Some useful results:-

Eq. of the Tangent at any point $P(x_1, y_1)$ to the parabola is

$$yy_1 = 2a(x + x_1)$$

Eq. of the Tangent at any point $P(x_1, y_1)$ to the ellipse is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

In Parabola $x = at^2, y = 2at$ is the parametric form and $(at^2, 2at) = T$

Exercise 6.2

Find equations of tangent and normal to each of the following curves at the indicated point (P-1 to 4)

Q # 1

$$y^2 = 4ax \quad \text{at } (a, -2a)$$

The given eq. is

$$y^2 = 4ax \quad \text{——— (1)}$$

Diff. w.r.t. 'x'

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

Now

$$\left.\frac{dy}{dx}\right|_{(a,-2a)} = \frac{2a}{-2a} = -1$$

Eq. of tangent at any point $P(x_1, y_1)$ is

$$y - y_1 = \left.\frac{dy}{dx}\right|_P (x - x_1)$$

Now

Eq. of tangent at $(a, -2a)$ is

$$(y + 2a) = -1(x - a)$$

$$y + 2a = -x + a$$

$$x + y + a = 0$$

Is the required eq. of Tangent.

Now Eq. of the normal at any point $P(x_1, y_1)$ is

$$y - y_1 = -\frac{dx}{dy}(x - x_1)$$

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Eq. of the normal at $(a, -2a)$ is

$$(y+2a) = -\frac{1}{-1}(x-a)$$

$$x - y - 3a = 0$$

Is the required eq. of the normal.

Q#2 $xy = c^2$ at $(ct, \frac{c}{t})$

The given eq. is

$$xy = c^2$$

Diff. w.r.t. 'x'

$$y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = -y \quad \Rightarrow \quad \frac{dy}{dx} = -y/x$$

Now $$\left.\frac{dy}{dx}\right|_{(ct, \frac{c}{t})} = -\frac{c/t}{ct} = -\frac{c}{t}\cdot\frac{1}{ct} = -\frac{1}{t^2}$$

Eq. of the tangent at $(ct, \frac{c}{t})$ is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - tc = -x + ct$$

$$x + yt^2 - 2ct = 0$$

$$x + yt^2 = 2ct$$

Eq. of Normal at $(ct, \frac{c}{t})$ is

$$y - \frac{c}{t} = t^2(x - ct)$$

$$yt - c = t^3x - ct^4$$

$$xt^3 - yt - ct^4 + c = 0$$

$$xt^2 - y = \frac{ct^4 - c}{t}$$

$$xt^3 - yt = c(t^4 - 1)$$

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Q.3. $x(x^2+y^2)-ay^2=0$ at $x=\frac{a}{2}$

The eq. is

$$x(x^2+y^2)-ay^2=0 \quad \text{———} (i)$$

$$x^3+xy^2-ay^2=0 \quad \text{———} (ii)$$

Diff. w.r.t. 'x'

$$3x^2+y^2+2xy\frac{dy}{dx}-2ay\frac{dy}{dx}=0$$

$$3x^2+y^2+(2xy-2ay)\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{3x^2+y^2}{2xy-2ay} \quad \text{———} (iii)$$

For value of y we put the value of x in eq.(ii)

$$\left(\frac{a}{2}\right)^3+\left(\frac{a}{2}\right)y^2-ay^2=0$$

$$\frac{a^3}{8}+\frac{a}{2}y^2-ay^2=0$$

$$\frac{a^3}{8}-ay^2/2=0$$

$$\frac{a^3}{8}=ay^2/2$$

$$y^2=\frac{a^2}{4}$$

$$\Rightarrow y=\pm\frac{a}{2}$$

$\Rightarrow$ we have to find the eq. of tangent and normal at $(a/2, a/2)$ and $(a/2, -a/2)$.

Now we find the eq. of Tangent and Normal at $(a/2, a/2)$

$$(iii) \Rightarrow \quad \left.\frac{dy}{dx}\right|_{(a/2,\, a/2)} = -\frac{3(a/2)^2+(a/2)^2}{2(a/2)(a/2)-2a(a/2)}$$

$$= -\frac{3a^2/4+a^2/4}{a^2/2-a^2}$$

$$= -\frac{\frac{3a^2+a^2}{4}}{\frac{a^2-2a^2}{2}} = -\frac{4a^2/4}{-a^2/2}$$

$$= a^2\times\frac{2}{a^2} = 2$$

Eq. of Tangent :- $y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$

$$(y - a/2) = 2(x - a/2)$$

$$y - a/2 = 2x - a$$

$$\Rightarrow \quad 2x - y + a/2 - a = 0$$

$$\Rightarrow \quad 2x - y - a/2 = 0$$

$$\Rightarrow \quad 4x - 2y - a = 0$$

Is the required Eq. of Tangent.

Eq. of Normal :- $y - y_1 = -\left(\frac{dx}{dy}\right)_P (x - x_1)$

$$y - a/2 = -\frac{1}{2}(x - a/2)$$

$$2y - a = -x + a/2$$

$$x + 2y - \frac{3a}{2} = 0$$

$$2x + 4y - 3a = 0$$

Is the required Eq. of Normal.

Eq. of Tangent and Normal at $(a/2, -a/2)$

iii) $\Rightarrow$

$$\left(\frac{dy}{dx}\right)_{(a/2, -a/2)} = -\frac{3(a/2)^2 + (-a/2)^2}{2(a/2)(-a/2) - 2a(-a/2)}$$

$$= -\frac{\frac{3a^2 + a^2}{4}}{-a^2/2 + a^2} = -\frac{\frac{4a^2}{4}}{\frac{-a^2 + 2a^2}{2}}$$

$$= -a^2 \times \frac{2}{a^2}$$

$$= -2$$

Eq. of Tangent: $y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$

$$(y + a/2) = -2(x - a/2)$$

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$\times$ by (2)

$$2y + a = -4x + 2a$$

$$4x + 2y - a = 0$$

Hence the Eq. of Tangent at $(a/2, -a/2)$ is

$$4x + 2y - a = 0$$

Eq. of Normal: $y - y_1 = -\dfrac{1}{\left(\frac{dy}{dx}\right)_p}(x - x_1)$

$$(y + a/2) = 2(x - a/2)$$

$$y + a/2 = 2x - a$$

$$2x - y - a - a/2 = 0$$

$$2x - y - 3a/2 = 0$$

$$4x - 2y = 3a$$

Is the required Eq. of Normal.

Q#4. $c^2(x^2 + y^2) = x^2y^2$ at $\left(\dfrac{c}{\cos\theta}, \dfrac{c}{\sin\theta}\right)$

$$c^2x^2 + c^2y^2 = x^2y^2$$

Dividing both sides by $x^2 \cdot y^2 \cdot c^2$, we have

$$\frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{c^2}$$

$$x^{-2} + y^{-2} = c^{-2} \quad \text{at} \quad (c/\cos\theta, c/\sin\theta) \quad \text{—— (1)}$$

Diff. w.r.t. x

$$-2x^{-3} - 2y^{-3}\frac{dy}{dx} = 0 \Rightarrow -x^{-3} = y^{-3}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-3}}{y^{-3}} = -\frac{y^3}{x^3}$$

$$\left(\frac{dy}{dx}\right)_{(c/\cos\theta,\, c/\sin\theta)} = -\frac{(c/\sin\theta)^3}{(c/\cos\theta)^3} = -\frac{c^3}{\sin^3\theta} \times \frac{\cos^3\theta}{c^3}$$

$$(\text{say}) \quad m = -\frac{\cos^3\theta}{\sin^3\theta}$$

Eq. of the tangent: $y - y_1 = m(x - x_1)$

$$y - \frac{c}{\sin\theta} = -\frac{\cos^3\theta}{\sin^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

$$y\sin^3\theta - c\sin^2\theta = -\cos^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

$$y\sin^3\theta - c\sin^2\theta = c\cos^2\theta - x\cos^3\theta$$
$$x\cos^3\theta + y\sin^3\theta = c(\cos^2\theta + \sin^2\theta)$$
$$x\cos^3\theta + y\sin^3\theta = c$$

Is the required Eq. of Tangent.

Eq. of Normal: $y - y_1 = -\frac{1}{m}(x - x_1)$

$$\left(y - \frac{c}{\sin\theta}\right) = \frac{\sin^3\theta}{\cos^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

$$y\cos^3\theta - \frac{c\cos^3\theta}{\sin\theta} = \sin^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

$$y\cos^3\theta - \frac{c\cos^3\theta}{\sin\theta} = x\sin^3\theta - \frac{c\sin^3\theta}{\cos\theta}$$

$$x\sin^3\theta - y\cos^3\theta = \frac{c\sin^3\theta}{\cos\theta} - \frac{c\cos^3\theta}{\sin\theta}$$

$$x\sin^3\theta - y\cos^3\theta = \frac{c\sin^4\theta - c\cos^4\theta}{\cos\theta\sin\theta} \quad *$$

$$x\sin^3\theta - y\cos^3\theta = -\frac{2c\cos 2\theta}{2\sin\theta\cos\theta}$$

$$x\sin^3\theta - y\cos^3\theta = -\frac{2c\cos 2\theta}{\sin 2\theta} = -2c\cot 2\theta$$

$$x\sin^3\theta - y\cos^3\theta + 2c\cot\theta = 0$$

Is the required eq. of Normal.

$*$
$$\frac{c\sin^4\theta - c\cos^4\theta}{\cos\theta \cdot \sin\theta} = \frac{c(\sin^4\theta - \cos^4\theta)}{\cos\theta\sin\theta}$$

$$= \frac{c\left[(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)\right]}{\cos\theta\sin\theta}$$

$$= \frac{c(\sin^2\theta - \cos^2\theta)}{\cos\theta\sin\theta}$$

$$= -\frac{c(\cos\theta\cos\theta - \sin\theta\sin\theta)}{\cos\theta\sin\theta} = \frac{c\cos(\theta+\theta)}{\cos\theta\sin\theta}$$

$$= -\frac{2c\cos 2\theta}{2\cos\theta\sin\theta} = -\frac{2c\cos 2\theta}{\sin 2\theta} = -2c\cot 2\theta$$

Written by
Shahid Javed

Find the points where the tangent is // to the x-axis and where it is // to the y-axis for each of the given curves. (P5 - 7):

Q#5.
$$x^3 + y^3 = a^3 \quad \text{——— (i)}$$

Diff. w.r.t. 'x'

$$3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x^2}{y^2} \quad \text{——— (ii)}$$

If the Tangent is // to x-axis then $\frac{dy}{dx} = 0$

$$\Rightarrow \quad -\frac{x^2}{y^2} = 0$$

$$\Rightarrow \quad x = 0$$

Put in (i)

$$y^3 = a^3$$

$$\Rightarrow y = a$$

Hence the tangent is // to x-axis at $(0, a)$

Now if the tangent is // to y-axis then $\frac{dy}{dx} = \infty$

$$\Rightarrow \quad -\frac{x^2}{y^2} = \infty$$

$$\Rightarrow \quad y^2 = 0$$

$$\Rightarrow \quad y = 0$$

Put in (i)

$$x^3 = a^3$$

$$\Rightarrow \quad x = a$$

Hence the tangent will // to y-axis at $(a, 0)$

(i) #6. $x^3+y^3=3axy$ ——— (i)

Diff. w.r.t. 'x'

$$3x^2+3y^2\frac{dy}{dx}=3a\left[y+x\frac{dy}{dx}\right]$$

$$3x^2+3y^2\frac{dy}{dx}=3ay+3ax\frac{dy}{dx}$$

$$(3y^2-3ax)\frac{dy}{dx}=3ay-3x^2$$

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$$

For Tangent Parallel to x-axis

$$\frac{dy}{dx}=0$$

$$\frac{ay-x^2}{y^2-ax}=0$$

$$ay-x^2=0$$

$$ay=x^2$$

$$y=x^2/a \quad \text{——— (ii)}$$

Put in (i)

$$x^3+\left(\frac{x^2}{a}\right)^3=3ax\cdot\frac{x^2}{a}$$

$$x^3+\frac{x^6}{a^3}=3x^3$$

$$\frac{x^6}{a^3}=2x^3$$

$$x^3=2a^3$$

$$x=2^{1/3}a$$

Put in (ii)

$$y=\frac{(2^{1/3})^2a^2}{a}$$

$$y=(2)^{2/3}a$$

$\therefore$ The tangent is parallel to x-axis at $(2^{1/3}a,\ 4^{1/3}a)$

Now for tangent Parallel to y-axis.

$$\frac{dy}{dx} = \infty$$

$$\frac{ay - x^2}{y^2 - ax} = \infty$$

$$\Rightarrow \quad y^2 - ax = 0$$

$$\Rightarrow \quad y^2 = ax$$

$$\Rightarrow \quad x = y^2/a \quad \text{———} \quad (iii)$$

Put in (i)

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\,\frac{y^2}{a}\cdot y$$

$$\frac{y^6}{a^3} + y^3 = 3y^3 \Rightarrow \quad \frac{y^6}{a^3} = 2y^3$$

$$\Rightarrow \quad y^3 = 2a^3$$

$$\Rightarrow \quad y = 2^{1/3}.a \quad \text{———} \quad (iv)$$

put in (iii)

$$\Rightarrow \quad x = \frac{(2^{1/3}.a)^2}{a}$$

$$x = 4^{1/3}.a$$

Hence tangent will parallel to y-axis at $(4^{1/3}a, 2^{1/3}a)$

Q#7. $$25x^2 + 12xy + 4y^2 = 1 \quad \text{———} \quad (i)$$

Diff. w.r.t 'x'

$$50x + 12y + 12x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$(12x + 8y)\frac{dy}{dx} = -\frac{50x + 12y}{1}$$

$$\frac{dy}{dx} = -\frac{50x + 12y}{12x + 8y}$$

The tangent will parallel to x-axis if

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad -\frac{50x + 12y}{12x + 8y} = 0 \Rightarrow \quad 50x + 12y = 0$$

$$\Rightarrow \quad 25x + 6y = 0$$

$$x = -\frac{6}{25}y \quad \text{——— (ii)}$$

Put in (i)

$$25\left(-\frac{6}{25}y\right)^2 + 12\left(-\frac{6}{25}y\right)y + 4y^2 = 1$$

$$\frac{36}{25}y^2 - \frac{72}{25}y^2 + 4y^2 = 1$$

$$y^2\left(\frac{36 - 72 + 100}{25}\right) = 1$$

$$y^2\left(\frac{64}{25}\right) = 1$$

$$y^2 = \frac{25}{64}$$

$$y = \pm\frac{5}{8} \quad \text{——— (iii)}$$

Put in (ii)

$$x = -\frac{6}{25}\left(\frac{5}{8}\right) \quad , \quad x = -\frac{6}{25}\left(-\frac{5}{8}\right)$$

$$x = -\frac{3}{20} \quad , \quad x = \frac{3}{20}$$

Hence the tangent is parallel to x-axis at $(-3/20, 5/8)$ and $(3/20, -5/8)$.

Now the tangent will parallel to y-axis if.

$$\frac{dy}{dx} = \infty$$

$$-\frac{50x + 12y}{12x + 8y} = \infty$$

$$\Rightarrow \quad 12x + 8y = 0$$

$$3x + 2y = 0$$

$$y = -\frac{3x}{2} \quad \text{——— (iv)}$$

Put in (i)

$$25x^2 + 12x\left(-\frac{3x}{2}\right) + 4\left(-\frac{3}{2}x\right)^2 = 1$$

Written by
Shahid Javed

$$25x^2 - 18x^2 + 9x^2 = 1$$
$$16x^2 = 1$$
$$x^2 = \frac{1}{16}$$

$$x = \pm \frac{1}{4} \quad \text{——— (v)}$$

Put in (iv)

$$y = -\frac{3}{2}\left(\frac{1}{4}\right) = -\frac{3}{8}$$

$$y = -\frac{3}{2}\left(-\frac{1}{4}\right) = \frac{3}{8}$$

Hence the tangent is parallel to y-axis at
$\left(\frac{1}{4}, -3/8\right)$, $\left(-1/4, 3/8\right)$

Q#8. If $P = x\cos\theta + y\sin\theta$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$$

Prove that $p^n = (a\cos\theta)^n + (b\sin\theta)^n$.

Proof:-

The given eq. is $\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$ ——— (i)

Diff. w.r.t. 'x'

$$\frac{n}{n-1}\left(\frac{x}{a}\right)^{\frac{n}{n-1}-1} \cdot \frac{1}{a} + \frac{n}{n-1}\left(\frac{y}{b}\right)^{\frac{n}{n-1}-1} \cdot \frac{1}{b}\frac{dy}{dx} = 0$$

$$\frac{n}{n-1}\left[\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{n-n+1}{n-1}} + \frac{1}{b}\left(\frac{y}{b}\right)^{\frac{n-n+1}{n-1}}\frac{dy}{dx}\right] = 0$$

$$\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}} + \frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}\frac{dy}{dx} = 0$$

$$\frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}\frac{dy}{dx} = -\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}$$

$$\frac{dy}{dx} = -\frac{\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}}{\frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}}$$

Let $P(x_1, y_1)$ be any point on the curve.

Then

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}{\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}}$$

Thus the eq. of the tangent at $P(x_1, y_1)$ is given by

$$y - y_1 = \frac{dy}{dx}\Big|_P (x - x_1)$$

$$\Rightarrow y - y_1 = -\frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}{\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}}(x - x_1)$$

Multiplying by $\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}$

$$\Rightarrow \left(\frac{y}{b}\right)\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}} - \left(\frac{y_1}{b}\right)\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}} = -\left(\frac{x}{a}\right)\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} + \left(\frac{x_1}{a}\right)\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}$$

$$\left(\frac{y}{b}\right)\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}} - \left(\frac{y_1}{b}\right)^{\frac{n}{n+1}} = -\left(\frac{x}{a}\right)\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} + \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}}$$

$$\Rightarrow \frac{x}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} + \frac{y}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}} = \left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}$$

$$\frac{x}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} + \frac{y}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}} = 1 \quad \text{———} \quad (2) \qquad \text{from } (1)$$

But $\quad p = x\cos\theta + y\sin\theta \quad$ ——— (3)

Comparing (2) and (3)

Comparing form for $a_1x_1 + b_1y_1 + c_1 = 0$

$a_2x_2 + b_2y_1 + c_2 = 0$

i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}{\cos\theta} = \frac{\frac{1}{b}\left(\frac{y_1}{b}\right)^{\frac{1}{n-1}}}{\sin\theta} = \frac{1}{p} \quad \text{———} \quad (4)$$

$$\Rightarrow \frac{\frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}}}{\cos\theta} = \frac{1}{p} \Rightarrow \frac{1}{a}\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} = \frac{\cos\theta}{p}$$

$$\Rightarrow p\left(\frac{x_1}{a}\right)^{\frac{1}{n-1}} = a\cos\theta$$

$$p^n\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} = (a\cos\theta)^n \quad \text{———} \quad (5)$$

and Similarly

$$p^n\left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = (b\sin\theta)^n \quad \text{———} \quad (6)$$

$(5)+(6) \Rightarrow$

$$p^n\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + p^n\left(\frac{y_1}{b}\right)^{\frac{n}{n-1}} = (a\cos\theta)^n + (b\sin\theta)^n$$

$$p^n\left[\left(\frac{x_1}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y_1}{b}\right)^{\frac{n}{n-1}}\right] = (a\cos\theta)^n + (b\sin\theta)^n$$

$$p^n(1) = (a\cos\theta)^n + (b\sin\theta)^n$$

$$\Rightarrow \quad p^n = (a\cos\theta)^n + (b\sin\theta)^n$$

Is as required.

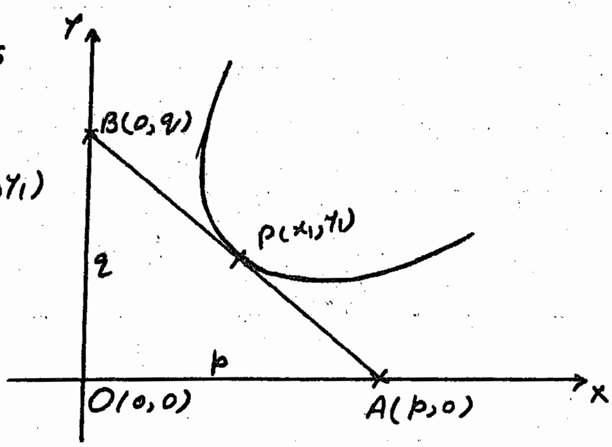
Q#9. The tangent at any point on the curve $x^3+y^3=2a^3$ makes intercepts p and q on the coordinate axes. Show that

$$p^{-3/2} + q^{-3/2} = 2^{-1/2}a^{-3/2}$$

Soln.

The given eq. of the curve is

$$x^3+y^3 = 2a^3 \quad \text{------} \quad (A)$$



Eq. of the tangent at any pt $P(x_1,y_1)$ will be

$$x_1^2 x + y_1^2 y = 2a^3$$

$\because$ $A(p,0)$ lies on the tangent

$$\therefore \quad p x_1^2 + 0 = 2a^3$$

$$x_1^2 = \frac{2a^3}{p}$$

$$x_1 = \frac{2^{1/2}a^{3/2}}{p^{1/2}}$$

$$x_1^3 = \frac{2^{3/2}a^{9/2}}{p^{3/2}} \quad \text{------} \quad (i)$$

Also $B(0,q)$ lies on the tangent

$$\therefore \quad 0 + y_1^2 q = 2a^3$$

$$y_1^2 = \frac{2a^3}{q} \Rightarrow y_1 = \frac{2^{1/2}a^{3/2}}{q^{1/2}}$$

$$\Rightarrow \quad y_1^3 = \frac{2^{3/2}a^{9/2}}{q^{3/2}} \quad \text{------} \quad (ii)$$

(i) + (ii) $\Rightarrow$

$$x_1^3+y_1^3 = \frac{2^{3/2}a^{9/2}}{p^{3/2}} + \frac{2^{3/2}a^{9/2}}{q^{3/2}}$$

$$2a^3 = 2^{3/2}a^{9/2}\left(1/p^{3/2} + 1/q^{+3/2}\right) \qquad \text{from (A)}$$

$$\Rightarrow \quad p^{-3/2}+q^{-3/2} = 2^{-1/2}a^{-3/2}$$

As required.

<u>Alternative Method</u>

The given curve is

$$x^3+y^3 = 2a^3 \quad \text{———— (i)}$$

Eq. of the tangent on the curve at any point $P(x_1, y_1)$ is

$$x.x_1^2 + y.y_1^2 = 2a^3 \quad \text{———— (ii)}$$

<u>Interception with x-axis</u>:

When the curve intercept with x-axis then $y=0$

(ii) $\Rightarrow$ $x.x_1^2 = 2a^3$

$\Rightarrow$ $x = \dfrac{2a^3}{x_1^2}$

$\because$ The curve makes intercept with x-axis at p

$\therefore$ $p = \dfrac{2a^3}{x_1^2} \Rightarrow x_1 = \dfrac{2^{1/2}a^{3/2}}{p^{1/2}}$

<u>Interception with y-axis.</u>

when the curve intercept with y-axis then $x=0$

(ii) $\Rightarrow$ $y.y_1^2 = 2a^3$

$y = \dfrac{2a^3}{y_1^2}$

$\because$ the curve makes intercept with y-axis at q

$\therefore$ $q = \dfrac{2a^3}{y_1^2} \Rightarrow y_1 = \dfrac{2^{1/2}a^{3/2}}{q^{1/2}}$

$\because$ $P(x_1, y_1)$ lies on the curve

$\therefore$ $x_1^3 + y_1^3 = 2a^3$

$$\left(\frac{2^{1/2}a^{3/2}}{p^{1/2}}\right)^3 + \left(\frac{2^{1/2}a^{3/2}}{q^{1/2}}\right)^3 = 2a^3$$

$$\frac{2^{3/2}a^{9/2}}{p^{3/2}} + \frac{2^{3/2}a^{9/2}}{q^{3/2}} = 2a^3$$

$$\Rightarrow \quad p^{-3/2} + q^{-3/2} = \frac{2a^3}{2^{3/2}a^{9/2}}$$

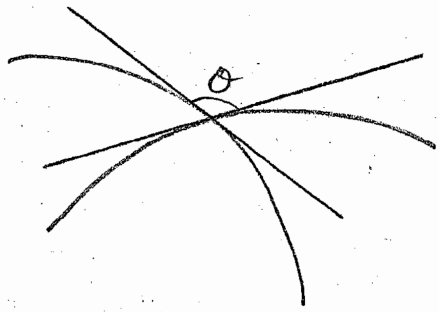
$$\Rightarrow \quad p^{-3/2} + q^{-3/2} = 2^{-1/2}a^{-3/2}$$

As required.

Angle of intersection of two curves: OR

Angle between the two curves:-

Def:- Angle between two curves at their common point of x^n is defined as the angle between the tangents at this point.



Find the angle of intersection of the given curves (10-12)

#10 The parabolas $y^2 = 4ax$ and $x^2 = 4by$ at the point other than $(0,0)$.

The given curves are

$$y^2 = 4ax \quad \text{———} \quad (i)$$

$$x^2 = 4by \quad \text{———} \quad (ii)$$

$$i) \Rightarrow \quad x = \frac{y^2}{4a} \quad \text{———} \quad (iii)$$

Put in (ii)

$$\left(\frac{y^2}{4a}\right)^2 = 4by$$

$$\frac{y^4}{16a^2} = 4by \Rightarrow y^4 = 64a^2by$$

$$y^4 - 64a^2by = 0 \Rightarrow y(y^3 - 64a^2b) = 0$$

$\Rightarrow \quad y = 0 \quad$ or $\quad y^3 - 64a^2b = 0 \Rightarrow y^3 = 64a^2b$

$$y = 4a^{2/3}b^{1/3}$$

$\Rightarrow \quad y = 0, \quad y = 4a^{2/3}b^{1/3}$

Case I. if $y = 0$ put in (iii)

$$\Rightarrow \quad x = 0$$

$\Rightarrow$ $(0,0)$ is the point of intersection.

Case II. if $y = 4a^{2/3}b^{1/3}$

$$\Rightarrow \quad x = (4a^{2/3}b^{1/3})^2/4a$$
$$= 16a^{4/3}b^{2/3}/4a$$
$$= 4a^{1/3}b^{2/3}$$

$\Rightarrow$ $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$ is the point of intersection.

By the given condition we find the angle of intersection at $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$.

Diff. (i) w.r.t. 'x'

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$m_1 = \left.\frac{dy}{dx}\right|_P = \frac{2a}{4a^{2/3}b^{1/3}}$$

$$m_1 = \frac{a^{1/3}}{2b^{1/3}}$$

Diff. (ii) w.r.t. 'x'

$$2x = 4b\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2b}$$

$$m_2 = \left.\frac{dy}{dx}\right|_P = \frac{4a^{1/3}b^{2/3}}{2b}$$

$$m_2 = \frac{2a^{1/3}}{b^{1/3}}$$

Let θ be the angle between the tangents at $(4a^{1/3}b^{2/3}, 4a^{2/3}, b^{1/3})$

Then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$$

$$\tan\theta = \left|\frac{\dfrac{a^{1/3}}{2b^{1/3}} - \dfrac{2a^{1/3}}{b^{1/3}}}{1 + \dfrac{a^{1/3}/2b^{1/3}}{} \cdot \dfrac{2a^{1/3}}{b^{1/3}}}\right|$$

Written by
Shahid Javed

$$\tan\theta = \left| \frac{\dfrac{a^{1/3} - 4a^{1/3}}{2b^{1/3}}}{1 + \dfrac{2a^{2/3}}{2b^{2/3}}} \right|$$

$$\tan\theta = \left| \frac{-\dfrac{3a^{1/3}}{2b^{1/3}}}{\dfrac{2b^{2/3} + 2a^{2/3}}{2b^{2/3}}} \right|$$

$$\tan\theta = \frac{3a^{1/3}}{2b^{1/3}} \cdot \frac{2b^{2/3}}{2b^{2/3} + 2a^{2/3}}$$

$$\tan\theta = \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}$$

$$\Rightarrow \theta = \tan^{-1}\left\{ \frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})} \right\}$$

Q#11 $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$

We have to find the angle of intersection between the curves

$$x^2 - y^2 = a^2 \quad \text{——— (i)}$$

$$x^2 + y^2 = a^2\sqrt{2} \quad \text{——— (ii)}$$

Let $p(x,y)$ be the pt. of intersection of (i) and (ii)

Diff. (i) w.r.t. 'x'

$$2x - 2y\frac{dy}{dx} = 0$$

$$m_1 = \frac{dy}{dx} = \frac{x}{y}$$

Diff. (ii) w.r.t. 'x'

$$2x + 2y\frac{dy}{dx} = 0$$

$$m_2 = \frac{dy}{dx} = -\frac{x}{y}$$

Let θ be the angle of intersection of (i) and (ii) then

$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\tan\theta = \left| \frac{x/y + x/y}{1 - x^2/y^2} \right|$$

$$\tan\theta = \frac{2\,x/y}{\frac{y^2-x^2}{y^2}} = \frac{2xy}{y^2-x^2} \quad \text{———— (A)}$$

Now we find the coordinate (x, y) from (i) and (ii)

$$\text{i)} \Rightarrow \quad x^2 - y^2 = a^2 \Rightarrow x^2 = y^2 + a^2 \quad \text{———— (iii)}$$

(iii) Put in (ii)

$$y^2 + a^2 + y^2 = a^2\sqrt{2}$$

$$a^2 + 2y^2 = a^2\sqrt{2}$$

$$2y^2 = a^2(\sqrt{2}-1)$$

$$y^2 = \frac{a^2}{2}(\sqrt{2}-1) \quad \text{———— (iv)}$$

Put (iv) in (iii)

$$x^2 = \frac{a^2}{2}(\sqrt{2}-1) + a^2$$

$$x^2 = \frac{a^2}{2}(\sqrt{2}-1+2)$$

$$x^2 = \frac{a^2}{2}(\sqrt{2}+1) \quad \text{———— (v)}$$

$$(iv) \times (v) \Rightarrow \quad x^2y^2 = \frac{a^2}{2}(\sqrt{2}+1) \cdot \frac{a^2}{2}(\sqrt{2}-1)$$

$$x^2y^2 = \frac{a^4}{4}(2-1) = \frac{a^4}{4}$$

$$xy = \pm\left(\frac{a^2}{2}\right) \quad \text{———— (vi)}$$

Now $\quad$ i) $\Rightarrow \quad x^2 - y^2 = a^2$

$$y^2 - x^2 = -a^2 \quad \text{———— (vii)}$$

Put (vi) and (vii) in (A)

$$\Rightarrow \quad \tan\theta = \frac{2(\pm a^2/2)}{-a^2}$$

$$= \frac{\pm a^2}{-a^2}$$

$$= \frac{+a^2}{-a^2}, \quad \frac{-a^2}{-a^2}$$

$\Rightarrow \quad \tan\theta = -1 \quad$ and $\quad \tan\theta = 1$

$\Rightarrow \quad \theta = \tan^{-1}(\pm 1)$

$\Rightarrow \quad \theta = 45^{\circ}$

Q#12. $\quad y^2 = ax$ and $x^3 + y^3 = 3axy$

Here $\quad y^2 = ax \quad$ —— (i)

$x^3 + y^3 = 3axy \quad$ —— (ii)

i) $\Rightarrow$

$$x = \frac{y^2}{a} \quad \text{—— (iii)}$$

(iii) in (ii) $\Rightarrow$

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right)(y)$$

$$\frac{y^6}{a^3} + y^3 = 3y^3$$

$$\frac{y^6}{a^3} = 2y^3$$

$$y^6 = 2a^3y^3$$

$$y^6 = 2a^3y^3$$

$$y^6 - 2a^3y^3 = 0$$

$$y^3(y^3 - 2a^3) = 0$$

$\Rightarrow \quad y = 0 \quad , \quad y^3 = 2a^3$

$\quad\quad y = 0 \quad\quad y = 2^{1/3}a$

Two cases arise here.

Case I $\quad$ if $y = 0$ put in (iii)

$$x = 0$$

$\Rightarrow$ The point of intersection is $(0, 0)$.

Case II $\quad$ if $\quad y = 2^{1/3}a \quad$ put in (iii)

$$x = \frac{(2^{1/3}a)^2}{a}$$

$$x = \frac{2^{2/3}a^2}{a}$$

$$x = 2^{2/3}a$$

$\Rightarrow$ The point of intersection is $(2^{4/3}a, 2^{1/3}a)$

First we find the angle of intersection at $(0,0)$

Diff. (i) w.r.t. 'x'

$$2y\frac{dy}{dx} = a$$

$$\frac{dy}{dx} = \frac{a}{2y} \quad \text{------ (iv)}$$

$$m_1 = \left.\frac{dy}{dx}\right|_{(0,0)} = \infty$$

$$m_1 = \infty$$

Diff. (ii) w.r.t. 'x'

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

$$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \quad \text{------ (v)}$$

$$m_2 = \left.\frac{dy}{dx}\right|_{(0,0)} = \frac{0}{0}$$

i.e. The angle of intersection is undefined at $(0,0)$.
Now we find the angle of intersection at $(2^{2/3}a, 2^{1/3}a)$.

$$(iv) \Rightarrow \quad m_1 = \left.\frac{dy}{dx}\right|_P = \frac{a}{2\cdot 2^{1/3}a} = \frac{1}{2^{4/3}}$$

$$(v) \Rightarrow \quad m_2 = \left.\frac{dy}{dx}\right|_P = \frac{a(2^{1/3}a) - (2^{2/3}a)^2}{(2^{1/3}a)^2 - a(2^{2/3}a)}$$

$$= \infty$$

Let θ be the angle of intersection.

then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\tan\theta = \frac{m_2\left[\left(\frac{m_1}{m_2}\right)-1\right]}{m_2\left(\frac{1}{m_2}+m_1\right)}$$

$$= \frac{m_1/\infty - 1}{1/\infty + \frac{1}{2^{4/3}}}$$

$$\tan\theta = -\frac{1}{\frac{1}{2^{4/3}}} = -2^{4/3}$$

$$\theta = \tan^{-1}(-2^{4/3})$$

$$\theta = 68^\circ 21'$$

Is the required angle of intersection.

Q#13. Find the condition that the curves $ax^2+by^2=1$ and $a_1x^2+b_1y^2=1$ should intersect orthogonally.

Here $ax^2+by^2=1$ ——— (i)

$a_1x^2+b_1y^2=1$ ——— (ii)

Diff. (i) w.r.t. 'x'

$$2ax + 2by\frac{dy}{dx} = 0$$

$$m_1 = \frac{dy}{dx} = -\frac{ax}{by}$$

Diff. (ii) w.r.t. 'x'

$$2a_1x + 2b_1y\frac{dy}{dx} = 0$$

$$m_2 = \frac{dy}{dx} = -\frac{a_1x}{b_1y}$$

$\because$ the curves cut orthogonally

$$\therefore \quad m_1 \cdot m_2 = -1$$

$$-\frac{ax}{by}\cdot -\frac{a_1x}{b_1y} = -1$$

$$\frac{aa_1x^2}{bb_1y^2} = -1 \quad \text{——— (iii)}$$

(i) − (ii) $\Rightarrow$ $(a-a_1)x^2 + (b-b_1)y^2 = 0$

$$\frac{x^2}{y^2} = -\frac{(b-b_1)}{(a-a_1)} \quad \text{——— (iv)}$$

Put in (iii)

$$+\frac{aa_1}{bb_1}\cdot -\frac{(b-b_1)}{a-a_1} = -1$$

$$\frac{a a_1}{a-a_1} = \frac{b b_1}{b-b_1}$$

$$\Rightarrow \frac{a-a_1}{a a_1} = \frac{b-b_1}{b b_1}$$

$$\frac{a}{a a_1} - \frac{a_1}{a a_1} = \frac{b}{b b_1} - \frac{b_1}{b b_1}$$

$$\frac{1}{a_1} - \frac{1}{a} = \frac{1}{b} - \frac{1}{b}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$$

is the required condition.

Q# 14. Show that the pedal eq. of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

Soln.

Here the eq. of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——— (1)}$$

Diff. (1) w.r.t. 'x'

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x b^2}{a^2 2y}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

Let $P(x_1, y_1)$ be the point where the tangent is drawn.

Then $$\left.\frac{dy}{dx}\right|_P = -\frac{b^2 x_1}{a^2 y_1}$$

If p is the length of the $\perp$ ~~from~~ of the tangent from $O(0,0)$ then

$$p = \frac{|y_1 - x_1 y_1'|}{\sqrt{1 + y_1'^2}}$$

$$\Rightarrow \quad p = \frac{y_1 - x_1\left(-\frac{b^2x_1}{a^2y_1}\right)}{\sqrt{1+\left(-\frac{b^2x_1}{a^2y_1}\right)^2}}$$

$$p = \frac{y_1 + \frac{b^2x_1^2}{a^2y_1}}{\sqrt{\frac{a^4y_1^2 + b^4x_1^2}{a^4y_1^2}}} = \frac{\frac{a^2y_1^2 + b^2x_1^2}{a^2y_1}}{\frac{\sqrt{a^4y_1^2 + b^4x_1^2}}{a^2y_1}}$$

$$p = \frac{a^2y_1^2 + b^2x_1^2}{\sqrt{a^4y_1^2 + b^4x_1^2}} \qquad \text{———— (2)}$$

from (1)

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$$\Rightarrow \quad b^2x_1^2 + a^2y_1^2 = a^2b^2 \qquad \text{———— (3)}$$

Put in (2)

$$p = \frac{a^2b^2}{a^2b^2\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}}}$$

$$p^2 = \frac{1}{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}} \qquad \text{———— (A)}$$

We know that $r^2 = x_1^2 + y_1^2$ ———— (4)

$$\frac{r^2}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2}$$

$$\frac{r^2}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{y_1^2}{b^2}$$

$$\frac{r^2}{a^2} = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) + \left(\frac{y_1^2}{a^2} - \frac{y_1^2}{b^2}\right)$$

$$\frac{r^2}{a^2} = 1 + y_1^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)$$

$$\frac{r^2}{a^2} - 1 = y_1^2\left(1/a^2 - 1/b^2\right)$$

$$\frac{r^2 - a^2}{a^2} = y_1^2\left(1/a^2 - 1/b^2\right)$$

$$\frac{r^2-a^2}{a^2} = y_1^2\left(\frac{b^2-a^2}{a^2b^2}\right)$$

$$\frac{b^2(r^2-a^2)}{b^2-a^2} = y_1^2 \qquad \text{———— (5)}$$

Put in (4)

$$r^2 = x_1^2 + \frac{b^2(r^2-a^2)}{b^2-a^2}$$

$$x_1^2 = r^2 - \frac{b^2(r^2-a^2)}{b^2-a^2}$$

$$x_1^2 = \frac{r^2b^2 - r^2a^2 - b^2r^2 + b^2a^2}{b^2-a^2}$$

$$x_1^2 = \frac{a^2(b^2-r^2)}{b^2-a^2} \qquad \text{———— (6)}$$

$$A) \Rightarrow \quad \frac{1}{p^2} = \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{\frac{a^2(b^2-r^2)}{b^2-a^2}}{a^4} + \frac{\frac{b^2(r^2-a^2)}{b^2-a^2}}{b^4}$$

$$\frac{1}{p^2} = \frac{b^2-r^2}{a^2(b^2-a^2)} + \frac{r^2-a^2}{b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{b^2(b^2-r^2) + a^2(r^2-a^2)}{a^2b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{b^4 - b^2r^2 + a^2r^2 - a^4}{a^2b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{b^4 - a^4 + r^2(a^2-b^2)}{a^2b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{(b^2-a^2)(b^2+a^2) - r^2(b^2-a^2)}{a^2b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{[b^2+a^2-r^2](b^2-a^2)}{a^2b^2(b^2-a^2)}$$

$$\frac{1}{p^2} = \frac{b^2+a^2-r^2}{a^2b^2}$$

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

Is as required.

Q#15. Show that the pedal equation of the curve

$$c^2(x^2+y^2) = x^2y^2$$

is

$$\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

Soln. Here

$$c^2(x^2+y^2) = x^2y^2 \quad \text{——— (i)}$$

$\div$ by $c^2x^2y^2$ we have

$$\frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{c^2} \quad \text{——— (ii)}$$

Differentiating, we have

$$-\frac{2.1}{y^3}\frac{dy}{dx} + \left(-2\frac{1}{x^3}\right) = 0$$

$$-\frac{2.1}{y^3}\frac{dy}{dx} = \frac{2}{x^3}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2y^3}{2x^3}$$

$$\frac{dy}{dx} = -\frac{y^3}{x^3}$$

Let $P(x_1, y_1)$ be any point on the curve where the tangent drawn. Then the slope of the tangent at that point is given by

$$\left.\frac{dy}{dx}\right|_P = -\frac{y_1^3}{x_1^3}$$

If p is the length of the $\perp$ on the tangent from $O(0,0)$. Then

$$p = \frac{|y_1 - x_1 y_1'|}{\sqrt{1+y_1'^2}}$$

$$p = \frac{y_1 + x_1\left(y_1^3/x_1^3\right)}{\sqrt{1 + \left(-y_1^3/x_1^3\right)^2}}$$

$$p = \frac{y_1 + y_1^3/x_1^2}{\sqrt{1 + y_1^6/x_1^6}}$$

$$p = \frac{\dfrac{x_1^2 y_1 + y_1^3}{x_1^2}}{\sqrt{\dfrac{x_1^6 + y_1^6}{x_1^6}}} = \frac{\dfrac{x_1^2 y_1 + y_1^3}{x_1^2}}{\dfrac{\sqrt{x_1^6 + y_1^6}}{x_1^3}}$$

$$p = \frac{x_1^2 y_1 + y_1^3}{x_1^2} \cdot \frac{x_1^3}{\sqrt{x_1^6 + y_1^6}}$$

$$p = \frac{x_1^3 y_1 + x_1 y_1^3}{\sqrt{x_1^6 + y_1^6}}$$

$$p = \frac{x_1^3 y_1^3\left(1/y_1^2 + 1/x_1^2\right)}{\sqrt{x_1^6 y_1^6\left(1/y_1^6 + 1/x_1^6\right)}}$$

$$p = \frac{1/y_1^2 + 1/x_1^2}{\sqrt{1/y_1^6 + 1/x_1^6}}$$

$$p = \frac{1/y_1^2 + 1/x_1^2}{\sqrt{\left(1/y_1^2\right)^3 + \left(1/x_1^2\right)^3}}$$

By using $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$p = \frac{1/y_1^2 + 1/x_1^2}{\sqrt{\left(1/y_1^2 + 1/x_1^2\right)^3 - 3\left(1/x_1^2\right)\left(1/y_1^2\right)\left(\frac{1}{x_1^2} + \frac{1}{y_1^2}\right)}}$$

$$p = \frac{1/c^2}{\sqrt{\left(1/c^2\right)^3 - \frac{3}{x_1^2 y_1^2} \cdot 1/c^2}}$$

$$p^2 = \frac{1/c^4}{1/c^6 - \dfrac{3}{c^2 x_1^2 y_1^2}}$$

$$\Rightarrow \quad \frac{1}{p^2} = \frac{1/c^6 - \dfrac{3}{c^2 x_1^2 y_1^2}}{1/c^4}$$

$$\frac{1}{p^2} = c^4 \left[\frac{1}{c^6} - \frac{3}{c^2 x_1^2 y_1^2} \right]$$

$$\frac{1}{p^2} = \frac{1}{c^2} - \frac{3c^2}{x_1^2 y_1^2} \quad \text{——— (iii)}$$

We know that $r^2 = x_1^2 + y_1^2$

$$\frac{r^2}{x_1^2 y_1^2} = \frac{1}{y_1^2} + \frac{1}{x_1^2}$$

$$\frac{r^2}{x_1^2 y_1^2} = \frac{1}{c^2} \qquad \text{from (ii)}$$

$$\frac{c^2}{x_1^2 y_1^2} = \frac{1}{r^2}$$

Put in (iii), we have

$$\frac{1}{p^2} = \frac{1}{c^2} - \frac{3}{r^2}$$

$$\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

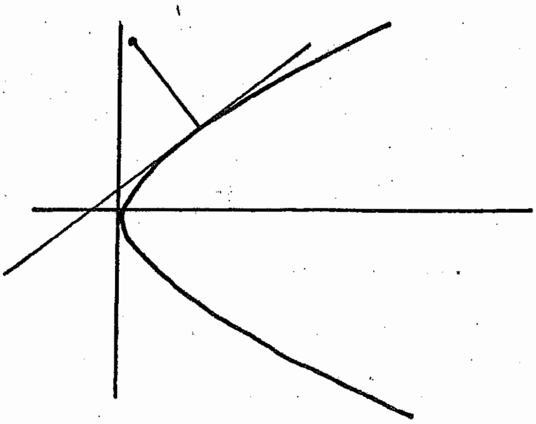
Is as required.

Q# 16. Show that from any point three normals can be drawn to a parabola $y^2 = 4ax$ and the sum of slopes of the three normals is zero.

Soln.

We know that the eq. of the normal to the parabola

$$y^2 = 4ax$$



is $\quad y = mx - 2am - am^3$

$\Rightarrow \quad am^3 + 2am - mx + y = 0$

$\quad am^3 + 0m^2 + (2a - x)m + y = 0$

$\because$ this eq. is cubic is in m

$\therefore$ We have three values of m from this eq.

m being slope of the normal we can say that three normals can be drawn to this parabola.

If m_1, m_2 and m_3 are the roots of this eq. then we know that

$$m_1+m_2+m_3 = -\frac{\text{Co-eff. of } m^2}{\text{Co-eff. of } m^3}$$

$$= -\frac{0}{a}$$

$$= 0$$

Hence the sum of the slopes of the three normals is zero.

$ax^2+bx+c=0$

If α & β are the roots then $\alpha+\beta = -\frac{b}{a}$

$\alpha\beta = c/a$

Similarly for

$ax^3+bx^2+cx+d=0$

$\alpha+\beta+\gamma = -b/a$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

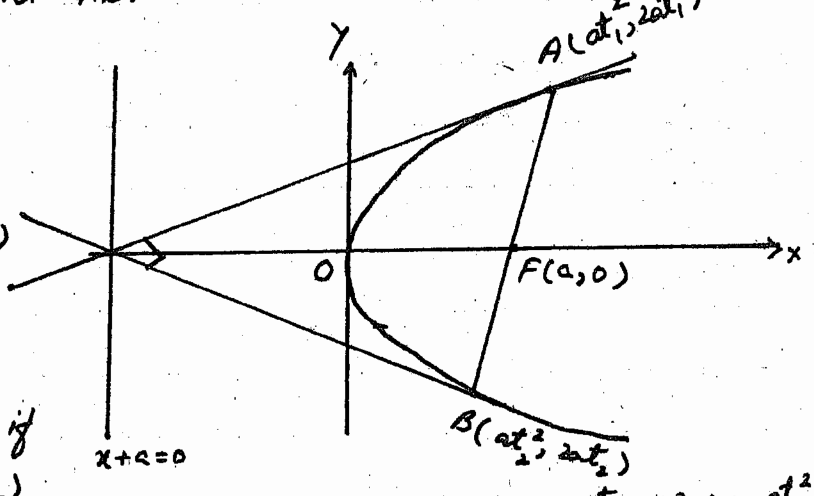
$\gamma\beta\gamma = -d/a$

Q#17. Show that tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix. (To understand this question see Example 8).

Soln. Let the focal chord ($\overline{AB}$) of the parabola

$y^2 = 4ax$ —— (i)

having the focus at $F(a,0)$. Let $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ be extrimities of the focal chord.



Then we know that if $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ be the extrimilies of a focal chord then

$$t_1 \cdot t_2 = -1 \quad \text{—— (ii)}$$

$y.2at_1 = 2a(x+at_1^2)$

$t_1 y = x+at_1^2$

Using $yy_1 = 2a(x+x_1)$

Also the equations of the tangents at A and B are

$t_1 y = x+at_1^2 \Rightarrow x - t_1 y + at_1^2 = 0$ —— (iii)

$t_2 y = x+at_2^2 \Rightarrow x - t_2 y + at_2^2 = 0$ —— (iv)

Now

$$m_1 = \text{Slope of the tangent at } A = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}$$

$$m_1 = -\frac{1}{-t_1} = \frac{1}{t_1}$$

$$m_2 = \text{slope of the tangent at } B = -\frac{1}{-t_2} = \frac{1}{t_2}$$

Now $\quad m_1 \cdot m_2 = \frac{1}{t_1} \cdot \frac{1}{t_2} = \frac{1}{t_1 t_2}$

$m_1 m_2 = -1 \qquad$ from (ii')

$\Rightarrow$ the tangent at A is $\perp$ to the tangent at B.

Multiplying (iii) by t_2

$$t_2 x - t_1 t_2 y + a t_1^2 t_2 = 0 \quad \text{——— (v)}$$

Multiplying (iv) by t_1

$$t_1 x - t_1 t_2 y + a t_1 t_2^2 = 0 \quad \text{——— (vi)}$$

(v) $-$ (vi) $\Rightarrow$

$$(t_2 - t_1)x + a(t_1^2 t_2 - t_1 t_2^2) = 0$$
$$(t_2 - t_1)x + a t_1 t_2 (t_1 - t_2) = 0$$
$$(t_2 - t_1)x - a t_1 t_2 (t_2 - t_1) = 0$$
$$(x - a t_1 t_2)(t_2 - t_1) = 0$$

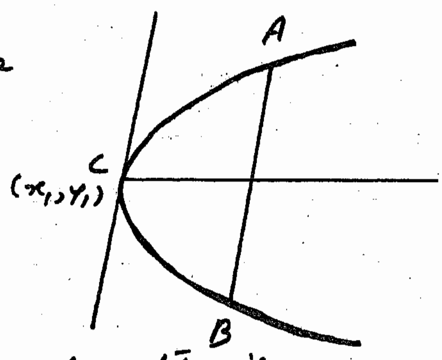
$\Rightarrow \quad x - a t_1 t_2 = 0$

$\Rightarrow \quad x - a(-1) = 0 \qquad$ from (ii')

$\Rightarrow \quad x + a = 0$

Hence the tangents at the ends of a focal chord of a parabola intersect at right angle on the directrix.

Q#18. (a) Show that the tangent at the vertex of a diameter of a parabola is parallel to the chords bisected by the diameter.



Soln. Let us consider the case of the parabola $\quad y^2 = 4ax$ ——— (i)

Let $\overline{AB}$ be the one of the $\parallel$ chords whose eq. is

$$y = mx + c$$

We know that the equation of the diameter is

$y = \frac{2a}{m}$ ——— (ii) where m is the slope of the chord.

If the pt. $C(x_1, y_1)$ lies on the diameter, then

$$y_1 = \frac{2a}{m} \quad \text{——— (A)}$$

$$\Rightarrow \quad m = \frac{2a}{y_1} \quad \text{——— (B)}$$

which is the slope of $\overline{AB}$.

We know that the equation of tangent at any point of the parabola is

$$yy_1 = 2a(x + x_1)$$

Now eq. of the tangent at $C(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

$$yy_1 = 2ax + 2ax_1$$

$$2ax - yy_1 - 2ax_1 = 0 \quad \text{——— (iii)}$$

Let m_1 = slope of the tangent at $C = -\dfrac{\text{Coeff. of } x}{\text{Coeff. of } y}$

$$m_1 = -\frac{2a}{-y_1} = \frac{2a}{y_1} \quad \text{——— (iv)}$$

Eq. B and (iv) $\Rightarrow$

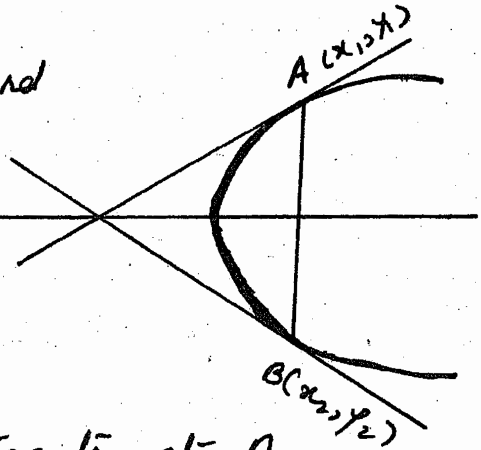
$$m = m_1$$

$\Rightarrow$ Tangent at C is parallel to the chord $\overline{AB}$.

(b) Prove that the tangents at the ends of any chord of a parabola meet on the diameter which bisects the chord.

<u>Soln.</u> Let us consider the chord $\overline{AB}$ of the parabola

$$y^2 = 4ax \quad \text{——— (i)}$$



Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2) be respectively.

Now equations of the tangents at A and B will

$$yy_1 = 2a(x + x_1)$$

$$yy_2 = 2(x + x_2)$$

Subtracting

$$yy_1 - yy_2 = 2a(x + x_1) - 2a(x + x_2)$$

$$y(y_1-y_2) = 2ax + 2ax_1 - 2ax - 2ax_2$$

$$y(y_1-y_2) = 2a(x_1-x_2)$$

$$\Rightarrow \quad y = \frac{2a(x_1-x_2)}{(y_1-y_2)}$$

$$\Rightarrow \quad y = 2a.\frac{x_2-x_1}{y_2-y_1} \quad \text{—— (ii)}$$

If m is the slope of $\overline{AB}$, Then

$$m = \frac{y_2-y_1}{x_2-x_1}$$

$$\Rightarrow \quad \frac{1}{m} = \frac{x_2-x_1}{y_2-y_1}$$

Put in (ii)

$$\Rightarrow \quad y = \frac{2a}{m}$$

which is the eq. of the diameter.

Hence the tangents at the ends of any chord of a parabola meet on the diameter which bisect the chord.

Q #19. Find the condition that straight line

$$lx + my + n = 0$$

may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Also find the coordinates of the point of contact.

Soln.

Here Line $= lx + my + n = 0$ —— (1)

and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —— (2)

We know that the eq. of the tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{—— (3)}$$

$\because$ (1) and (3) are the tangents at the same point

$\therefore$ Comparing (1) and (3)

$$\frac{\frac{x_1}{a^2}}{l} = \frac{\frac{y_1}{b^2}}{m} = \frac{-1}{n}$$

$$\frac{x_1}{a^2 l} = \frac{y_1}{b^2 m} = -\frac{1}{n} \quad \text{—— (4)}$$

(4) $\Rightarrow$
$$\frac{x_1}{a^2 l} = -\frac{1}{n}$$
$$x_1 = -\frac{a^2 l}{n}$$

and
$$\frac{y_1}{b^2 m} = -\frac{1}{n}$$
$$y_1 = -\frac{b^2 m}{n}$$

Hence the point of contact is $\left(-\frac{a^2 l}{n}, -\frac{b^2 m}{n}\right)$

$\because$ the point lie on the line

$\therefore$
$$l\left(-\frac{a^2 l}{n}\right) + m\left(-\frac{b^2 m}{n}\right) + n = 0$$
$$-\frac{a^2 l^2}{n} + \left(-\frac{b^2 m^2}{n}\right) + n = 0$$
$$\frac{-a^2 l^2 - m^2 b^2}{n} + n = 0$$
$$\Rightarrow \quad n = \frac{a^2 l^2 + m^2 b^2}{n}$$
$$n^2 = a^2 l^2 + m^2 b^2$$

is the required condition.

Q # 20. Show that the condition that normals at that points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

Soln. The given eq. of the ellipse is
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——— (1)}$$

We know that the eq. of the normal at (x_1, y_1) is
$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 - b^2$$

$\times$ by $x_1 y_1$
$$a^2 x y_1 + b^2 y x_1 = (a^2 - b^2) x_1 y_1 \quad \text{——— (2)}$$

Similarly the equations of the normals at (x_2, y_2) and (x_3, y_3) are

$$a^2 x y_2 + b^2 y x_2 = (a^2 - b^2)(x_2 y_2) \quad \text{———} \quad (3)$$

and

$$a^2 x y_3 + b^2 y x_3 = (a^2 - b^2)(x_3 y_3) \quad \text{———} \quad (4)$$

$\Rightarrow$ The condition is

$$\begin{vmatrix} a^2 y_1 & b^2 x_1 & (a^2-b^2) x_1 y_1 \\ a^2 y_2 & b^2 x_2 & (a^2-b^2)\, x_2 y_2 \\ a^2 y_3 & b^2 x_3 & (a^2-b^2)(x_3 y_3) \end{vmatrix} = 0$$

$$a^2 b^2 (a^2-b^2) \begin{vmatrix} y_1 & x_1 & x_1 y_1 \\ y_2 & x_2 & x_2 y_2 \\ y_3 & x_3 & x_3 y_3 \end{vmatrix} = 0$$

$$-a^2 b^2 (a^2-b^2) \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

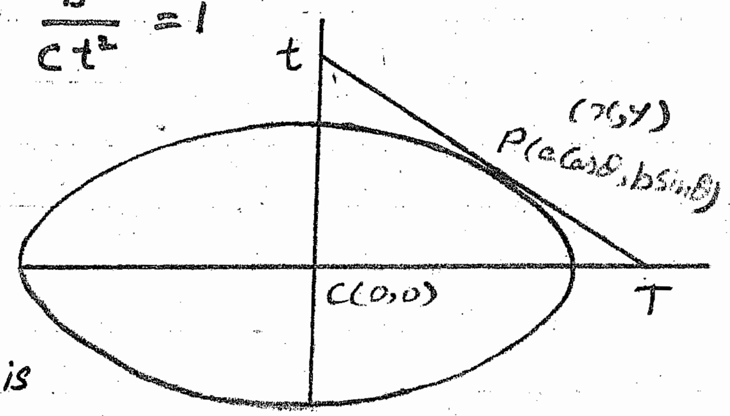
$$\Rightarrow \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

Is the required condition.

Q# 21 If a Tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre C, meets the major and minor axes in T and t prove that

$$\frac{a^2}{CT^2} + \frac{b^2}{Ct^2} = 1$$

Soln:



Here eq. of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{———} \quad (1)$$

eq. of the tangent at any point of the ellipse is

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1 \quad \text{———} \quad (2)$$

intersection of tangent with major Axis:-

$\Rightarrow \quad y = 0$

Put in (2)

$$\frac{x x_1}{a^2} + 0 = 1$$

$$\Rightarrow \quad x_1 = \frac{a^2}{x} \quad \Rightarrow \quad x = \frac{a^2}{x_1}$$

$\Rightarrow$ The coordinates of T are $\left(\frac{a^2}{x_1}, 0\right)$.

X^n of tangent with minor axis :-

$\Rightarrow \quad x = 0$

put in (2)

$$0 + \frac{y y_1}{b^2} = 1$$

$$\Rightarrow \quad y = \frac{b^2}{y_1}$$

$\Rightarrow$ The coordinates of t are $\left(0, \frac{b^2}{y_1}\right)$.

Now

$$CT = \left(\frac{a^2}{x_1} - 0\right) + (0 - 0) \quad \Rightarrow \quad CT = \frac{a^2}{x_1}$$

squaring $\quad (CT)^2 = \dfrac{a^4}{x_1^2}$ ——— (3)

and $\quad ct = \dfrac{b^2}{y_1}$

$(ct)^2 = \dfrac{b^4}{y_1^2}$ ——— (4)

3) $\Rightarrow \quad \dfrac{x_1^2}{a^2} = \dfrac{a^2}{CT^2}$ ——— (5)

4) $\Rightarrow \quad \dfrac{y_1^2}{b^2} = \dfrac{b^2}{ct^2}$ ——— (6)

3 + 4 $\Rightarrow$

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{a^2}{CT^2} + \frac{b^2}{ct^2}$$

$$\Rightarrow \quad \frac{a^2}{CT^2} + \frac{b^2}{ct^2} = 1$$

is as required.

Q#22. Show that the locus of the point of intersection of tangents at two points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\lambda$$

where 2λ is the difference of the eccentric angles of two points.

Soln. Here Ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ——— (1)

Difference in ecentric angles $=2\lambda$

Target : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\sec^2\lambda$

Let $P=(a\cos\theta, b\sin\theta)$, $Q=(a\cos\theta', b\sin\theta')$

be any pts. on (1) with θ and θ' as the ecentric angles of P and Q. Then eq. of the tangent at $P(a\cos\theta, b\sin\theta)$ to the ellipse is

$$\frac{x\cdot a\cos\theta}{a^2}+\frac{y.\, b\sin\theta}{b^2}=1 \quad \text{———} \quad (2)$$

$$\frac{\cos\theta}{a}x+\frac{\sin\theta}{b}.y=1$$

$$x\frac{\cos\theta}{a}+\frac{\sin\theta}{b}y-1=0 \quad \text{———} \quad (3)$$

Like wise the equation of tangent at Q

$$\frac{\cos\theta'}{a}x+\frac{\sin\theta'}{b}y-1=0 \quad \text{———} \quad (4)$$

Intersection of the tangents

$$\frac{x/a}{-\sin\theta+\sin\theta'}=\frac{-y/b}{-\cos\theta+\cos\theta'}=\frac{1}{\cos\theta\sin\theta'-\cos\theta'\sin\theta}$$

$$\Rightarrow \frac{x}{a}=\frac{\sin\theta'-\sin\theta}{\sin(\theta'-\theta)} \; ; \; \frac{y}{b}=\frac{\cos\theta-\cos\theta'}{\sin(\theta'-\theta)}$$

$$\cos\theta\frac{x}{a}+\sin\theta\frac{y}{b}-1=0$$
$$\cos\theta'\frac{x}{a}+\sin\theta'\frac{y}{b}-1=0$$

$\because \theta'-\theta=2\lambda$

$$\therefore \frac{x^2}{a^2}=\frac{(\sin\theta'-\sin\theta)^2}{\sin^2 2\lambda} \quad \text{———} \quad (5)$$

and

$$\frac{y^2}{b^2}=\frac{(\cos\theta-\cos\theta')^2}{\sin^2 2\lambda} \quad \text{———} \quad (6)$$

$(5) + (6) \Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(\sin\theta' - \sin\theta)^2}{\sin^2 2\lambda} + \frac{(\cos\theta - \cos\theta')^2}{\sin^2 2\lambda}$$

$$= \frac{(\sin\theta' - \sin\theta)^2 + (\cos\theta - \cos\theta')^2}{\sin^2 2\lambda}$$

$$= \frac{\sin^2\theta' + \sin^2\theta - 2\sin\theta\sin\theta' + \cos^2\theta + \cos^2\theta' - 2\cos\theta\cos\theta'}{\sin^2 2\lambda}$$

$$= 2 - 2(\cos\theta'\cos\theta + \sin\theta'\sin\theta) / \sin^2 2\lambda$$

$$= \frac{2(1 - \cos(\theta' - \theta))}{\sin^2 2\lambda}$$

$$= \frac{2(1 - \cos 2\lambda)}{(2\sin\lambda\cos\lambda)^2}$$

$$= \frac{2 \cdot 2\sin^2\lambda}{4\sin^2\lambda\cos^2\lambda}$$

$$= \frac{1}{\cos^2\lambda}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\lambda$$

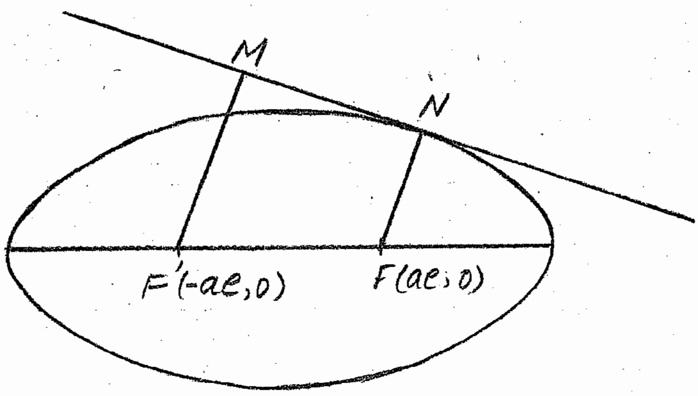
Is as required.

#23. Show that locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle and product of the ~~[illegible]~~ lengths of perpendiculars is equal to square on the semi-minor axis.

Soln.

Here ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



We know that the square of the tangent at any point on the ellipse can be written in the form

$$y = mx + \sqrt{a^2m^2+b^2} \quad \text{———— (i)}$$

$\because$ Slope of the tangent $= m$

$\therefore$ Slope of the perpendicular to the tangent $= -\frac{1}{m}$

Eq. of the perpendicular from $F(ae, 0)$ to the tangent is

$$(y-0) = -\frac{1}{m}(x-ae)$$

$$\Rightarrow \quad my = -x + ae$$

$$\Rightarrow \quad x + my = ae \quad \text{———— (ii)}$$

Elimination of m

$$1) \Rightarrow \quad y - mx = \sqrt{a^2m^2+b^2} \quad \text{———— (iii)}$$

Squaring and adding (ii) and (iii)

$$(x+my)^2 + (y-mx)^2 = a^2e^2 + (\sqrt{a^2m^2+b^2})^2$$

$$\Rightarrow \quad x^2 + m^2y^2 + 2xmy + y^2 + m^2x^2 - 2ymx = a^2e^2 + a^2m^2 + b^2$$

$$(1+m^2)x^2 + (1+m^2)y^2 = a^2(e^2+m^2) + b^2$$

$$(1+m^2)(x^2+y^2) = a^2m^2 + a^2e^2 + a^2 - a^2e^2 \qquad \because b^2 = a^2 - a^2e^2$$

$$(1+m^2)(x^2+y^2) = a^2m^2 + a^2$$

$$(1+m^2)(x^2+y^2) = a^2(1+m^2)$$

$$x^2 + y^2 = a^2 \quad \text{———— (iv)}$$

Let Product of $\perp$s FN and $F'M$ on the tangent $= b^2$

Then $i) \Rightarrow \quad xm - y + \sqrt{a^2m^2+b^2} = 0$

Now $|FN|$ = Distance of F from the tangent

$$= \frac{|mae - 0 + \sqrt{a^2m^2+b^2}|}{\sqrt{m^2+1}}$$

$$= \frac{|mae + \sqrt{a^2m^2+b^2}|}{\sqrt{m^2+1}}$$

Similarly

$$|F'M| = \frac{|-mae + \sqrt{a^2m^2+b^2}|}{\sqrt{1+m^2}}$$

$$|FN| \cdot |F'M| = \frac{(mae + \sqrt{a^2m^2+b^2})(\sqrt{a^2m^2+b^2} - mae)}{m^2+1}$$

$$|FN| \cdot |F'M| = \frac{a^2m^2 + b^2 - m^2a^2e^2}{m^2+1}$$

~~for ellipse~~

$$= \frac{a^2m^2 + b^2 + m^2(b^2 - a^2)}{1+m^2}$$

$$= \frac{a^2m^2 + b^2 + m^2b^2 - a^2m^2}{1+m^2}$$

$$= \frac{b^2(1+m^2)}{1+m^2}$$

$b^2 = a^2 - a^2e^2$
$b^2 - a^2 = -a^2e^2$
and $a^2e^2 = a^2 - b^2$
for ellipse

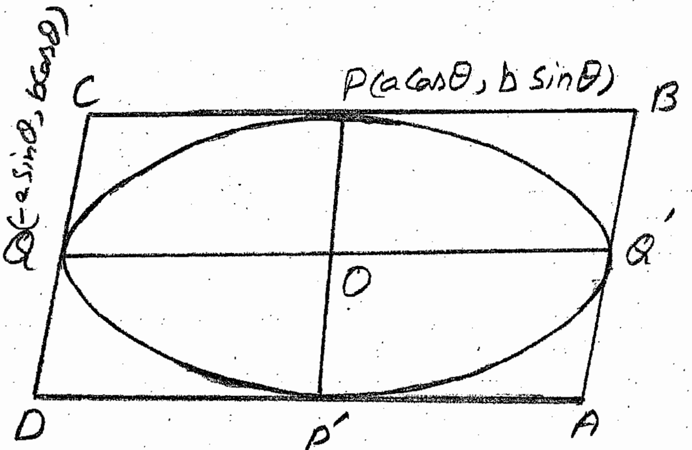
$|FN| \cdot |F'M| = b^2$

(iv) is a circle having the centre at $O(0,0)$ and radius a and is the auxiliary circle, and the product of the lengths of perpendiculars is equal to the square of the semi minor axis.

Q #24. Prove that the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

Soln.

Let POP' and QOQ' be the conjugate diameter then



$P = (a\cos\theta, b\sin\theta)$

$Q = (-a\sin\theta, b\cos\theta)$

ABCD is a //gram due to the tangent drawn at the pts. P, Q, P' and Q'.

Now Area of the //gram ABCD = 4[Area of the //gram OPCQ] (1)

Now Vector area of the //gram OPCQ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta & b\cos\theta & 0 \\ a\cos\theta & b\sin\theta & 0 \end{vmatrix}$

$$= \hat{k}(-ab\sin^2\theta - ab\cos^2\theta)$$

$$= -ab(\sin^2\theta + \cos^2\theta)\hat{k}$$

$$= -ab\,\hat{k}$$

Required area of the //gram OPCQ $= \sqrt{(-ab)^2}$

$$= ab$$

Put in (i)

$\Rightarrow$ Area of the ||gram ABCD $= 4(ab)$

$$= 4ab$$

$$= \text{Constant.}$$

Hence the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

Q#25. The hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ are said to conjugate to each other. If e and e' are eccentricities of a hyperbola and it's conjugate, prove that

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Soln: The given equations of the hyperbolas are

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{———} \quad (i)$$

and

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{———} \quad (ii)$$

If e is the eccentricity of (i) then we know that

$$a^2 + b^2 = a^2e^2$$

$$\Rightarrow \quad b^2 = a^2e^2 - a^2$$

$$b^2 = a^2(e^2 - 1) \quad \text{———} \quad (iii)$$

Now (ii) $\Rightarrow$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

If e' is the eccentricity of this hyperbola

Then $a^2 = b^2(e'^2 - 1)$ ——— (iv)

(iii) $\times$ (iv)

$$a^2b^2 = a^2b^2(e^2 - 1)(e'^2 - 1)$$

$$\frac{a^2b^2}{a^2b^2} = (e^2 - 1)(e'^2 - 1)$$

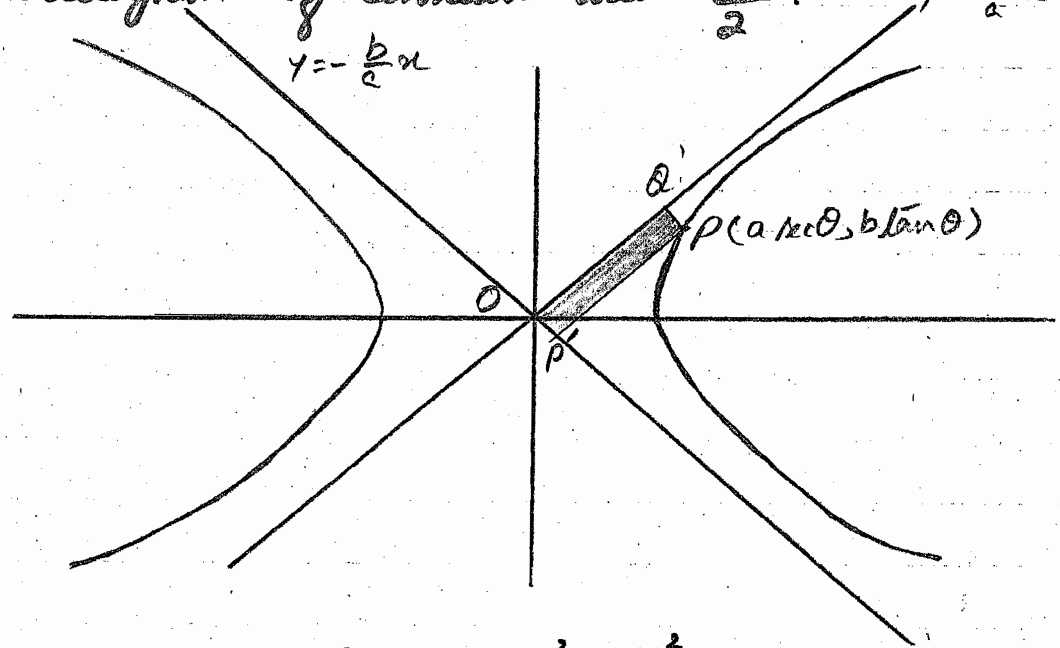
$$1 = e^2 e'^2 - e^2 - e'^2 + 1$$
$$0 = e^2 e'^2 - e^2 - e'^2$$
$$e^2 + e'^2 = e^2 e'^2$$
$\div$ by $e^2 (e')^2$, we have

$$\frac{1}{(e')^2} + \frac{1}{e^2} = 1$$

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

Is as required.

Q#26. Show that the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the lines drawn from any point on the hyperbola parallel to the asymptotes form a parallelogram of constant area $\frac{ab}{2}$.



Soln.

Here hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ——— (1)
and eq. of the asymptote is $y = \pm \frac{b}{a} x$
let $P = (a\sec\theta, b\tan\theta)$ be any pt. on the hyperbola.
Let the line drawn at P ∥ to the asymptotes meets the asymptotes ~~at~~ at P' and Q.
Eq. of line through P or drawn from P ∥ to the

asymptote $y = \frac{b}{a}x$

$\because$ The asymptote and line drawn from P are $\parallel$.

$\therefore$ The slope of the asymptote and line are equal

i.e the slope of line $= \frac{b}{a}$

Now Eq. of the line drawn from P is

$$y - b\tan\theta = \frac{b}{a}(x - a\sec\theta) \quad \text{——— (2)}$$

Also eq. of the asymptote is

$$y = -\frac{b}{a}x \quad \text{——— (3)}$$

P' is the pt. of intersection of (2) and (3)

Put (3) in (2)

$$\Rightarrow -\frac{b}{a}x - b\tan\theta = \frac{b}{a}x - b\sec\theta$$

$$b\sec\theta - b\tan\theta = \frac{b}{a}x + \frac{b}{a}x$$

$$b(\sec\theta - \tan\theta) = 2\frac{b}{a}x$$

$$\Rightarrow \quad x = \frac{a}{2}(\sec\theta - \tan\theta). \quad \text{——— (4)}$$

Put (4) in (3)

$$y = -\frac{b}{a}\cdot\frac{a}{2}(\sec\theta - \tan\theta)$$

$$y = -\frac{b}{2}(\sec\theta - \tan\theta)$$

Hence the coordinates of P' are

$$\frac{a}{2}(\sec\theta - \tan\theta), \ -\frac{b}{2}(\sec\theta - \tan\theta)$$

Now

vector area of the $\parallel^{am}$ $OP'PQ$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a\sec\theta & b\tan\theta & 0 \\ \frac{a}{2}(\sec\theta - \tan\theta) & -\frac{b}{2}(\sec\theta - \tan\theta) & 0 \end{vmatrix}$

$$= \left[-\frac{ab}{2}(\sec^2\theta - \sec\theta\tan\theta) - \frac{ab}{2}(\sec\theta\tan\theta - \tan^2\theta)\right]\hat{k}$$

$$= \left[-\frac{ab}{2}(\sec^2\theta - \sec\theta\tan\theta + \sec\theta\tan\theta - \tan^2\theta)\right]\hat{k}$$

$$= -\frac{ab}{2}\hat{k}$$

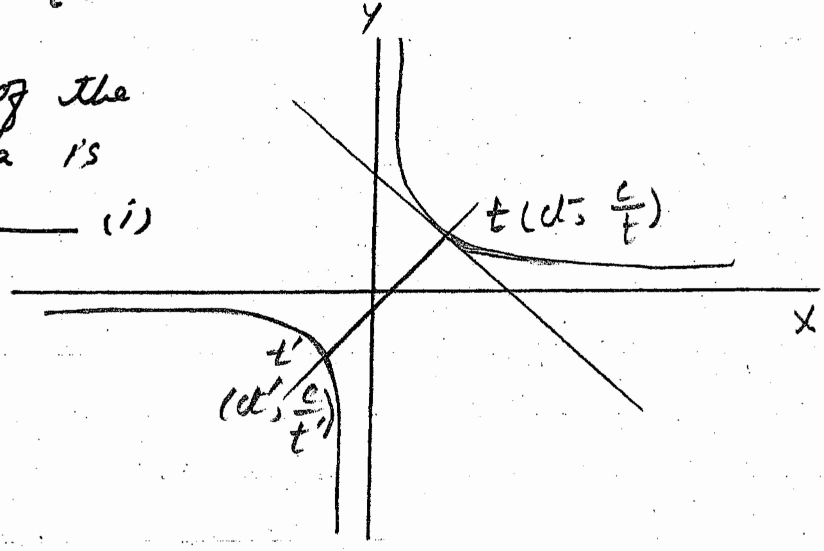
Hence area of the $\parallel^{am}$ $OP'PQ$ $= \sqrt{\left(-\frac{ab}{2}\right)^2} = \frac{ab}{2}$

i.e the area of $\parallel^{am}$ $OP'PQ = \frac{ab}{2}$

Is as required.

Q#27. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at the point 't'' such that $t^3 t' = -1$

Note: The point $(ct, \frac{c}{t})$ on $xy = c^2$ is reffered to as as the point 't'.



Soln: The equation of the rectangular hyperbola is

$$xy = c^2 \quad \text{—— (i)}$$

we find the eq. of normal at the pt. $t(ct, \frac{c}{t})$.

Here

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\frac{dy}{dt}\Big|_t = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

Now the equation of the normal

$$y - y_1 = -\frac{1}{\frac{dy}{dx}\Big|_t}(x - x_1)$$

$$\Rightarrow \quad y - \frac{c}{t} = -\frac{1}{-\frac{1}{t^2}}(x - ct)$$

$$y - \frac{c}{t} = t^2(x - ct)$$

$$t^2 y - ct = t^4(x - ct)$$

$$ty - c = t^3 x - ct^4$$

$$t^3 x - ty + c - ct^4 = 0 \quad \text{—— (i)}$$

if this normal meets the hyperbola at $t'(ct', \frac{c}{t'})$ then

$$t^3 . ct' - t . \frac{c}{t'} + c - ct^4 = 0$$

$$t^3 c t'^2 - tc + t'c - ct^4 t' = 0$$

$$ct'^2t^3 + ct' - ct't^4 - ct = 0$$

$$ct'(t't^3+1) - ct(t't^3+1) = 0$$

$$(ct' - ct)(t't^3+1) = 0$$

$$c(t'-t)(t't^3+1) = 0$$

$$\Rightarrow \quad t't^3+1 = 0 \qquad \because t' \neq t$$

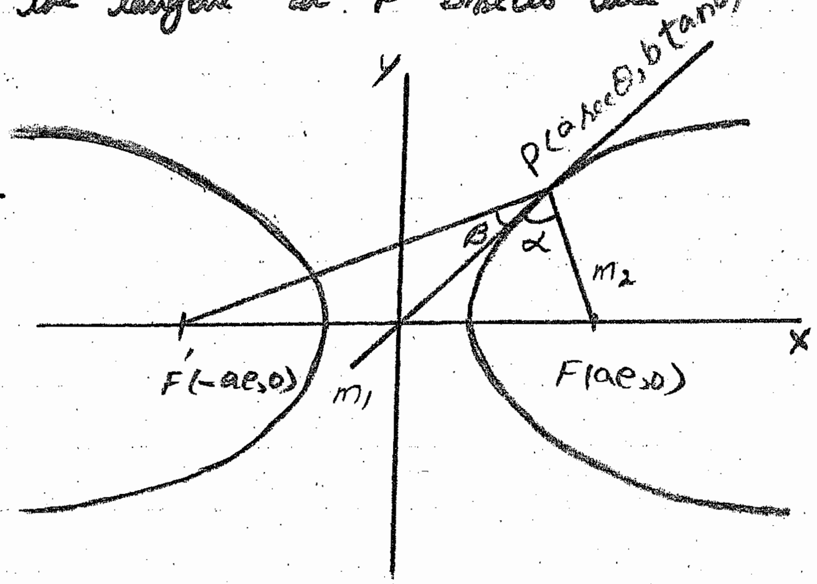
$$t't^3 = -1$$

$$\text{or} \quad t^3t' = -1$$

Is as required.

Q# 28. Prove that if P is any point on a hyperbola with foci F and F', the tangent at P bisects the angle $F'PF$.

Sol.



Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{------} \quad (i)$$

having the pt.

$P(a\sec\theta, b\tan\theta)$ on it.

(i) $\Rightarrow$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$dy/dx = -\frac{b^2x}{a^2y}$$

$$m_1 = \text{slope of the tangent at } P = (dy/dx)_P$$

$$m_1 = -\frac{b^2 a\sec\theta}{a^2 b\tan\theta}$$

$$m_1 = \frac{b\sec\theta}{a\tan\theta} \quad \text{------} \quad (ii)$$

$$m_2 = \text{slope of } \overline{PF} = \frac{y_2-y_1}{x_2-x_1} = \frac{b\tan\theta - 0}{a\sec\theta - ae}$$

$$m_2 = \frac{b\tan\theta}{a\sec\theta - ae}$$

$$\text{Now} \quad \tan\alpha = \left|\frac{m_2 - m_1}{1 + m_2m_1}\right|$$

$$\tan\alpha = \frac{\dfrac{b\tan\theta}{a\sec\theta - ae} - \dfrac{b\sec\theta}{a\tan\theta}}{1 + \dfrac{b\tan\theta}{a\sec\theta - ae}\cdot\dfrac{b\sec\theta}{a\tan\theta}}$$

$$\tan\alpha = \frac{\dfrac{ab\tan^2\theta - b\sec\theta(a\sec\theta - ae)}{(a\sec\theta - ae)(a\tan\theta)}}{\dfrac{(a\sec\theta - ae)(a\tan\theta) + b\tan\theta\, b\sec\theta}{(a\sec\theta - ae)(a\tan\theta)}}$$

$$\tan\alpha = \frac{ab\tan^2\theta - ab\sec^2\theta + abe\sec\theta}{a^2\sec\theta\tan\theta - a^2e\tan\theta + b^2\tan\theta\sec\theta}$$

$$\tan\alpha = \frac{ab(\tan^2\theta - \sec^2\theta) + abe\sec\theta}{(a^2+b^2)\sec\theta\tan\theta - a^2e\tan\theta}$$

$$\tan\alpha = \frac{(-1)ab + abe\sec\theta}{a^2e^2\tan\theta\sec\theta - a^2e\tan\theta} \qquad \because \text{for hyperbola } a^2+b^2 = a^2e^2$$

$$\tan\alpha = \frac{ab(e\sec\theta - 1)}{a^2e\tan\theta(e\sec\theta - 1)}$$

$$\tan\alpha = \frac{b}{ae\tan\theta}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{b}{ae\tan\theta}\right)$$

Similarly for β, only replace ae by $-ae$ to get

$$\beta = \tan^{-1}\left(\frac{b}{ae\tan\theta}\right)$$

$$\Longrightarrow \quad \alpha = \beta$$

which is the required.

Q# 29. Find an equation of a tangent to the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
in the form
$$\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1$$
Show that the product of lengths of perpendiculars on it from the foci is constant.

Soln.

The given hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\because$ $P(a\cosh\theta, b\sinh\theta)$ pt. lies on it.

$\therefore$ Eq. of tangent at P is given by

$$\frac{x\,x_1}{a^2} - \frac{y\,y_1}{b^2} = 1$$

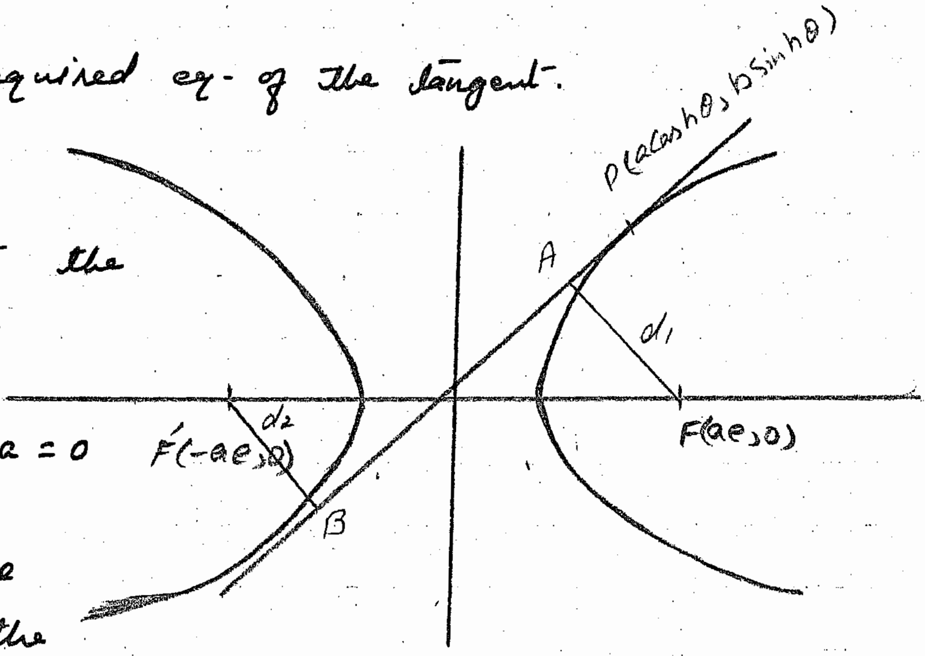
$$\frac{x}{a^2}\,a\cosh\theta - \frac{y}{b^2}\,b\sinh\theta = 1$$

$$\Rightarrow \frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1 \qquad \text{—— (i)}$$

is the required eq. of the tangent.

Now let d_1 and d_2 are the perpendicular distances from F, F' to the tangent line, then from (i)

$bx\cosh\theta - ay\sinh\theta - ba = 0$



Now

$d_1 = |FA| =$ Distance of $F(ae, 0)$ from the tangent line.

$$d_1 = \frac{|bae\cosh\theta - 0 - ab|}{\sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta}}$$

Now $d_2 = |FB| =$ Distance of $F(-ae, 0)$ from the tangent line

$$d_2 = \frac{|-bae\cosh\theta - 0 - ab|}{\sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta}}$$

Hence $$d_1 = \frac{ab(e\cosh\theta - 1)}{\sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta}}$$

and

$$d_2 = \frac{ab(e\cosh\theta + 1)}{\sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta}}$$

Now

$$d_1 d_2 = \frac{ab(e\cosh\theta - 1) \cdot ab(e\cosh\theta + 1)}{\sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta} \cdot \sqrt{b^2\cosh^2\theta + a^2\sinh^2\theta}}$$

$$= \frac{a^2b^2(e^2\cosh^2\theta - 1)}{b^2\cosh^2\theta + a^2\sinh^2\theta} \Rightarrow \frac{a^2b^2(e^2\cosh^2\theta - 1)}{b^2\cosh^2\theta + a^2(\cosh^2\theta - 1)}$$

$$= \frac{a^2b^2(e^2\cosh^2\theta - 1)}{b^2\cosh^2\theta + a^2\cosh^2\theta - a^2} \Rightarrow \frac{a^2b^2(e^2\cosh^2\theta - 1)}{(a^2+b^2)\cosh^2\theta - a^2}$$

$$= \frac{a^2b^2(e^2\cosh^2\theta - 1)}{a^2e^2\cosh^2\theta - a^2}$$

$$= \frac{a^2b^2(e^2\cosh^2\theta - 1)}{a^2(e^2\cosh^2\theta - 1)}$$

$$= b^2$$

$=$ Constant.

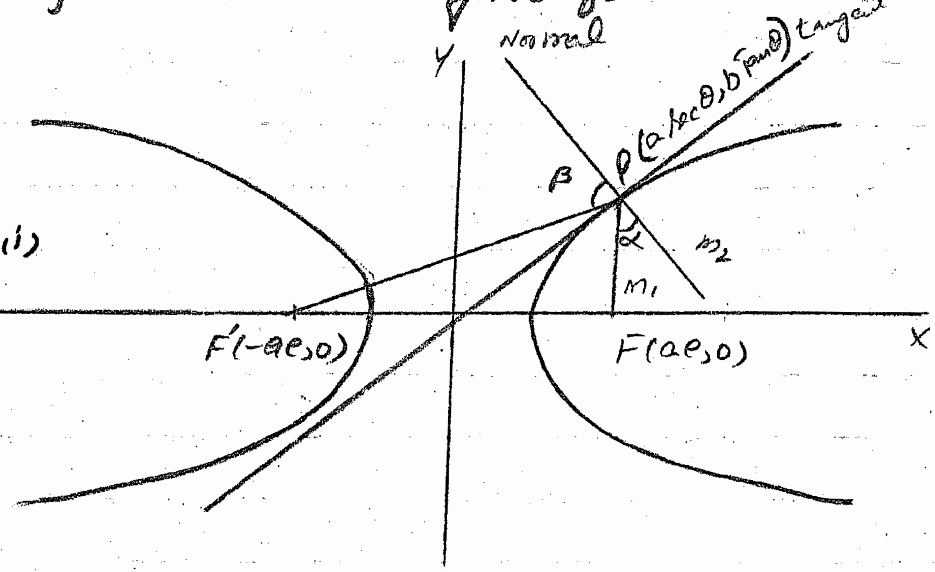
Is as required.

Q # 30. Find an equation of a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the form

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2+b^2.$$

Prove that the normal is external bisector of the angle between the focal distances of it's foot.

Soln:



The eq. of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{——— (i)}$$

Diff. (i) w.r.t to 'x'

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\left.\frac{dy}{dx}\right|_P = \frac{b^2 a\sec\theta}{a^2 b\tan\theta} = \frac{b}{a\sin\theta}$$

Hence the slope of the normal is $-\frac{a\sin\theta}{b}$ and equation of the normal at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\frac{\sin\theta}{\cos\theta}$$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\tan\theta$$

$$by + ax\sin\theta = a^2\tan\theta + b^2\tan\theta$$

$$by + ax\sin\theta = (a^2+b^2)\tan\theta$$

$$\frac{by}{\tan\theta} + \frac{ax\sin\theta}{\frac{\sin\theta}{\cos\theta}} = a^2+b^2$$

$$\frac{by}{\tan\theta} + ax\sin\theta.\frac{\cos\theta}{\sin\theta} = a^2+b^2$$

$$\frac{by}{\tan\theta} + \frac{ax}{\sec\theta} = a^2+b^2$$

Is the required eq. of normal.

[Additional Work.]

Question. Define homogeneous equation of Degree 2. Prove that this eq. always represents two straight lines. Find the angle b/w these two straight lines also deduce a condition so that the straight lines are

i) Parallel

ii) perpendicular.

Solution.

Def. The eq. of the form

$$ax^2 + 2hxy + by^2 = 0$$

where a, h and b are not simultaneously zero, is called homogeneous eq. of Degree 2.

Now we show that this eq. always represents two straight lines.

$\because$ Homogeneous eq. of Degree 2 is

$$ax^2 + 2hxy + by^2 = 0 \quad \text{——— (1)}$$

$$\Rightarrow \quad by^2 + 2hxy + ax^2 = 0$$

$\div$ by x^2 $\quad b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$

which is quadratic in $\frac{y}{x}$. So by using quadratic formula, we have

$$\frac{y}{x} = \frac{-2h \pm \sqrt{4h^2 - 4ba}}{2b}$$

$$\frac{y}{x} = \frac{-2h \pm 2\sqrt{h^2 - ab}}{2b}$$

$$\frac{y}{x} = \frac{-h \pm \sqrt{h^2 - ab}}{b}$$

$$y = \frac{-h \pm \sqrt{h^2 - ab}}{b} \cdot x$$

So the given homogeneous eq. of Degree 2 will represents two straight lines which are given by

$$y = \frac{-h+\sqrt{h^2-ab}}{b}\cdot x \qquad \text{———— (2)}$$

and

$$y = \frac{-h-\sqrt{h^2-ab}}{b}\cdot x \qquad \text{———— (3)}$$

Now from (2) and (3) it is clear that the slopes of the lines are

$$m_1 = \frac{-h+\sqrt{h^2-ab}}{b}$$

$$m_2 = \frac{-h-\sqrt{h^2-ab}}{b}$$

if θ is the required angle then we know that

$$\tan\theta = \frac{|m_1 - m_2|}{1+m_1 m_2}$$

$$= \frac{-h+\sqrt{h^2-ab}/b - \dfrac{-h-\sqrt{h^2-ab}}{b}}{1+\left(\dfrac{-h+\sqrt{h^2-ab}}{b}\right)\left(\dfrac{-h-\sqrt{h^2-ab}}{b}\right)}$$

$$= \frac{\dfrac{-h+\sqrt{h^2-ab}+h+\sqrt{h^2-ab}}{b}}{\dfrac{b^2+(-h+\sqrt{h^2-ab})(-h-\sqrt{h^2-ab})}{b^2}}$$

$$= \frac{2\sqrt{h^2-ab}}{b}\times\frac{b^2}{b^2+(-h+\sqrt{h^2-ab})(-h-\sqrt{h^2-ab})}$$

$$= \frac{2b\sqrt{h^2-ab}}{b^2+h^2-(h^2-ab)}$$

$$= \frac{2b\sqrt{h^2-ab}}{b^2+h^2-h^2+ab}$$

$$= \frac{2b\sqrt{h^2-ab}}{b(a+b)}$$

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} \qquad \text{———— (4)}$$

Hence $\theta = \tan^{-1}\left(\dfrac{2\sqrt{h^2-ab}}{a+b}\right)$

Now if $\theta = 0$

$\Rightarrow \tan\theta = 0$

put in (4)

$$\Rightarrow \quad \frac{2\sqrt{h^2-ab}}{a+b} = 0$$

$$\sqrt{h^2-ab} = 0$$

$$h^2 = ab$$

Is the required condition for lines to be parallel.

Now if $\theta = 90^\circ$

$$\Rightarrow \tan 90^\circ = \infty$$

Put in (v)

$$\frac{2\sqrt{h^2-ab}}{a+b} = \infty$$

$$\Rightarrow \quad a+b = 0$$

is the required condition for perpendicularity.

<u>Question.</u> (Theorem 6.15)

Show that the general Eq. of 2nd in x and y always represents a conic section.

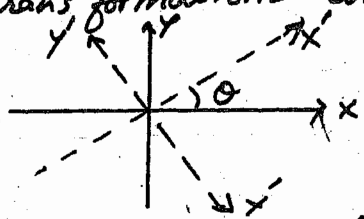
<u>Ans</u>: We know that the general Eq. of 2nd degree in x, y is given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{——— (1)}$$

where a, h and b are not simultaneously zero.

if the axis of coordinates rotate through an angle θ. Then we know that the eqs. of transformations are

$$X = x'\cos\theta - y'\sin\theta$$
$$Y = x'\sin\theta + y'\cos\theta$$



So from (1) we have

$$a(x'\cos\theta - y'\sin\theta)^2 + 2h(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 2g(x'\cos\theta - y'\sin\theta) + 2f(x'\sin\theta + y'\cos\theta) + c = 0$$

$$a(x'^2\cos^2\theta + y'^2\sin^2\theta - 2x'y'\cos\theta\sin\theta) + 2h(x'^2\cos\theta\sin\theta + x'y'\cos^2\theta - x'y'\sin^2\theta - y'^2\sin\theta\cos\theta) + b(x'^2\sin^2\theta + y'^2\cos^2\theta + 2x'y'\sin\theta\cos\theta) + 2gx'\cos\theta - 2gy'\sin\theta + 2fx'\sin\theta + 2fy'\cos\theta + c = 0$$

$$a x'^2 \cos^2\theta + a y'^2 \sin^2\theta - 2x'y' a \cos\theta \sin\theta + 2h x'^2 \cos\theta \sin\theta + 2h x'y' \cos^2\theta - 2h x'y' \sin^2\theta$$
$$- 2h y'^2 \sin\theta \cos\theta + b x'^2 \sin^2\theta + b y'^2 \cos^2\theta + 2 b x' \sin^2\theta + b y'^2 \cos^2\theta + 2b x'y' \sin\theta \cos\theta$$
$$+ 2g\, x' \cos\theta - 2g y' \sin\theta + 2 f x' \sin\theta + 2f y' \cos\theta + C = 0$$
$$x'^2 (a \cos^2\theta + 2h \cos\theta \sin\theta + b \sin^2\theta) + 2x'y' (b \sin\theta \cos\theta - h \sin^2\theta + h \cos^2\theta$$
$$- a \sin\theta \cos\theta) + y'^2 (a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta) + 2x' (g \cos\theta + f \sin\theta)$$
$$+ 2y' (f \cos\theta - g \sin\theta) + C = 0$$

$\Rightarrow$

$$x'^2 (a \cos^2\theta + 2h \cos\theta \sin\theta + b \sin^2\theta) + 2 x'y' [h (\cos^2\theta - \sin^2\theta) - (a-b) \sin\theta \cos\theta]$$
$$+ y'^2 (a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta) + 2x' (g \cos\theta + f \sin\theta) + 2y' [f \cos\theta$$
$$- g \sin\theta] + C = 0 \quad \text{——— (2)}$$

Now

Put
$$\left.\begin{aligned} A &= a \cos^2\theta + 2h \cos\theta \sin\theta + b \sin^2\theta \\ H &= h(\cos^2\theta - \sin^2\theta) - (a-b) \sin\theta \cos\theta \\ B &= a \sin^2\theta - 2h \sin\theta \cos\theta + b \cos^2\theta \\ G &= g \cos\theta + f \sin\theta \\ F &= f \cos\theta - g \sin\theta \end{aligned}\right\} \text{——— (3)}$$

So (2) can be written as
$$A x'^2 + 2H x'y' + B y'^2 + 2G x' + 2F y' + C = 0 \quad \text{——— (4)}$$

If we vanish term involving $x'y'$ in (2) then we have to set $\{h (\cos^2\theta - \sin^2\theta) - (a-b) \sin\theta \cos\theta\} = 0$

$\times$ by (2) we have
$$2h (\cos^2\theta - \sin^2\theta) - (a-b)\, 2\sin\theta \cos\theta = 0$$
$$2h (\cos 2\theta) - (a-b) \sin 2\theta = 0$$
$$2h \cos 2\theta = (a-b) \sin 2\theta$$
$$\Rightarrow \quad \frac{\sin 2\theta}{\cos 2\theta} = \frac{2h}{a-b}$$
$$\Rightarrow \quad \operatorname{Tan} 2\theta = \frac{2h}{a-b}$$

And then (4) can be written as
$$A x'^2 + B y'^2 + 2G x' + 2F y' + C = 0 \quad \text{——— (5)}$$

Written by
Shahid Javed

$$Ax'^2 + 2Gx' + By'^2 + 2Fy' = -C$$

$$A\left(x'^2 + \frac{2G x'}{A}\right) + B\left(y'^2 + \frac{2F}{B}y'\right) = -C$$

$$A\left(x'^2 + \frac{2G}{A}x' + \frac{G^2}{A^2} - \frac{G^2}{A^2}\right) + B\left(y'^2 + \frac{2F}{B}y' + \frac{F^2}{B^2} - \frac{F^2}{B^2}\right) = -C$$

$$A\left[\left(x' + \frac{G}{A}\right)^2 - \frac{G^2}{A^2}\right] + B\left[\left(y' + \frac{F}{B}\right)^2 - \frac{F^2}{B^2}\right] = -C$$

$$A\left(x' + \frac{G}{A}\right)^2 - \frac{G^2}{A} + B\left(y' + \frac{F}{B}\right)^2 - \frac{F^2}{B} = -C$$

$$A\left(x' + \frac{G}{A}\right)^2 + B\left(y' + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - C \quad \text{——— (6)}$$

<u>Case (1):</u> if $AB \neq 0$

$\Rightarrow \quad A \neq 0, \; B \neq 0$

So from (6), we have

$$AX^2 + BY^2 = C \quad \text{——— (7)}$$

where $X = x' + \frac{G}{A}$, $Y = y' + \frac{F}{B}$

and

$$C = \frac{G^2}{A} + \frac{F^2}{B} - c$$

(7) $\div$ by C, we have

$$\frac{X^2}{C/A} + \frac{Y^2}{C/B} = 1$$

$$\frac{X^2}{(\sqrt{C/A})^2} + \frac{Y^2}{(\sqrt{C/B})^2} = 1 \quad \text{——— (8)}$$

Now may discuss the following cases.

<u>Case (a):</u> If C/A and C/B are +ve and also

$C/A = C/B$ then (8) will represent a circle of the form

$$x^2 + y^2 = a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

Case (b). If $\frac{c}{A}$ and $\frac{c}{B}$ both are +ve and $\frac{c}{A} \neq \frac{c}{B}$ then (8) represent an ellipse.

Case (C) If both $\frac{c}{A}$ and $\frac{c}{B}$ are −ve then (8) will represent an imaginary ellipse.

Case (d) If $\frac{c}{A}$ and $\frac{c}{B}$ are of opposite sign then (8) will represent a hyperbola.

Case (E). If $c = 0$ then (8) can be written as

$$Ax^2 + By^2$$

which is homogeneous equation of Degree 2 will represent two straight line

Case 2. If $AB = 0$ then $A = 0$, $B = 0$ or $A = 0$, $B \neq 0$ or $A \neq 0$, $B = 0$

now we discuss there cases.

Case (A) both A and can not be zero because if so then from (5)

$$0 + 0 + 2G'x + 2Fy' + C = 0$$

$$2Gx' + 2Fy' + C = 0$$

which is linear and so it is a contradiction.

Case (B). If $A \neq 0$, $B \neq 0$ then (5) can be written as

$$A x'^2 + 0 + 2G x' + 2Fy' + C = 0$$

$$A\left(x'^2 + \frac{2G}{A}x'\right) = -2Fy' - C$$

$$A\left(x'^2 + \frac{2G}{A}x' + \frac{G^2}{A^2} - \frac{G^2}{A^2}\right) = -2Fy' - C$$

$$A\left[\left(x' + \frac{G}{A}\right)^2 - \frac{G^2}{A^2}\right] = -2Fy' + (-C)$$

$$A\left(x' + \frac{G}{A}\right)^2 - \frac{G^2}{A} = -2Fy' - C$$

$$A\left(x' + \frac{G}{A}\right)^2 = -2Fy' + \frac{G^2}{A} - C$$

$$A\left(x' + \frac{G}{A}\right)^2 = -2Fy' + \frac{G^2 - AC}{A}$$

$$A\left(x' + \frac{G}{A}\right)^2 = -2F\left(y' - \frac{G^2 - AC}{2FA}\right)$$

$$\left(x' + \frac{G}{A}\right)^2 = -\frac{2F}{A}\left(y' - \frac{G^2 - AC}{2FA}\right)$$

$$\left(x' + \frac{G}{A}\right)^2 = -\frac{4F}{2A}\left(y' - \frac{G^2 - AC}{2AF}\right)$$

$$\Rightarrow \quad X^2 = -\frac{4F}{2A} Y$$

where

$$X = x' - \frac{G}{A}, \quad Y = y' - \frac{G^2 - AC}{2AF}$$

which represent a parabola.

Case (2).

If $A \neq 0$, $B = 0$ in addition to this F is also equal to zero.

So $5) \Rightarrow \quad A x'^2 + 2Gx' + C = 0$

then this will represent two straight lines.

Case 3. If $A = 0$, $B \neq 0$ Then

$$5) \Rightarrow \quad 0 + By'^2 + 2Gx' + 2Fy' + C = 0$$

$$By'^2 + 2Fy' = -2Gx' - C$$

$$B\left(y'^2 + \frac{2F}{B}y'\right) = -2Gx' - C$$

$$B\left(y'^2 + \frac{2F}{B}y' + \frac{F^2}{B^2} - \frac{F^2}{B^2}\right) = -2Gx' - C$$

$$B\left[\left(y' + \frac{F}{B}\right)^2 - \frac{F^2}{B^2}\right] = -2Gx' - C$$

$$B\left(y' + \frac{F}{B}\right)^2 - \frac{F^2}{B} = -2Gx' - C$$

$$B\left(y' + \frac{F}{B}\right)^2 = -2Gx' + \frac{F^2}{B} - C$$

$$B\left(y' + \frac{F}{B}\right)^2 = -2Gx' + \frac{F^2 - BC}{B}$$

$$B\left(y' + \frac{F}{B}\right)^2 = -2G\left[x' - \frac{F^2 - BC}{2BG}\right]$$

$$\Rightarrow \quad \left(y' + \frac{F}{B}\right)^2 = -\frac{2G}{B}\left[x' - \frac{F^2 - BC}{2BG}\right]$$

$$\left(y' + \frac{F}{B}\right)^2 = -\frac{4G}{2B}\left[x' - \frac{F^2 - BC}{2BG}\right]$$

$$\Rightarrow \quad Y^2 = -4\frac{G}{2B} X$$

where $Y = y' + \frac{F}{B}$, $X = x' - \frac{F^2 - BC}{2BG}$

which represents a parabola.

Case 4. If $A=0$, $B\neq 0$ and $G=0$ then

(5) $\Rightarrow$ $\quad By'^2 + 2Fy' + C = 0$

Then this represents two straight lines and it is quadratic in y'.

From above discussion what ever the case the general equation of 2nd degree always represent a conic section. In above article we may discuss the notes.

Note 1.

$$H^2 - AB = 0$$

$$h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta - (a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)$$

$$= h\cos^2\theta - h\sin^2\theta - a\sin\theta\cos\theta + b\sin\theta\cos\theta - a^2\cos^2\theta\sin^2\theta - 2ha\cos^3\theta\sin\theta - ab\cos^4\theta - 2ha\cos\theta\sin^3\theta + 4h^2\sin^2\theta\cos^2\theta - 2hb\cos^3\theta\sin\theta$$

$$- ba\sin^4\theta + 2bh\sin^3\theta\cos\theta - b^2\sin^2\theta\cos^2\theta$$

$$= h^2 - ab$$

Similarly we may show that

$$A + B = a + b$$

these are called invariants of rotation.

Note 2. The invariants of rotation provide a rule to identify the conic which is as follows.

i) If $h^2 - ab > 0$

Then conic will hyperbola

ii) If $h^2 - ab < 0$

Then conic will an ellipse

iii) If $h^2 - ab = 0$

Then conic will be a parabola.

Theorem. Consider that $F_1(c,0)$; $F_2(-c,0)$ are the foci of an ellipse s.t. the sum of the distances of all points on the ellipse to the foci is $2a$ then prove that equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Proof: Consider that $p(x,y)$ is a pt. on the ellipse then given that

$$|PF_1|+|PF_2| = 2a \qquad \text{—— I}$$

$$|PF_1| = \sqrt{(x-c)^2+(y-0)^2}$$

$$|PF_2| = \sqrt{(x+c)^2+(y-0)^2}$$

Put in (1)

$$\sqrt{(x-c)^2+(y-0)^2}+\sqrt{(x+c)^2+y^2} = 2a$$

$$\sqrt{x^2+c^2-2cx+y^2}+\sqrt{x^2+c^2+2cx+y^2} = 2a$$

$$\sqrt{x^2+c^2-2cx+y^2} = 2a-\sqrt{x^2+c^2+2cx+y^2}$$

$$x^2+c^2-2cx+y^2 = \left[2a-\sqrt{x^2+c^2+2cx+y^2}\right]^2$$

$$x^2+c^2-2cx+y^2 = 4a^2+x^2+c^2+2cx+y^2-2a.2\sqrt{x^2+c^2+2cx+y^2}$$

$$-2cx = 4a^2+2cx-2(2a)\sqrt{x^2+c^2+2cx+y^2}$$

$$-2cx-4a^2-2cx = -4a\sqrt{x^2+c^2+2cx+y^2}$$

$$-4(cx+a^2) = -4a\sqrt{x^2+c^2+2cx+y^2}$$

$$cx+a^2 = a\sqrt{x^2+c^2+2cx+y^2}$$

$$c^2x^2+a^4+2a^2cx = a^2x^2+a^2c^2+2ca^2x+a^2y^2$$

$$a^2x^2-c^2x^2+a^2y^2+a^2c^2-a^4=0$$

$$(a^2-c^2)x^2+a^2y^2 = a^4-a^2c^2$$

$$(a^2-c^2)x^2+a^2y^2 = a^2(a^2-c^2)$$

$\div$ by $(a^2-c^2)a^2$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad \because a^2-c^2=b^2$$

Theorem. Consider that $F_1(c,0)$ and $F_2(-c,0)$ are the foci of a hyperbola s.t. difference between the distances from any arbitrary pt. on this hyperbola to the foci is $2a$. Then prove that eq. of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Proof: Prove your self.

Questions.

Find the equation of the tangent and equation of the normal to the following curves at the given pts.

i) $x^2 + y^2 = a^2$ at (x_1, y_1)

ii) $y^2 = 4ax$ at (x_1, y_1)

iii) $y^2 = 4ax$ at $(at^2, 2at)$

iv) $x^2 = 4ay$ at (x_1, y_1)

v) $x^2 = 4ay$ at $(2at, at^2)$

vi) $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1)

vii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1)

viii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\sin\theta, b\cos\theta)$

ix) $x^2/a^2 + y^2/b^2 = 1$ at $(a\cos\theta, b\sin\theta)$

x) $x^2/a^2 - y^2/b^2 = 1$ at (x_1, y_1)

xi) $x^2/a^2 - y^2/b^2 = 1$ at $(a\sec\theta, b\tan\theta)$

xii) $x^2/a^2 - y^2/b^2 = 1$ at $(a\cosh\theta, b\sinh\theta)$

Questions.

Find the equation of the tangent and normal at the given pts.

i) $x = at^2$, $y = 2at$

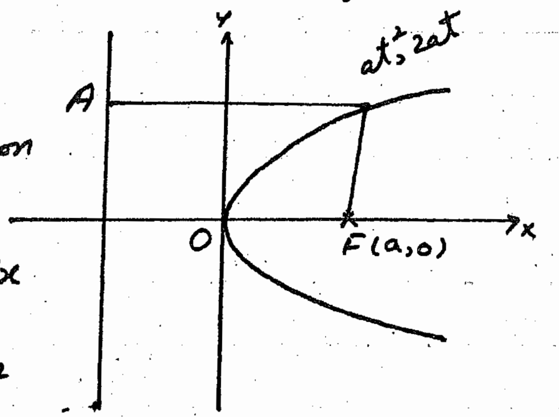
ii) $x = a\cos\theta$, $y = b\sin\theta$

iii) $x = a\sec\theta$, $y = b\tan\theta$

iv) $x = a\cosh\theta$, $y = b\sinh\theta$

Theorem.

Prove distance of a pt. P on the parabola from its focus is the same as distance of P from the directrix i.e. $|PF| = |PA|$



Proof:-

Let $P(at^2, 2at)$ be a pt. on the parabola $y^2 = 4ax$ having the focus $F(a,0)$ and directrix $x + a = 0$.

Then $|PF|^2 = (at^2 - a)^2 + (2at - 0)^2$

$= a^2t^4 + a^2 - 2a^2t^2 + 4a^2t^2$

$= a^2t^4 + a^2 + 2a^2t^2$

$= (a + at^2)^2$

$\Rightarrow |PF| = a + at^2$ ——— (1)

$|PA|$ = Distance of the pt. $P(at^2, 2at)$ from the line $x + a = 0$

$$= \frac{|at^2 + a|}{\sqrt{1^2 + 0^2}} \qquad \left| \; \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \right.$$

$|PA| = a + at^2$ ——— (2)

(1) and (2) $\Rightarrow$

$|PF| = |PA|$.

As required.

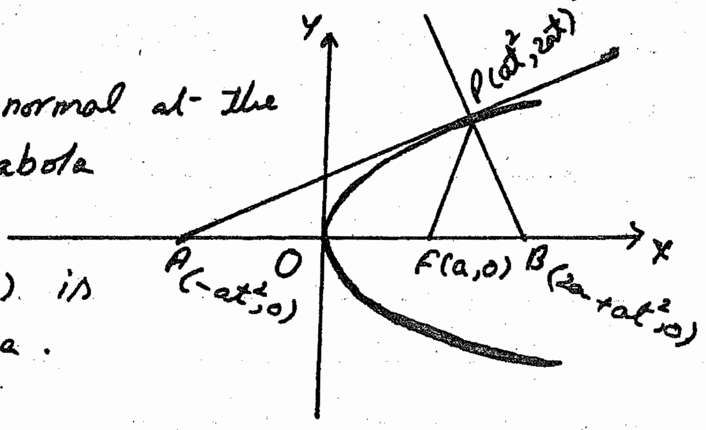
Theorem.

If the tangent and the normal at a pt. P of a parabola meets x-axis at A and B respectively then prove that

$|FP| = |FA| = |FB|$ where F is the focus.

Proof:-

Let the tangent and normal at the pt. $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ meet x-axis at A and B resp. $F(a,0)$ is the focus of the parabola.



We know that equation of the tangent at P is

$$x - ty + at^2 = 0$$ (See exp. 8)

For the coordinates of A put $y=0$ then

$$x - 0 + at^2 = 0$$
$$x = -at^2$$

$\Rightarrow$ Coordinates of $A(-at^2, 0)$

Similarly equation of the normal at P is

$$tx + y - 2at - at^3 = 0$$

For the coordinates of B put $y=0$, then

$$tx + 0 - 2at - at^3 = 0$$
$$tx = 2at + at^3$$
$$x = 2a + at^2$$

$\Rightarrow$ Coordinates of B are $(2a + at^2, 0)$

Now

$$|FP| = \sqrt{(at^2 - a)^2 + (2at - 0)^2}$$
$$= a + at^2 \quad \text{——— (1)}$$
$$|FA| = \sqrt{(a + at^2)^2 + (0-0)^2}$$
$$= a + at^2 \quad \text{——— (2)}$$
$$|FB| = \sqrt{(2a + at^2 - a)^2 + (0-0)^2}$$
$$= \sqrt{(a + at^2)^2}$$
$$= a + at^2 \quad \text{——— (3)}$$

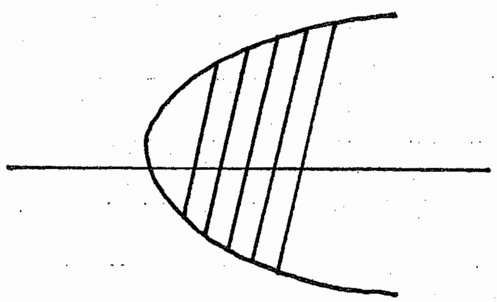
(1), (2) and (3) $\Rightarrow$

$$|FP| = |FA| = |FB|$$

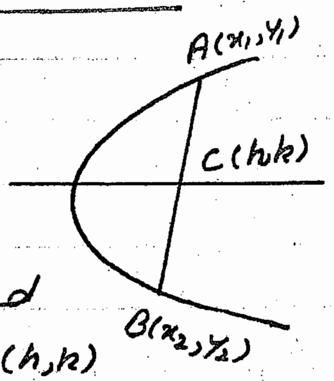
as required.

Diameter of a Parabola.

The locus of the middle pts. of parallel chords of a parabola is called the Diameter of the parabola.



Equation of the Diameter of the parabola.



Consider the parabola

$$y^2 = 4ax \quad \text{—— (1)}$$

Let AB be one of the parallel chords. We suppose that the coordinates of A and B are (x_1,y_1) and (x_2,y_2) resp. Let $C(h,k)$ be the mid point of this chord.

Consider that the equation of this chord is

$$y = mx + C$$

$$\Rightarrow \quad mx = y - C$$

$$x = \frac{y-C}{m} \quad \text{—— (2)}$$

Put in (1)

$$y^2 = 4a \cdot \frac{y-C}{m}$$

$$my^2 = 4ay - 4aC$$

$$my^2 - 4ay - 4aC = 0$$

which is quadratic in y thus if y_1 and y_2 are the roots of this eq. then

$$y_1 + y_2 = -\frac{-4a}{m}$$

$$y_1 + y_2 = \frac{4a}{m}$$

$$\frac{y_1+y_2}{2} = \frac{2a}{m} \quad \text{—— (3)}$$

$ax^2+bx+C=0$
$x_1 + x_2 = -\frac{b}{a}$

$\because$ $C(h,k)$ is the mid pt. of AB

$\therefore$ $h = \frac{x_1+x_2}{2}$ and $k = \frac{y_1+y_2}{2}$

Put in (3)

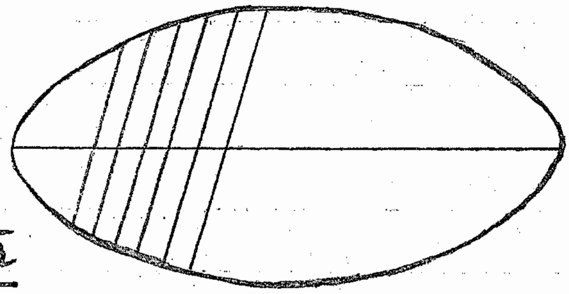
$$k = \frac{2a}{m}$$

Hence equation of the diameter

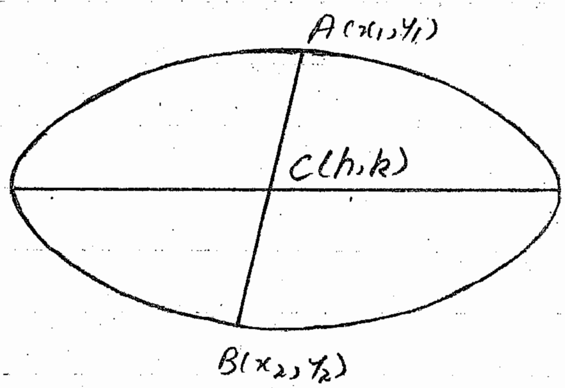
$$y = \frac{2a}{m}$$

Diameter of an ellipse :-

The locus of the middle pts. of parallel chords of an ellipse is called diameter of the ellipse.



Equation of the Diameter of an Ellipse :-



Consider that $\overline{AB}$ is one of the parallel chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —— (1)

Let A and B has coordinates (x_1, y_1) and (x_2, y_2) resp.

$C(h,k)$ is the mid point of $\overline{AB}$. Then $h = \frac{x_1+x_2}{2}$ and $k = \frac{y_1+y_2}{2}$ —— (2)

we consider that Eq. of the chord AB is

$$y = mx + C \quad —— (3)$$

Put in (1)

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$

$$b^2x^2 + a^2m^2x^2 + a^2c^2 + 2a^2cmx - a^2b^2 = 0$$

$$(b^2+a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$$

which is quadratic in x.

$\because$ If x_1 and x_2 are the roots of the equation then

$$x_1 + x_2 = -\frac{2mca^2}{b^2+a^2m^2}$$

$$\frac{x_1+x_2}{2} = -\frac{mca^2}{b^2+a^2m^2}$$

$$\Rightarrow \quad h = -\frac{mca^2}{b^2+a^2m^2} \quad —— (4)$$

$\because$ $C(h,k)$ lies on AB whose equation is

$$y = mx + C$$

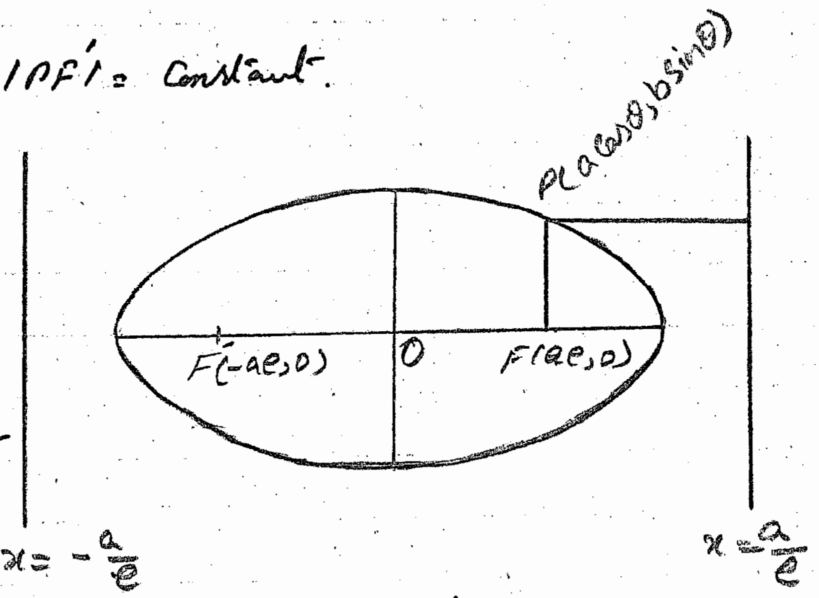
Then $$K = mh + C$$
$$K - mh = C$$
Put this value of C in (4)
$$\Rightarrow \quad h = -\frac{ma^2(K - mh)}{b^2 + a^2m^2}$$
$$h(b^2 + a^2m^2) = -ma^2(k - mh)$$
$$hb^2 + ha^2m^2 = -ma^2k + m^2a^2h$$
$$hb^2 = -ma^2k$$
$$K = -\frac{hb^2}{ma^2}$$
Thus the equation of the diameter is
$$y = \frac{-b^2}{ma^2}x$$

Theorem.

1) Prove that distance of a pt. P on an ellipse from the focus $= e$ times its distance from the corresponding directrix.

2) Also prove that
$$|PF| + |PF'| = \text{Constant}.$$

Proof:-



1) Let the pt. $P(a\cos\theta, b\sin\theta)$ on the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{——— (1)}$$
having the focus $(ae, 0)$ and $F'(-ae, 0)$. Centre at $O(0,0)$ and $x = \pm\frac{a}{e}$ its directrix.

Now $$|PF|^2 = (ae - a\cos\theta)^2 + (0 - b\sin\theta)^2$$
$$= a^2e^2 + a^2\cos^2\theta - 2a^2e\cos\theta + b^2\sin^2\theta$$
$$= a^2\cos^2\theta + a^2e^2 - 2a^2e\cos\theta + b^2(1 - \cos^2\theta)$$
$$= a^2\cos^2\theta + a^2e^2 - 2a^2e\cos\theta + b^2 - b^2\cos^2\theta$$
$$= (a^2 - b^2)\cos^2\theta + b^2 + a^2e^2 - 2a^2e\cos\theta \quad \text{——— (2)}$$
$$= (a^2 - b^2)\cos^2\theta + a^2 - 2a^2e\cos\theta \qquad \because a^2 - b^2 = a^2e^2$$
$$= a^2e^2\cos^2\theta + a^2 - 2a^2e\cos\theta \qquad a^2 = b^2 + a^2e^2$$
$$= a^2(e^2\cos^2\theta + 1 - 2e\cos\theta)$$

$$|PF|^2 = a^2(e\cos\theta - 1)^2$$

$$\Rightarrow |PF| = a(e\cos\theta - 1) \quad \text{------ (3)}$$

Now equation of the directrix is

$$x = \frac{a}{e}$$

$$ex = a$$

$$ex - a = 0 \quad \text{------ (4)}$$

Now $|PA|$ = Distance of the pt. P from the directrix.

$$|PA| = \frac{|ea\cos\theta - a|}{\sqrt{e^2 + 0^2}}$$

$$|PA| = \frac{a(e\cos\theta - 1)}{e}$$

$$e|PA| = a(e\cos\theta - 1)$$

$$e|PA| = |PF| \qquad \text{by (3)}$$

$$\Rightarrow |PF| = e|PA|$$

Hence distance of pt. P on an ellipse from the focus = e times its distance from the corresponding directrix.

2).

$$\because |PF|^2 = a^2(e\cos\theta - 1)^2 \quad \text{------ (5)}$$

$$\Rightarrow |PF|^2 = a^2(1 - e\cos\theta)^2$$

$$|PF| = a(1 - e\cos\theta) \quad \text{------ (6)}$$

Similarly $|PF'| = a(1 + e\cos\theta)$ ------ (7)

(7) + (6) $\Rightarrow$

$$|PF| + |PF'| = a(1 - e\cos\theta) + a(1 + e\cos\theta)$$

$$= a(1 - e\cos\theta + 1 + e\cos\theta)$$

$$= 2a$$

$$= \text{Const.}$$

= Length of the major axis of the ellipse.

Question.

Find the locus of the intersection of normals to the parabola $y^2=4ax$ inclined at right angle to each other.

Solution.

The equation of the parabola is

$$y^2=4ax \quad \text{—— (1)}$$

then the equation of the normal is

$$y = mx-2am-am^3$$

if the pt. $P(x_1,y_1)$ lies on it then

$$y_1 = mx_1-2am-am^3$$

$$\Rightarrow \quad am^3+0m^2+(2a-x_1)m+y_1=0 \quad \text{—— (2)}$$

if $ax^3+bx^2+cx+d=0$ and x_1,x_2,x_3 be the roots, then

$$x_1+x_2+x_3=(-1)^1\frac{b}{a}$$

$$x_1x_2+x_1x_3+x_2x_3=(-1)^2\frac{c}{a}$$

$$x_1x_2x_3=(-1)^3\frac{d}{a}$$

Now if m_1, m_2 and m_3 are the roots then

$$m_1m_2m_3=(-1)^3\frac{y_1}{a} \quad \text{—— (3)}$$

Since given that two of the normals are $\perp$

$\therefore \quad m_1m_2=-1$ put in (3)

$$-m_3=-\frac{y_1}{a}$$

$$\Rightarrow \quad m_3=\frac{y_1}{a}$$

But m_3 is the root of (2)

$$\therefore \quad a\cdot\frac{y_1^3}{a^3}+(2a-x_1)\frac{y_1}{a}+y_1=0$$

$$\frac{y_1^3}{a^2}+(2a-x_1)\frac{y_1}{a}+y_1=0$$

$$y_1^3+(2a-x_1)ay_1+a^2y_1=0$$

$$y_1^3+2a^2y_1-ax_1y_1+a^2y_1=0$$

$$y_1^3+3a^2y_1-ax_1y_1=0$$

Thus the locus of the pt. (x_1,y_1) is

$$y^3+3a^2y-axy=0$$

$$y^3+3a^2y-axy=0$$

$$y(y^2+3a^2-ax)=0$$

$$\Rightarrow \quad y=0, \quad y^2+3a^2-ax=0$$

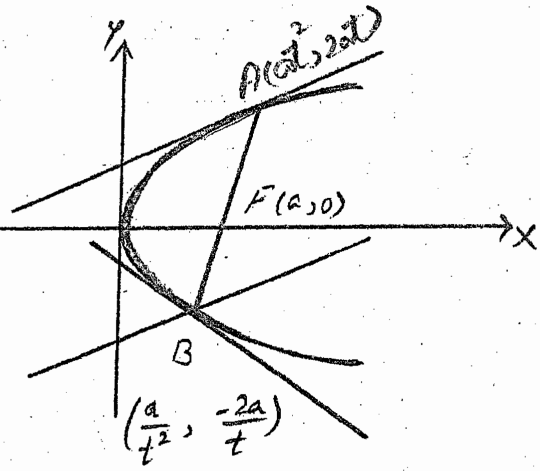
$$\Rightarrow \quad y=0, \quad y^2=ax-3a^2$$

$$\Rightarrow \quad y=0, \quad y^2=a(x-3a)$$

as required.

Question. Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

Solution.



Consider that the focal chord AB of the parabola $y^2 = 4ax$ —— (1)
We know that the coordinates of the extremities are as shown.

Diff. (1), we have

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \text{——— (2)}$$

Now

m_1 = slope of the tangent at A

$$m_1 = \frac{dy}{dx}\Big|_{(at^2, 2at)} = \frac{2a}{2at}$$

$$m_1 = \frac{1}{t} \quad \text{——— (3)}$$

Now m_2 = slope of the tangent at B

$$m_2 = \frac{dy}{dx}\Big|_{\left(\frac{a}{t^2}, -\frac{2a}{t}\right)} = \frac{2a}{-\frac{2a}{t}}$$

$$m_2 = -t \quad \text{——— (4)}$$

Let m_3 = slope of the normal at B

$$= -\frac{1}{m_2}$$

$$= -\frac{1}{-t} = \frac{1}{t} \quad \text{——— (5)}$$

From (3) and (5)

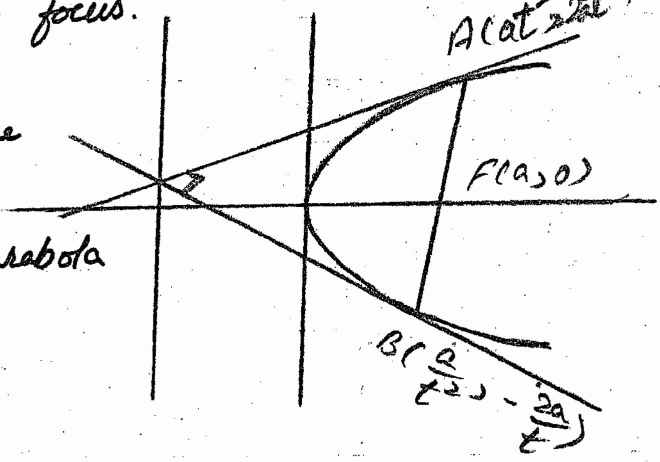
$$m_1 = m_3$$

$\Rightarrow$ The tangent at one extrimity of a focal chord of a parabola is parallel to the normal at the other extrimity.

Question.

Prove that ⊥or tangent to a parabola intersect on the directrix and the chord joining the pts. of contact passes through the focus.

Solution.



We know that the equation of the tangent to the parabola $y^2 = 4ax$ is

$$ty = x + at^2$$

$$x - ty = at^2$$

$$x - ty - at^2 = 0 \quad \text{——— (1)}$$

$$m_1 = \text{slope of the tangent} = -\frac{\text{Co-eff. of } x}{\text{Co-eff. of } y}$$

$$= -\frac{1}{-t}$$

$$= \frac{1}{t}$$

Now m_2 = slope of the tangent ⊥or to (1)

$$= -t$$

and so the equation of this tangent is

$$x + \frac{1}{t}y + \frac{a}{t^2} = 0 \qquad (\text{replace by } -\tfrac{1}{t})$$

$$t^2x + ty + a = 0 \quad \text{——— (2)}$$

(1)+(2)

$$x - ty + at^2 = 0$$

$$t^2x + ty + a = 0$$

$$x(1+t^2) + a(1+t^2) = 0$$

$$\Rightarrow (1+t^2)(x+a) = 0$$

$\Rightarrow$ $x + a = 0$ which is the equation of the directrix.

Hence the ⊥or tangent to a parabola intersect on directrix.

Now points of contact are

$A(at^2, 2at)$ and $B\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

Eq. of the line passing through A and B is

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$$

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

Putting the values

$$y-2at = \frac{-\frac{2a}{t} - 2at}{\frac{a}{t^2} - at^2}(x-at^2)$$

$$y-2at = \frac{\frac{-2a-2at^2}{t}}{\frac{a-at^4}{t^2}}(x-at^2)$$

$$\Rightarrow \quad y-2at = \frac{-2a(1+t^2)}{t}\cdot\frac{t^2}{a(1-t^4)}(x-at^2)$$

$$y-2at = \frac{-2(1+t^2)t}{(1-t^2)(1+t^2)}(x-at^2)$$

$$y-2at = \frac{2t}{1-t^2}(x-at^2) \quad \text{———} \ (4)$$

Now the focus is $(a,0)$ put in (4)

$$0-2at = -\frac{2t}{1-t^2}(a-at^2)$$

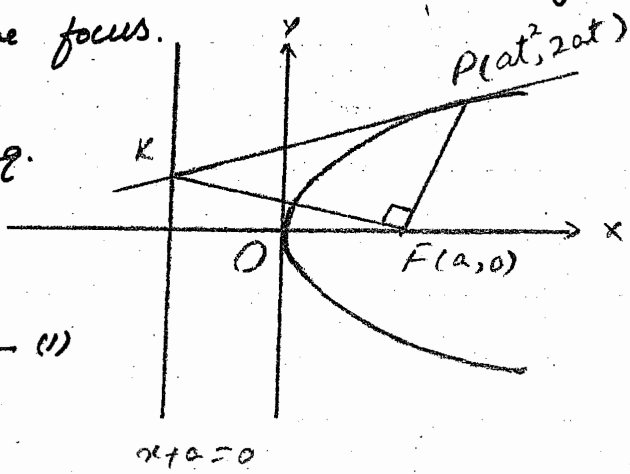
$$-2at = -\frac{2t}{1-t^2}a(1-t^2)$$

$$-2at = -2at$$

$\Rightarrow$ The chord joining A and B passes through the focus.

Question. If the tangent at P of a parabola meets the directrix at K, then prove that PFK is right angle, where F is the focus.

Solution



We know that the eq. of the tangent to the Parabola at P is

$$ty = x + at^2 \quad \text{———} \ (1)$$

but at K, $x = -a$

put in (1)

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$$-a - ty + at^2 = 0$$
$$yt = at^2 - a$$
$$y = \frac{a(t^2-1)}{t} \Rightarrow$$ the coordinate of the Point K.

$$\left(-a, \left(\frac{at^2-a}{t}\right)\right)$$

Now m_1 = slope of $\overline{PF} = \frac{0-2at}{a-at^2} = \frac{-2at}{a-at^2}$

also

$$m_2 = \text{slope of } \overline{FK} = \frac{\frac{at^2-a}{t} - 0}{-a-a}$$
$$= \frac{at^2-a}{-2at}$$
$$= \frac{-a+at^2}{-2at} = \frac{a-at^2}{2at}$$

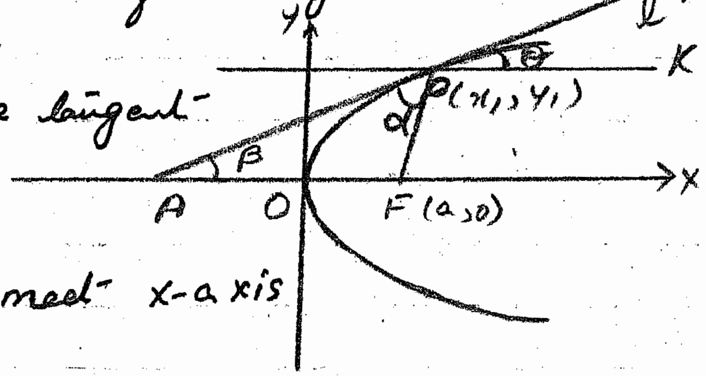
Consider

$$m_1 \cdot m_2 = \frac{-2at}{a-at^2} \times \frac{a-at^2}{2at}$$
$$= -1$$
$$\Rightarrow \overline{PF} \perp \overline{KP}$$
$$P\hat{F}K = 90^\circ$$

as required.

Question. If l is the tangent line at $P(x_1, y_1)$ of a parabola $y^2 = 4ax$ and if k is a line through P $\parallel$ to x-axis. Show that the measure of the angle b/w k and l is equal to the measure of the angle between l and $\overline{PF}$

Solution.



Consider that the tangent at the pt. (x_1, y_1) of the parabola

$$y^2 = 4ax$$ meet x-axis at A. Now

$$|PF| = |FA| \quad (\because \text{Reflective prop.})$$
$$\Rightarrow \triangle AFP \text{ is isoceless}$$
$$\Rightarrow \alpha = \beta \quad \text{———} \quad (1)$$

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Also by elementry geometry

$$\beta = \theta$$

Put in (1) $\Rightarrow$

$$\alpha = \theta$$

as required

Question.

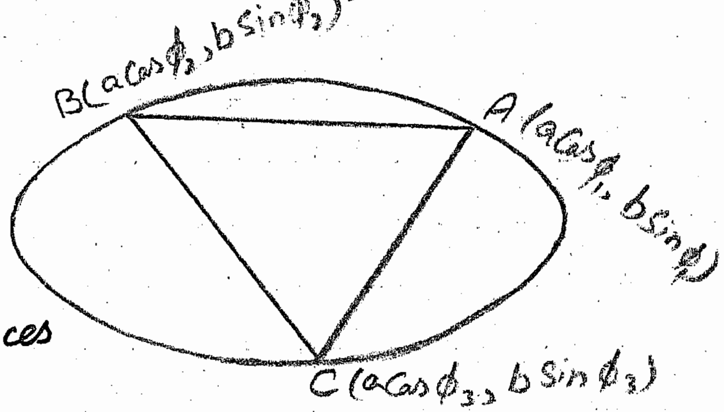
If ϕ_1, ϕ_2 and ϕ_3 are the ecentric angles of vertices of a Δ inscribed in an ellipse, fin the Area.

Solution.

Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If A, B, C are the vertices of the said triangle, the ecentric angles of the vertices are given.



$\Rightarrow$ The coordinates of these values will be as shown.

$\therefore$ Area of ΔABC is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} a\cos\phi_1 & b\sin\phi_1 & 1 \\ a\cos\phi_2 & b\sin\phi_2 & 1 \\ a\cos\phi_3 & b\sin\phi_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[a\cos\phi_1\{b\sin\phi_2 - b\sin\phi_3\} - b\sin\phi_1\{a\cos\phi_2 - a\cos\phi_3\} + 1\{ab\cos\phi_2\sin\phi_3 - ab\sin\phi_2\cos\phi_3\}\right]$$

simplify to get result.

[Additional work is over]