the equation of the hyperbola.

2) The shape of this hyperbola is shown.

3) $O(0,0)$ is called centre of the hyperbola.

4) $A(a,0)$ and $A'(-a,0)$ are called vertices of the hyperbola.

5) $F(ae,0)$ and $F'(-ae,0)$ (foci) are called foci of the hyperbola.



6) The line segments $\overline{BC}$ and $\overline{DE}$ are called latus recta of the hyperbola.

7) The $P(a\sec\theta,\ b\tan\theta)$ lies on the hyperbola.

8) The equation $x=a\sec\theta$, $y=b\tan\theta$ are called parametric equations of the hyperbola.

9) Similarly the pt. $(a\cosh\theta,\ b\,\mathrm{sech}\,\theta)$ lies on the hyperbola.

10) The equations $x=a\cosh\theta$, $y=b\,\mathrm{sech}\,\theta$ are called parametric equations of the Hyperbola.

11) Hyperbola obeys the relation

$$a^2e^2=a^2+b^2$$

12) The ratio of the distance of any pt. P from the directrix to the distance of P from corresponding focus is called eccentricity of the hyperbola and is denoted by e.

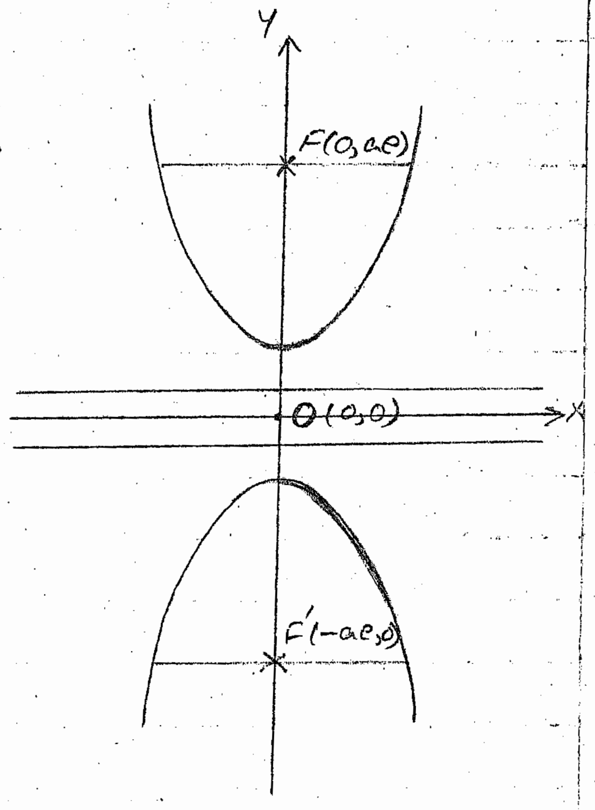
13) For hyperbola $e > 1$

14) The lines $x = \pm \frac{a}{e}$ are called directrices of the hyperbola.

15) The lines $y = \pm \frac{b}{a} x$ are called asymptotes of the hyperbola

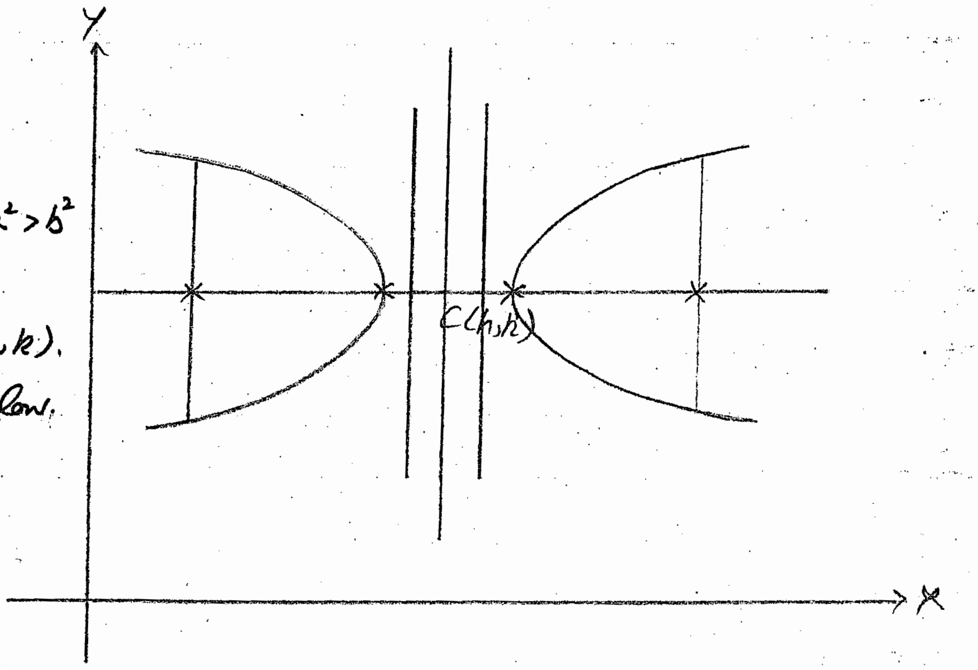
<u>Case 2.</u>

If the equation of the hyperbola is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ where $a^2 > b^2$.
Then the shape of the hyperbola is shown.



<u>Case 3.</u>

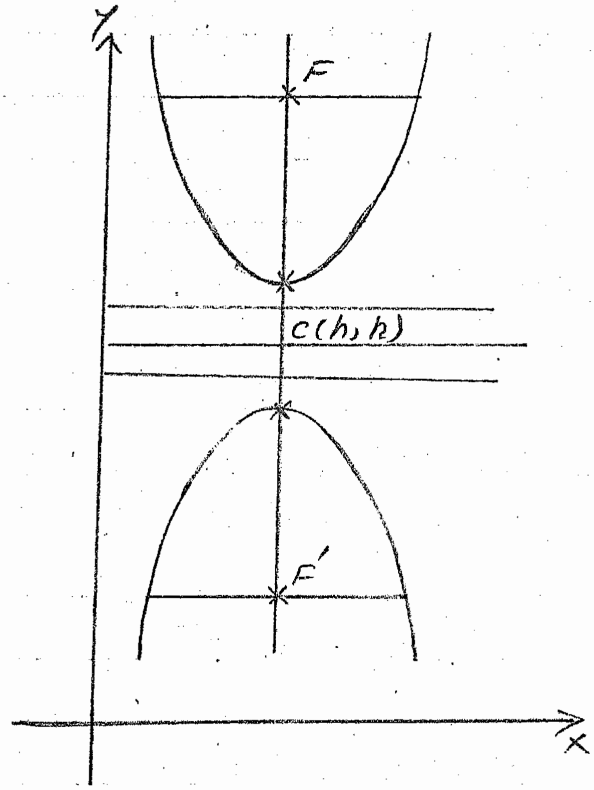
If the equation of the hyperbola is
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; $a^2 > b^2$
Then centre of the hyperbola is at (h,k).
shape is shown below.



If the equation of the hyperbola is

$$\frac{(x-h)^2}{b^2} - \frac{y-k}{a^2} = 1 \; ; \; a^2 > b^2$$

Then the centre of the hyperbola is at (h,k). The shape is shown.



Some Useful Results to Recognize the Conic:

i) The Conic is a parabola if $e=1$

ii) The Conic is an ellipse if $e<1$

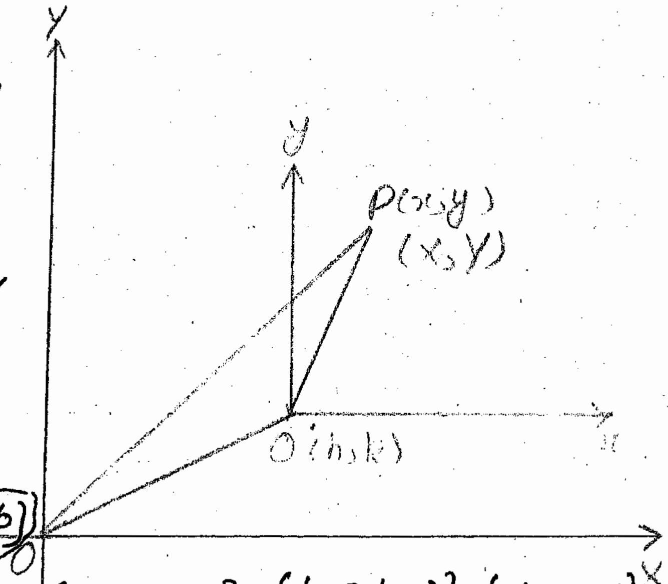
iii) The Conic is a hyperbola if $e>1$

Translation of axis.

Let (h,k) be the coordinates of O' w.r.t. 'O' and let (x,y) and (X,Y) be the coordinates of P with respect to old and new axes.

By Using vector addition.

$$\overrightarrow{OP} = \overrightarrow{OO'} + \overrightarrow{O'P}$$



~~$\Rightarrow [X-0, Y-0] = [h-0, k-0] + [x-0, y-0]$~~

~~$\Rightarrow [X, Y] = [h+x, k+y]$~~

~~$\Rightarrow X = h+x \; , \; Y = k+y$~~

~~$\Rightarrow X = h+x \; , \; Y = k+y$~~

$$\Rightarrow [x-0, y-0] = [h-0, k-0] + [X-0, Y-0]$$

$$\Rightarrow [x, y] = [h+X, k+Y]$$

$$\Rightarrow x = h+X \; , \; y = k+Y$$

$$\Rightarrow X = x-h \; , \; Y = y-k$$

Pair of Lines

The most general equation of 2nd degree in x,y is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{———} \ (1)$$

where a, b and h are not simultaneously 0.

Test for 2nd degree equation.

To represent a pair of st lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Homogeneous 2nd Degree equation

$$ax^2 + 2hxy + by^2 = 0 \quad \text{———} \ (1)$$

This eq. is known as homogenious 2nd degree eq.

Dividing (1) by x^2

$$a + 2h\frac{y}{x} + \frac{by^2}{x^2} = 0$$

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \quad \text{———} \ (2)$$

$\because$ It is Quadratic in $\frac{y}{x}$

$$\therefore \quad \frac{y}{x} = \frac{-2h \pm \sqrt{(2h)^2 - 4ab}}{2b}$$

$$\frac{y}{x} = \frac{-2h \pm 2\sqrt{h^2 - ab}}{2b}$$

$$\frac{y}{x} = \frac{-h \pm \sqrt{h^2 - ab}}{b}$$

We observe that the eq. can be converted into two straight lines.

Angle between the Straight Lines

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0 \qquad \text{from eq. ②}$$

Let $\frac{y}{x} = m$

Then $bm^2 + 2hm + a = 0$

Let m_1 and m_2 be two values of m

$$m_1 + m_2 = -\frac{2h}{b}$$

$$m_1 m_2 = \frac{a}{b}$$

$ax^2 + bx + c = 0$

Sum of roots, $S = -\frac{b}{a}$

Product of roots, $P = \frac{c}{a}$

Let 'θ' be the angle between the lines then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(-2h/b)^2 - 4(a/b)}}{1 + a/b}$$

$$= \frac{\sqrt{4h^2/b^2 - 4\,a/b}}{\frac{b+a}{b}}$$

$$= \frac{\frac{2\sqrt{h^2 - ab}}{b}}{\frac{b+a}{b}}$$

$$= \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$(a+b)^2 - 4ab = (a-b)^2$

Deduction.

For $\parallel$ lines $\theta = 0$

$$\frac{2\sqrt{h^2-ab}}{a+b} = 0$$

$$\Rightarrow \quad 2\sqrt{h^2-ab} = 0$$

$$\Rightarrow \quad h^2 - ab = 0$$

$$h^2 = ab$$

For perpendicular Lines $\theta = \frac{\pi}{2}$

$$\frac{2\sqrt{h^2-ab}}{a+b} = \infty$$

$$\Rightarrow \quad \frac{2\sqrt{h^2-ab}}{\infty} = a+b$$

$$\Rightarrow \quad 0 = a+b$$

e.g $\quad x^2 + 2xy - y^2 = 0$

Here $a = 1$, $b = -1$

$$a + b = 0$$

Some Useful Results to Recognize the Conic.

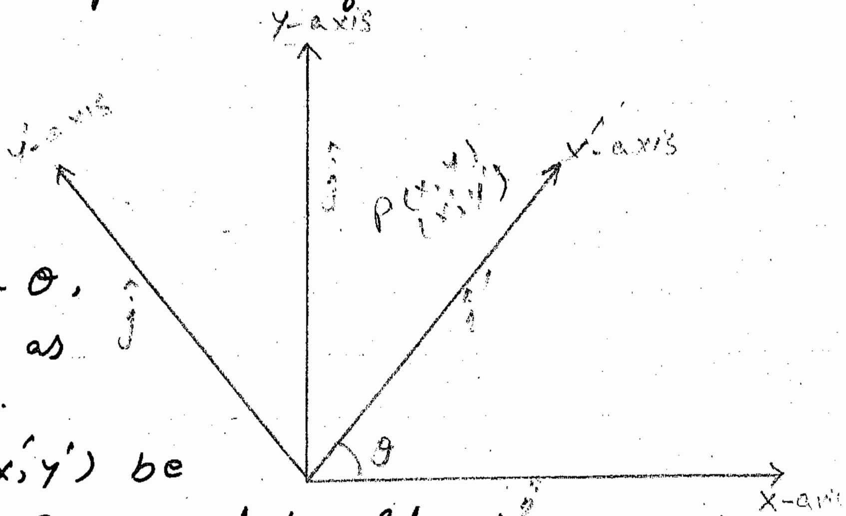
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The Conic is hyperbola if $h^2 - ab > 0$

The Conic is an ellipse if $h^2 - ab < 0$

The Conic is a parabola if $h^2 - ab = 0$

Rotation of Axes.



Let the axes (x-axis, y-axis) be rotated through an angle θ, name the new axes as x'-axis and y'axis.

Let (x, y) and (x', y') be the Coordinates of P refered to old and new axes.

Then $\quad \overline{OP} = x\hat{i} + y\hat{j}$ ——— ①

$\overline{OP'} = x'\hat{i}' + y'\hat{j}'$ ——— ②

$\because$ The unit vector $\hat{i}'$ makes an angle θ with the positive direction of x-axis.

$\therefore \quad \hat{i}' = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ ——— ③

Like wise

$$\hat{j}' = \cos(90+\theta)\,\hat{i} + \sin(90+\theta)\,\hat{j}$$

$$\hat{j}' = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \quad \text{——— ④}$$

③ and ④ in ②

$$\overline{OP} = x'[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}] + y'[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}]$$

$$\overline{OP} = x'\cos\theta\,\hat{i} + x'\sin\theta\,j + (-y'\sin\theta\,\hat{i}) + y'\cos\theta\,\hat{j}$$

$$\overline{OP} = (x'\cos\theta\,i - y'\sin\theta\,i) + (x'\sin\theta\,\hat{j} + y'\cos\theta\,j).$$

$$x\,i + y\,\hat{j} = (x'\cos\theta - y'\sin\theta)\,i + (x'\sin\theta + y'\cos\theta)\,j$$

Comparing Co-efficient of i and j

$$x = x'\cancel{\cos}\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

Where the matrix is called the Matrix of rotation.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

In order to remove the product term (x, y) from the equation of Conic we make use of the subtitution brought in due to rotation and apply the formula

$$\tan 2\theta = \frac{2h}{a-b}.$$

Asymptotes of a Hyperbola.

$$x^2/a^2 - y^2/b^2 = 1$$

An asymptote for the curve $y = f(x)$ is a straight line $y = g(x)$ such that

$$\lim_{\substack{x \to \infty \\ \text{or } x \to -\infty}} (f(x) - g(x)) = 0$$

Theorem.

The hyperbola $x^2/a^2 - y^2/b^2 = 1$ Lies asymptotes $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$.

Proof:

$$x^2/a^2 - y^2/b^2 = 1$$

$$\Rightarrow y^2/b^2 = x^2/a^2 - 1$$

$$\Rightarrow y^2/b^2 = \frac{x^2 - a^2}{a^2}$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

$$\Rightarrow y = \pm \frac{b}{a}(\sqrt{x^2 - a^2}) \quad \text{——— (1)}$$

from $\lim_{x \to \infty} (f(x) - g(x))$

$$\lim_{x \to \infty} \left(\frac{b}{a}\sqrt{x^2 - a^2} - \frac{b}{a}x \right)$$

given line $y = \frac{b}{a}x$

$$= \frac{b}{a}\left[\lim_{x \to \infty} \sqrt{x^2 - a^2} - x \right]$$

$$= \frac{b}{a}\left[\lim_{x \to \infty} \sqrt{x^2 - a^2} - x \right]$$

$$= \frac{b}{a}\left[\lim_{x \to \infty} \frac{(\sqrt{x^2 - a^2} - x)(\sqrt{x^2 - a^2} + x)}{(\sqrt{x^2 - a^2} + x)} \right]$$

$$= \frac{b}{a}\left[\lim_{x \to \infty} -\frac{a^2}{\sqrt{x^2 - a^2} + x} \right]$$

$$= \frac{b}{a}\left[-\frac{a^2}{\infty} \right] = 0$$

In Second quadrant from (1)

$$y = -\frac{b}{a}\sqrt{x^2-a^2} \qquad \because y = -\frac{b}{a}x$$

$$y = -\frac{b}{a}x \quad , \quad y = -\frac{b}{a}\sqrt{x^2-a^2}$$

Now $\lim\limits_{x \to -\infty}\left(-\frac{b}{a}\sqrt{x^2-a^2} + \frac{b}{a}x\right)$

$$= \frac{b}{a}\lim_{x \to -\infty}\left(x - \sqrt{x^2-a^2}\right)$$

$$= \frac{b}{a}\lim_{x \to -\infty}\left[\frac{(x-\sqrt{x^2-a^2})(x+\sqrt{x^2-a^2})}{x+\sqrt{x^2-a^2}}\right]$$

$$= \frac{b}{a}\lim_{x \to -\infty}\frac{a^2}{x+\sqrt{x^2-a^2}}$$

$$= \frac{b}{a}\,\frac{a^2}{\infty}$$

$$= 0$$

Hence the hyperbola $x^2/a^2 - y^2/b^2 = 1$ lies the asymptotes $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$.