Techniques of Integration

EXERCISE 4.3

Integration By Parts means 4 brackets in such a way that:

$(\ )(\ ) - \int (\ )(\ )dx$

$(F1)\ (integral\ of\ F2)$

$- \int (derivative\ of\ F1)(integral\ of\ F2)dx$

Or

$(F1)\left(\int F2.dx\right)$

$- \int (F1')\left(\int F2.dx\right)dx$

Q No. 1 $I = \int x sec^2 x dx$

Applying By Parts

$$I = (x)(tanx) - \int (1)(tanx)dx$$

$$I = xtanx - \int tanxdx$$

$$I = xtanx - \ln(secx) + c$$

Q No. 2 $I = \int x csc^2 x dx$

$$I = (x)(-cotx) - \int (1)(-cotx)dx$$

$$I = -xcotx + \int cotxdx$$

$$I = -xcotx + \ln(sinx) + c$$

Q No. 3 $I = \int x^n lnx dx$

$$I = (lnx)\left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^{n+1}}{n+1}\right) dx$$

$$I = (lnx)\left(\frac{x^{n+1}}{n+1}\right) - \frac{1}{n+1}\int x^n dx$$

$$I = \frac{x^{n+1}(lnx)}{n+1} - \frac{1}{n+1}.\frac{x^{n+1}}{n+1}$$

$$I = \frac{x^{n+1}(lnx)}{n+1} - \frac{x^{n+1}}{(n+1)^2} + c$$

Q No. 4 $I = \int x^2 \tan^{-1} x\, dx$

$$I = (\tan^{-1} x)\left(\frac{x^{2+1}}{2+1}\right) - \int \left(\frac{1}{x^2+1}\right)\left(\frac{x^{2+1}}{2+1}\right) dx$$

$$I = (\tan^{-1} x)\left(\frac{x^3}{3}\right) - \frac{1}{3}\int \left(\frac{x^3}{x^2+1}\right) dx$$

In $\frac{x^3}{x^2+1}$ we use long division and get $x - \frac{x}{x^2+1}$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int \left(x - \frac{x}{x^2+1}\right) dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}\int xdx + \frac{1}{3}\int \left(\frac{x}{x^2+1}\right) dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{1}{3}.\frac{x^2}{2} + \frac{1}{3}.\frac{1}{2}\int \left(\frac{2x}{x^2+1}\right) dx$$

$$I = \frac{x^3}{3}(\tan^{-1} x) - \frac{x^2}{6} + \frac{1}{6}\ln(x^2+1) + c$$

Q No.5 $I = \int sec^3 x dx$

$$I = \int secx.sec^2 xdx$$

$$I = (secx)(tanx) - \int (secx.tanx)(tanx)dx$$

$$I = secxtanx - \int secx.(tan^2 x)dx$$

$$I = secxtanx - \int secx.(sec^2 x - 1)dx$$

$$I = secxtanx - \int sec^3 xdx + \int secxdx$$

Here we have,

$$I = secxtanx - I + \ln(secx + tanx)$$

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$$I + I = secxtanx + \ln(secx + tanx)$$

$$2I = secxtanx + \ln(secx + tanx)$$

$$I = \frac{1}{2}secxtanx + \frac{1}{2}\ln(secx + tanx) + c$$

Q No. 6 $I = \int csc^3 x dx$

$$I = \int cscx.csc^2 x dx$$

$$I = (cscx)(-cotx) - \int (-cscxcotx)(-cotx)dx$$

$$I = -cscxcotx - \int cscx(cot^2 x)dx$$

$$I = -cscxcotx - \int cscx(csc^2 x - 1)dx$$

$$I = -cscxcotx - \int csc^3 x dx + \int cscxdx$$

$$I = cscxcotx - I + \ln(cscx - cotx)$$

$$I + I = -cscxcotx + \ln(cscx - cotx)$$

$$2I = -cscxcotx + \ln(cscx - cotx)$$

$$I = \frac{-1}{2}cscxcotx + \frac{1}{2}\ln(cscx - cotx) + c$$

Q No. 7 $I = \int \frac{x - sinx}{1 - cosx} dx$

In trigonometry we write:

$\sin^2\left(\frac{x}{2}\right) = \frac{1-cosx}{2}$ and $sinx = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

So,

$$I = \int \frac{x - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)} dx$$

$$I = \int \frac{x}{2\sin^2\left(\frac{x}{2}\right)} dx - \int \frac{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)} dx$$

$$I = \frac{1}{2}\int xcsc^2\left(\frac{x}{2}\right)dx - \int \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx$$

$$I = \frac{1}{2}\int xcsc^2\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

$$I = \frac{1}{2}\int xcsc^2\left(\frac{x}{2}\right)dx - 2\ln\left(sin\frac{x}{2}\right) \quad \text{------(1)}$$

$$I_1 = \int xcsc^2\left(\frac{x}{2}\right)dx$$

$$I_1 = (x)\left(-2\cot\frac{x}{2}\right) - \int (1)\left(-2\cot\frac{x}{2}\right)dx$$

$$I_1 = -2x\cot\frac{x}{2} + 2\int \cot\frac{x}{2}dx$$

$$I_1 = -2x\cot\frac{x}{2} + 2.2\ln(sinx)$$

$$I_1 = -2x\cot\frac{x}{2} + 4\ln(sinx)$$

Hence,

$$I = \frac{1}{2}\left[-2x\cot\frac{x}{2} + 4\ln\left(sin\frac{x}{2}\right)\right] - 2\ln\left(sin\frac{x}{2}\right)$$

$$I = -x\cot\frac{x}{2} + 2\ln\left(sin\frac{x}{2}\right) - 2\ln\left(sin\frac{x}{2}\right) + c$$

$$I = -x\cot\frac{x}{2} + c$$

Alternative method (after step 4):

$$I = \frac{1}{2}\int xcsc^2\left(\frac{x}{2}\right)dx - \int \frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx$$

$$I = \frac{1}{2}\int xcsc^2\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

Applying by parts on first

$$I = \frac{1}{2}\left[(x)\left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right) - \int (1)\left(\frac{-\cot\frac{x}{2}}{\frac{1}{2}}\right)dx\right] - \int \cot\left(\frac{x}{2}\right)dx$$

$$I = -x\cot\frac{x}{2} + \int \cot\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

$$I = -xcot\frac{x}{2} + c$$

Q No. 8 $I = \int x\sin^{-1}x\, dx$

$$I = (\sin^{-1}x)\left(\frac{x^2}{2}\right) - \int \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{x^2}{2}\right)dx$$

$$I = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}dx$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \left[\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right]dx$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \left[\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right]dx$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{1-x^2}}$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x - \frac{1}{2}\sin^{-1}x$$

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} - \frac{1}{4}\sin^{-1}x + c$$

Q No. 9 $I = \int x^3\sqrt{x^2+1}\, dx$

$$I = \int x^2.x\sqrt{x^2+1}\,dx$$

$$I = \int (x^2+1-1).x\sqrt{x^2+1}\,dx$$

$$I=\int(x^2+1).x\sqrt{x^2+1}\,dx-\int x\sqrt{x^2+1}\,dx$$

$$I=\int(x^2+1)^{1+\frac{1}{2}}.x\,dx-\int(x^2+1)^{\frac{1}{2}}x\,dx$$

$$I=\frac{1}{2}\int(x^2+1)^{\frac{3}{2}}.2x\,dx-\frac{1}{2}\int(x^2+1)^{\frac{1}{2}}.2x\,dx$$

$$I=\frac{1}{2}.\frac{(x^2+1)^{\frac{3}{2}+1}}{\frac{3}{2}+1}-\frac{1}{2}.\frac{(x^2+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c$$

$$I=\frac{1}{2}.\frac{(x^2+1)^{\frac{5}{2}}}{\frac{5}{2}}-\frac{1}{2}.\frac{(x^2+1)^{\frac{3}{2}}}{\frac{3}{2}}+c$$

$$I=\frac{(x^2+1)^{\frac{5}{2}}}{5}-\frac{(x^2+1)^{\frac{3}{2}}}{3}+c$$

Alternative method of QNo. 9 is at the end of the exercise (page9)

Q No. 10 $I=\int e^x\frac{1+xlnx}{x}\,dx$

$$I=\int e^x\frac{1}{x}dx+\int e^x\,lnxdx$$

Applying by parts on first angle,

$$I=(e^x)(lnx)-\int(e^x)(lnx)dx+\int e^x\,lnxdx$$

$$I=e^xlnx+c$$

Q No. 11 $I=\int e^x\frac{1-sinx}{1-cosx}dx$

$$I=\int e^x.\left(\frac{1-sinx}{1-cosx}\times\frac{1+cosx}{1+cosx}\right)dx$$

$$I=\int e^x.\left(\frac{1-sinx+cosx-sinxcosx}{1-cos^2x}\right)dx$$

$$I=\int e^x.\left(\frac{1-sinx+cosx-sinxcosx}{sin^2x}\right)dx$$

$$I=\int e^x\left(\frac{1}{sin^2x}-\frac{sinx}{sin^2x}+\frac{cosx}{sin^2x}-\frac{sinxcosx}{sin^2x}\right)dx$$

$$I=\int e^x\,(csc^2x-cscx+cscxcotx-cotx)dx$$

$$I=\int e^xcsc^2x\,dx-\int e^x\,cscx\,dx+\int e^xcscxcotx\,dx-\int e^x\,cotx\,dx$$

Hence our integral becomes as follows,

We will apply By Parts technique upon 1st and 3rd integral:

$$I_1=\int e^xcsc^2x\,dx$$

$$=(e^x)(-cotx)-\int(e^x)\,(-cotx)\,dx$$

$$=-e^xcotx+\int e^xcotxdx \quad \text{------- (1)}$$

And

$$I_3=\int e^xcscxcotx\,dx$$

$$=(e^x)(-cscx)-\int(e^x)\,(-cscx)\,dx$$

$$=-e^xcscx+\int e^xcscxdx \quad \text{------- (2)}$$

Putting values in I we get:

$$I=-e^xcotx-e^xcscx+c$$

Q No. 12 $I=\int\tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

$$Put\ \tan^{-1}\sqrt{\frac{1-x}{1+x}}=z\ ---\rightarrow(1)$$

$$\sqrt{\frac{1-x}{1+x}}=tanz$$

$$\frac{1-x}{1+x}=tan^2z$$

$$(1-x)=tan^2z(1+x)$$

$$1-x=tan^2z+xtan^2z$$

$$1-tan^2z=x+xtan^2z$$

$$1-tan^2z=x(1+tan^2z)$$

Multiply D[r] and N[r] by cos^2z

$$\frac{1-tan^2z}{1+tan^2z}=x\quad\Rightarrow\quad\frac{cos^2z-sin^2z}{cos^2z+sin^2z}=x$$

$$\Rightarrow\quad\frac{cos2z}{1}=x\quad\Rightarrow\ cos2z=x\qquad ---\rightarrow(2)$$

Diff. w.r.t x

$$-2sin2zdz=dx$$

$$I=\int z.(-2sin2z)dz$$

$$I=-2\int zsin2z\,dz$$

Applying integration by parts

$$I=-2(z)\left(-\frac{cos2z}{2}\right)+2\int(1)\left(-\frac{cos2z}{2}\right)dz$$

$$I=zcos2z-\int cos2zdz$$

$$I=zcos2z-\frac{sin2z}{2}+c$$

Re-back substitution, using eqs. (1) & (2)

$$I=\tan^{-1}\sqrt{\frac{1-x}{1+x}}.x-\frac{\sqrt{1-x^2}}{2}+c$$

Q No. 13 $I=\int\sin^{-1}\sqrt{\frac{x}{x+a}}dx$

$$Put\ \sin^{-1}\sqrt{\frac{x}{x+a}}=z-----\rightarrow(1)$$

$$\sqrt{\frac{x}{x+a}}=sinz$$

$$\frac{x}{x+a}=sin^2z$$

$$x=sin^2z(x+a)$$

$$x=xsin^2z+asin^2z$$

$$x-xsin^2z=asin^2z$$

$$x(1-sin^2z)=asin^2z$$

$$xcos^2z=asin^2z$$

$$x=atan^2z---\rightarrow(2)$$

$$dx=2atanz\,sec^2z\,dz$$

Hence our integral become,

$$I=\int z\,(2atanz\,sec^2z)\,dz$$

$$I=2a\int ztanz\,sec^2z\,dz$$

$$I=2a(z)\left(\frac{tan^2z}{2}\right)-2a\int(1)\left(\frac{tan^2z}{2}\right)dz$$

$$I=aztan^2z-a\int tan^2z\,dz$$

$$I=aztan^2z-a\int(sec^2z-1)\,dz$$

$$I=aztan^2z-a\int sec^2zdz+a\int dz$$

$$I=aztan^2z-atanz+az+c$$

Re-back substitution, using eqs. (1) and (2)

$$I=a\frac{x}{a}\sin^{-1}\sqrt{\frac{x}{x+a}}-a\sqrt{\frac{x}{a}}+a\sin^{-1}\sqrt{\frac{x}{x+a}}+c$$

$$I=a\sin^{-1}\sqrt{\frac{x}{x+a}}-\sqrt{ax}+a\sin^{-1}\sqrt{\frac{x}{x+a}}+c$$

$$I=2a\sin^{-1}\sqrt{\frac{x}{x+a}}-\sqrt{ax}+c$$

Q No. 14 $I=\int e^{ax}\sin(bx+c)dx$

Using by parts formula,

$$I=(e^{ax})\left(\frac{-\cos(bx+c)}{b}\right)-\int(ae^{ax})\left(\frac{-\cos(bx+c)}{b}\right)dx$$

$$I=\frac{-e^{ax}\cos(bx+c)}{b}+\frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

Using by parts formula,

$$I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b}(e^{ax})\left(\frac{\sin(bx+c)}{b}\right)-\frac{a}{b}\int(ae^{ax})\left(\frac{\sin(bx+c)}{b}\right)dx$$

$$I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)-\frac{a^2}{b^2}\int e^{ax}\sin(bx+c)\,dx$$

$$I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)-\frac{a^2}{b^2}I$$

$$I+\frac{a^2}{b^2}I=-\frac{1}{b}e^{ax}\cos(bx+c)+\frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$\left(1+\frac{a^2}{b^2}\right)I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$\left(\frac{a^2+b^2}{b^2}\right)I = -\frac{1}{b}e^{ax}\cos(bx+c) + \frac{a}{b^2}e^{ax}\sin(bx+c)$$

$$I = -\frac{b}{a^2+b^2}e^{ax}\cos(bx+c) + \frac{1}{a^2+b^2}e^{ax}\sin(bx+c)$$

Q No. 15 $\boldsymbol{I = \int ln\left(x+\sqrt{1+x^2}\right)dx}$

$$I = \int 1.ln(x+\sqrt{1+x^2})dx$$

$$I = \left(ln(x+\sqrt{1+x^2})\right)(x) - \int \left(\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{(x+\sqrt{1+x^2})}\right)(x)dx$$

$$I = \left(ln(x+\sqrt{1+x^2})\right)(x) - \int \left(\frac{\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}}{(x+\sqrt{1+x^2})}\right)(x)dx$$

$$I = \left(ln(x+\sqrt{1+x^2})\right)(x) - \int \left(\frac{x}{\sqrt{1+x^2}}\right)dx$$

$$I = xln(x+\sqrt{1+x^2}) - \int (1+x^2)^{-\frac{1}{2}}xdx$$

$$I = xln(x+\sqrt{1+x^2}) - \frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}.2xdx$$

$$I = xln\left(x+\sqrt{1+x^2}\right) - \frac{1}{2}.\frac{(1+x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$I = xln\left(x+\sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

Q No. 16 $\boldsymbol{I = \int \frac{x^2+1}{(x+1)^2}e^x\,dx}$

$$I = \int \frac{x^2+1}{(x+1)^2}e^x\,dx$$

$$I = \int \frac{x^2+1+2x-2x}{(x+1)^2}e^x\,dx$$

$$I = \int \frac{(x+1)^2-2x}{(x+1)^2}e^x\,dx$$

$$I = \int e^x dx - \int \frac{2xe^x}{(x+1)^2}dx$$

$$I = e^x - \int \frac{2xe^x}{x^2+2x+1}dx$$

$$I = e^x - \int \frac{(2x+2-2)e^x}{(x+1)^2}dx$$

$$I = e^x - \int \frac{(2x+2)e^x}{(x+1)^2}dx + 2\int \frac{e^x}{(x+1)^2}dx$$

$$I = e^x - 2\int \frac{(x+1)e^x}{(x+1)^2}dx + 2\int \frac{e^x}{(x+1)^2}dx$$

$$I = e^x - 2\int \frac{e^x}{x+1}dx + 2\int \frac{e^x}{(x+1)^2}dx$$

Integrating first integral by parts,

$$I = e^x - 2\left(\frac{1}{x+1}\right)(e^x) + 2\int \frac{(-1)}{(x+1)^2}.(e^x)dx + 2\int \frac{e^x}{(x+1)^2}dx$$

$$I = e^x - \frac{2e^x}{x+1} - 2\int \frac{e^x}{(x+1)^2}dx + 2\int \frac{e^x}{(x+1)^2}dx$$

$$I = e^x - \frac{2e^x}{x+1} + c$$

Q No. 17 $\boldsymbol{I = \int cos(lnx)\,dx}$

$$I = (\cos(lnx))(x) - \int \frac{-\sin(lnx)}{x}(x)dx$$

$$I = x.\cos(lnx) + \int \sin(lnx)\, dx$$

Integrating again by parts,

$$I = x.\cos(lnx) + (sinlnx)(x) - \int \cos\frac{(lnx)}{x}(x)\, dx$$

$$I = x[\cos(lnx) + \sin(lnx)] - \int \cos(lnx)\, dx$$

$$I = x[\cos(lnx) + \sin(lnx)] - I$$

$$2I = x[\cos(lnx) + \sin(lnx)]$$

$$I = \frac{x}{2}[\cos(lnx) + \sin(lnx)]$$

Q No. 18 $\boldsymbol{I = \int \sqrt{x}\, e^{-\sqrt{x}} dx}$

$$Put\ \sqrt{x} = z \ \Rightarrow x = z^2 \ \Rightarrow \ dx = 2zdz$$

$$I = \int ze^{-z}.2zdz \ = 2\int z^2e^{-z}dz$$

Integrating by parts

$$I = 2(z^2)(-e^{-z}) - 2\int 2z.(-e^{-z})dz$$

$$I = -2z^2e^{-z} + 4\int ze^{-z}dz$$

Integrating by parts again

$$I = -2z^2e^{-z} + 4(z)(-e^{-z}) - 4\int (1)(-e^{-z})dz$$

$$I = -2z^2e^{-z} - 4ze^{-z} + 4\int e^{-z}dz$$

$$I = -2z^2e^{-z} - 4ze^{-z} - 4e^{-z} + c$$

Hence

$$I = -2xe^{-\sqrt{x}} - 4\sqrt{x}e^{-\sqrt{x}} - 4e^{-\sqrt{x}} + c$$

Q No. 19 $\boldsymbol{I = \int x^3e^{2x}\, dx}$

$$I = \int x^3e^{2x}dx$$

Integrating by parts,

$$I = (x^3)\left(\frac{e^{2x}}{2}\right) - \int (3x^2)\left(\frac{e^{2x}}{2}\right)dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{2}\int x^2e^{2x}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{2}(x^2)\left(\frac{e^{2x}}{2}\right) + \frac{3}{2}\int 2x.\frac{e^{2x}}{2}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\int xe^{2x}dx$$

$$= \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3}{2}(x)\left(\frac{e^{2x}}{2}\right) - \frac{3}{2}\int (1)\left(\frac{e^{2x}}{2}\right)dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3x}{4}e^{2x} - \frac{3}{4}\int e^{2x}dx$$

$$I = \frac{x^3}{2}e^{2x} - \frac{3x^2}{4}e^{2x} + \frac{3x}{4}e^{2x} - \frac{3}{4}.\frac{e^{2x}}{2} + c$$

Q No. 20 $\boldsymbol{I = \int x^5e^{x^3}dx}$

$$I = \int x^3\, x^2e^{x^3}dx$$

$$Put\ x^3 = z \qquad \Rightarrow \ 3x^2dx = dz \qquad \Rightarrow x^2dx = \frac{dz}{3}$$

Hence our integral becomes:

$$I = \int ze^z\frac{dz}{3} \quad = \frac{1}{3}\int ze^z\, dz$$

Apllying by parts,

$$I = \frac{1}{3}(z)(e^z) - \frac{1}{3}\int (1)(e^z)dx$$

$$I = \frac{ze^z}{3} - \frac{1}{3}\int e^z\, dz$$

$$I = \frac{ze^z}{3} - \frac{e^z}{3} + c$$

$$I = \frac{x^3e^{x^3}}{3} - \frac{e^{x^3}}{3} + c$$

Q No. 21 Show that

$$\int x^n tan^{-1}x\, dx = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}dx$$

Hence evaluate, $\int x^3 \tan^{-1} x\, dx$

Let,

$$I = \int x^n tan^{-1}x\, dx$$

Integrating by parts,

$$I = (tan^{-1}x)\left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{1}{1+x^2}\right)\left(\frac{x^{n+1}}{n+1}\right)dx$$

$$I = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}dx$$

As required.

Now, put n=3

$$I = \frac{x^{3+1}}{3+1}\tan^{-1}x - \frac{1}{3+1}\int \frac{x^{3+1}}{1+x^2}dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2}dx$$

By long division we get,

$$\frac{x^4}{1+x^2} = x^2 - 1 + \frac{1}{1+x^2}$$

So I becomes,

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \left(x^2 - 1 + \frac{1}{1+x^2}\right)dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2\, dx + \frac{1}{4}\int dx - \frac{1}{4}\int \frac{dx}{1+x^2}$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}.\frac{x^3}{3} + \frac{x}{4} - \frac{1}{4}\tan^{-1}x + c$$

Q No. 22 Find a reduction formula for $\int x^n e^{ax} dx$ and apply it to evaluate $\int x^3 e^{ax} dx$.

$$I = \int x^n e^{ax} dx$$

Applying by parts formula,

$$I = (x^n)\left(\frac{e^{ax}}{a}\right) - \int (nx^{n-1})\left(\frac{e^{ax}}{a}\right)dx$$

$$I = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\, dx$$

Which is the required reduction formula.

Now, put $n = 3$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^{3-1} e^{ax}\, dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\, dx$$

Again put $n - 2$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x^{2-1} e^{ax}\, dx\right]$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\int x\, e^{ax}\, dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}(x)\left(\frac{e^{ax}}{a}\right) - \frac{6}{a^2}\int (1)\left(\frac{e^{ax}}{a}\right)dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6}{a^3}\int e^{ax}dx$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6}{a^3}.\frac{e^{ax}}{a} + c$$

$$I = \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6xe^{ax}}{a^3} - \frac{6e^{ax}}{a^3} + c$$

Q No. 23 Find a reduction formula for $\int \sin^n x dx$ and $\int \cos^n x dx$ where n is a positive integer.

$$I = \int \sin^n x dx$$

We separate a single powerof sinx. As follows:

$$I = \int \sin^{n-1}x \, \sin x dx$$

Applying by parts formula

$$I = (\sin^{n-1}x)(-\cos x) - \int (n-1)(\sin^{n-2}x.\cos x)(-\cos x)dx$$

$$I = -\cos x.\sin^{n-1}x + (n-1)\int \sin^{n-2}x.\cos^2 x \, dx$$

$$I = -\cos x.\sin^{n-1}x + (n-1)\int \sin^{n-2}x.(1-\sin^2 x) \, dx$$

$$= -\cos x.\sin^{n-1}x + (n-1)\int \sin^{n-2}x dx - (n-1)\int \sin^n x \, dx$$

$$I = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx - (n-1)I$$

$$I + (n-1)I = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$(1+n-1)I = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$nI = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x dx$$

$$I = \frac{-\cos x \sin^{n-1}x}{n} + \frac{(n-1)}{n}\int \sin^{n-2}x dx$$

And,

$$I = \int \cos^n x dx$$

We separate a single powerof sinx. As follows:

$$I = \int \cos^{n-1}x \, \cos x dx$$

Applying by parts formula

$$I = (\cos^{n-1}x)(\sin x) - \int (n-1)(\cos^{n-2}x)(-\sin x)(\sin x)dx$$

$$I = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x.\sin^2 x \, dx$$

$$I = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x.(1-\cos^2 x) \, dx$$

$$= \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx - (n-1)\int \cos^n x \, dx$$

$$I = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx - (n-1)I$$

$$I + (n-1)I = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx$$

$$(1+n-1)I = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx$$

$$nI = \sin x.\cos^{n-1}x + (n-1)\int \cos^{n-2}x dx$$

$$I = \frac{\sin x.\cos^{n-1}x}{n} + \left(\frac{n-1}{n}\right)\int \cos^{n-2}x dx$$

Q No. 24 Find a reduction formula for $\int x^n \sin ax \, dx$, where $n > 1$ is an integer. Hence evaluate $\int x^4 \sin ax \, dx$.

$$I = \int x^n \sin ax$$

Integrating by parts,

$$I = (x^n)\left(\frac{-\cos ax}{a}\right) - \int (nx^{n-1})\left(\frac{-\cos ax}{a}\right)dx$$

$$I = \frac{-x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax dx$$

Again by parts,

$$I = \frac{-x^n \cos ax}{a} +$$

$$\frac{n}{a}(x^{n-1})\left(\frac{\sin ax}{a}\right) - \frac{n}{a}\int (n-1)x^{n-2}\left(\frac{\sin ax}{a}\right)dx$$

$$I=\frac{-x^{n}cosax}{a}+\frac{nx^{n-1}sinax}{a^{2}}-\frac{n(n-1)}{a^{2}}\int x^{n-2}sinaxdx$$

Which is the required reduction formula,

Put n=4 & a=4

$$I=\frac{-x^{4}cos4x}{4}+\frac{4x^{3}sin4x}{16}-\frac{4.3}{16}\int x^{2}sin4xdx$$

Now put n=2 and a=4,

$$I=\frac{-x^{4}cos4x}{4}+\frac{4x^{3}sin4x}{16}-\frac{4.3}{16}$$

$$\left[\frac{-x^{2}cos4x}{2}+\frac{2xcos4x}{4}-\frac{2}{4}\int sin4xdx\right]$$

$$I=\frac{-x^{4}cos4x}{4}+\frac{4x^{3}sin4x}{16}$$

$$-\frac{3}{4}\left[\frac{-x^{2}cos4x}{2}+\frac{2xcos4x}{4}-\frac{2}{4}\frac{cos4x}{4}\right]$$

$$I=\frac{-x^{4}cos4x}{4}+\frac{4x^{3}sin4x}{16}+\frac{3x^{2}cos4x}{8}+\frac{3xcos4x}{8}$$

$$-\frac{3cos4x}{32}+c$$

Q No. 25 Find a reduction formula for $\int x^{m}(lnx)^{n}dx, m\neq -1$ And n is an integer greater than 1. Hence evaluate, $\int x^{3}(lnx)^{2}dx$

$$I=\int x^{m}(lnx)^{n}dx$$

$$I=(lnx)^{n}\left(\frac{x^{m+1}}{m+1}\right)-\int n(lnx)^{n-1}\frac{1}{x}.\left(\frac{x^{m+1}}{m+1}\right)dx$$

$$I=\frac{x^{m+1}(lnx)^{n}}{m+1}-\frac{n}{m+1}\int x^{m}(lnx)^{n-1}dx$$

Which is the required reduction formula.
Now put m=3 and n=2

$$I=\frac{x^{3+1}(lnx)^{2}}{3+1}-\frac{2}{3+1}\int x^{3}(lnx)^{2-1}dx$$

$$I=\frac{x^{4}(lnx)^{2}}{4}-\frac{2}{4}\int x^{3}lnxdx$$

Integrating by parts,

$$I=\frac{x^{4}(lnx)^{2}}{4}-\frac{2}{4}\left[(lnx)\left(\frac{x^{4}}{4}\right)-\int\left(\frac{1}{x}\right)\left(\frac{x^{4}}{4}\right)dx\right]$$

$$I=\frac{x^{4}(lnx)^{2}}{4}-\frac{1}{8}x^{4}lnx+\frac{1}{8}\int x^{3}\,dx$$

$$I=\frac{x^{4}(lnx)^{2}}{4}-\frac{1}{8}x^{4}lnx+\frac{1}{8}.\frac{x^{4}}{4}+c$$

Alternative method Q No. 9 $I=\int x^{3}\sqrt{x^{2}+1}\,dx$

$$I=\int x^{2}.x\sqrt{x^{2}+1}\,dx$$

Put $\sqrt{x^{2}+1}=z \quad\Rightarrow\quad x^{2}+1=z^{2} \quad\Rightarrow\quad x^{2}=z^{2}-1$

$2xdx=2zdz \;\Rightarrow\; xdx=zdz$

So

$$I=\int(z^{2}-1)z.zdz$$

$$I=\int(z^{4}-z^{2})dz$$

$$I=\frac{z^{5}}{5}-\frac{z^{3}}{3}+c$$

$$I=\frac{(x^{2}+1)^{\frac{5}{2}}}{5}-\frac{(x^{2}+1)^{\frac{3}{2}}}{3}+c$$

Alternative method Q No. 11 $I=\int e^{x}\frac{1-sinx}{1-cosx}dx$

$$I=\int e^{x}\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2sin^{2}\frac{x}{2}}dx$$

$$I=\int e^{x}\left[\frac{1}{2sin^{2}\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2sin^{2}\frac{x}{2}}\right]dx$$

$$I=\int e^{x}csc^{2}\frac{x}{2}.\frac{1}{2}dx-\int e^{x}\cot\frac{x}{2}dx$$

Applying by parts on first integral

$$I=(e^{x})\left(-\cot\frac{x}{2}\right)-\int(e^{x})\left(-\cot\frac{x}{2}\right)dx-\int e^{x}\cot\frac{x}{2}dx$$

$$I=-e^{x}\cot\frac{x}{2}+\int e^{x}\cot\frac{x}{2}dx\int e^{x}\cot\frac{x}{2}dx$$

$$I=-e^{x}\cot\frac{x}{2}+c$$