Hence $f(x) = \ln(1+x)$ can be expanded into an infinite series.

By Maclaurin's theorem

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots\dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots$$

[Indeterminate Forms]

The form $\left(\frac{0}{0}\right)$:- Suppose that two functions f & ϕ satisfy the conditions of Cauchy's M.V.T on same interval. If $f(a) = \phi(a) = 0$

Then the expression $\frac{f(a)}{\phi(a)}$ is meaningless. But $\lim_{x \to a} \frac{f(x)}{\phi(x)}$ may exist.

The calculation of limits of this type is known as evaluating the indeterminate form $\left(\frac{0}{0}\right)$. e.g

$\frac{\sin x}{x}$ is meaningless at $x = 0$

But $\frac{\sin x}{x} \longrightarrow 1$ as $x \longrightarrow 0$

L' Hospital's Rule :-

Statement :-

(1) Let the functions f & ϕ are continuous on $[a, b]$

(2) f & ϕ are derivable in $]a, b[$.

(3) $f(a) = 0 = \phi(a)$ & $\phi'(x) \neq 0 \quad \forall x \in]a, b[$

(4) $\lim_{x \to a} \frac{f(x)}{\phi(x)} = l$

Then $\lim_{x \to a} \frac{f(x)}{\phi(x)} = l$

Proof:

Consider $\dfrac{f(x)}{\phi(x)} = \dfrac{f(x)-f(a)}{\phi(x)-\phi(a)}$ ∵ $f(a) = \phi(a) = 0$

∵ Since $f(x)$ & $\phi(x)$ are continuous on $[a,b]$ & derivable in $]a,b[$.

Hence applying Cauchys M.V.T we get

$\dfrac{f(x)}{\phi(x)} = \dfrac{f'(c)}{\phi'(c)}$, for some $c \in]a,x[$

If $x \to a$ then $c \to a$ also because $a < c < x$

Hence if $\lim\limits_{x \to a} \dfrac{f'(x)}{\phi'(x)} = l$, Then $\lim\limits_{c \to a} \dfrac{f'(c)}{\phi'(c)}$ exists & is equal to l.

So $\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)} = \lim\limits_{c \to a} \dfrac{f'(c)}{\phi'(c)}$.

$\lim\limits_{x \to a} \dfrac{f(x)}{\phi(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{\phi'(x)}$ as required

---

Exercise no. 33

Evaluate the given limit (Prob 1 – 48)

(1) $\lim\limits_{x \to 0} \dfrac{e^{x} - e^{-x}}{\sin x}$

Sol:- Let l be the required limit. Then

$$l = \lim_{x \to 0} \frac{e^{x} - e^{-x}}{\sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^{x} + e^{-x}}{\cos x} \qquad \text{By } \propto \text{ H. Rule}$$

$$\therefore \ l = \frac{1+1}{1} = 2.$$

---

(2) Let $l = \lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{\cos x - 1}$ $\left(\tfrac{0}{0} \text{ form}\right)$

$$\lim_{x \to 0} \frac{2x e^{x^2}}{-\sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\left[x \cdot 2x e^{x^2} + e^{x^2}\right]}{-\cos x}$$

$$= -\frac{2(1)}{1} \qquad \boxed{l = -2}$$

---

(3) $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x - \sin x}$

Sol:- Let l be the required limit, Then

$$l = \lim_{x \to 0} \frac{x - \tan x}{x - \sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \ l = \lim_{x \to 0} \frac{1 - \sec^2 x}{1 - \cos x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$l = \lim_{x \to 0} \frac{-2\sec x \cdot \sec x \tan x}{\sin x}$$

$$= \lim_{x \to 0} \frac{-2\sec^2 x \tan x}{\sin x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-2\left[\sec^2 x \cdot \sec^2 x + \tan x \cdot 2\sec x \sec x \tan x\right]}{\cos x}$$

$$= \lim_{x \to 0} \frac{-2\left[\sec^4 x + 2\sec^2 x \tan^2 x\right]}{\cos x}$$

$$= \frac{-2\left[1+0\right]}{1}$$

$$l = \boxed{-2}$$

---

(4) $\displaystyle \lim_{x \to \pi} \frac{\sin^2 x}{\cos 3x + 1}$

Sol :- Let l be required Limit, so

$$l = \lim_{x \to \pi} \frac{\sin^2 x}{\cos 3x + 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \pi} \frac{2\sin x \cos x}{-3\sin 3x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \pi} \frac{2\left[\cos x \cdot \cos x + \sin x(-\sin x)\right]}{-3\cos 3x \cdot 3}$$

$$= \lim_{x \to \pi} \frac{2\left[\cos^2 x - \sin^2 x\right]}{-9\cos 3x}$$

$$= \frac{2(1-0)}{-9(-1)} \qquad \boxed{l = \frac{2}{9}}$$

---

(5) $\displaystyle \lim_{x \to 0} \frac{\sin x}{\sqrt{1-\cos x}}$

Sol :- Let l be required limits. Then

$$l = \lim_{x \to 0} \frac{\sin x}{\sqrt{1-\cos x}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{2\sin x/2 \cos x/2}{\sqrt{2} \sin \frac{x}{2}}$$

$$= \lim_{x \to 0} \sqrt{2} \cos x/2$$

$$= \sqrt{2} \cos 0$$

$$\boxed{l = \sqrt{2}}$$

⑥ $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$

Sol:- Let l be the required limit

$$l = \lim_{x \to a} \frac{\sin x - \sin a}{x - a} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to a} \frac{\cos x}{1}$$

$$\boxed{l = \cos(a)}$$

⑦ $\lim_{x \to 0} \frac{\ln \cos x}{\ln \cos 3x}$

Sol:- Let l be required limit, Then

$$l = \lim_{x \to 0} \frac{\ln \cos x}{\ln \cos 3x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\frac{-\sin x}{\cos x}}{\frac{-3\sin 3x}{\cos 3x}}$$

$$= \lim_{x \to 0} \frac{-\tan x}{-3\tan 3x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-\sec^2 x}{-9\sec^2 3x} = \frac{-1}{-9} \Rightarrow \boxed{l = \frac{1}{9}}$$

(3) $\lim_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$

sol :- Let l be the required limit. Then

$$l = \lim_{x \to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + 2\sin x - e^{-x}}{x\cos x + \sin x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + \cos x + \cos x}$$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + 2\cos x}$$

$$= \frac{1+2+1}{2} = \frac{4}{2} \qquad \therefore \boxed{l = 2}$$

(4) $\lim_{x \to 0} \dfrac{\ln(1-x^2)}{\ln \cos x}$

sol:. Let l be required limit

so $$l = \lim_{x \to 0} \frac{\ln(1-x^2)}{\ln \cos x} \quad \left(\tfrac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{-2x}{1-x^2}}{-\frac{\sin x}{\cos x}}$$

$$= \lim_{x \to 0} \frac{2x\cos x}{(1-x^2)\sin x} \quad \left(\tfrac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\left[-x\sin x + \cos x\right]}{-2x\sin x + (1-x^2)\cos x} = \lim_{x \to 0} \frac{-2x\sin x + 2\cos x}{-2x\sin x + (1-x^2)\cos x}$$

$l = \frac{2}{1} = 2$ ——— Ans

(10) $\lim_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$

Sol:- Let l be req limit,

so $l = \lim_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{\sinh x + \sin x}{x \cos x + \sin x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \dfrac{\cosh x + \cos x}{-x \sin x + \cos x + \cos x}$

so $l = \dfrac{1+1}{1+1} = 1$ or $\boxed{l = 1}$

(11) $\lim_{x \to 1} \dfrac{1 - x + \ln x}{1 - \sqrt{2x - x^2}}$

Sol:- let l be required limit, so

$l = \lim_{x \to 1} \dfrac{1 - x + \ln x}{1 - \sqrt{2x - x^2}}$ $\left(\frac{0}{0}\right)$

$= \lim_{x \to 1} \dfrac{-1 + \frac{1}{x}}{-\frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)}$

$= \lim_{x \to 1} \dfrac{\frac{\cancel{1-x}}{x}}{-(2x - x^2)^{-1/2} \cdot \cancel{(1-x)}}$

$= \lim_{x \to 1} - \dfrac{(2x - x^2)^{1/2}}{x}$

$= -\frac{1}{1} = -1$

so $\boxed{l = -1}$

(12) $\lim_{x \to 0} \dfrac{x \cos x - \ln(1+x)}{x^2}$

let l be the req limit. Then

$$l = \lim_{x \to 0} \frac{x\cos x - \ln(1+x)}{x^2} \quad \left(\frac{0}{0}\right)$$ 65

$$= \lim_{x \to 0} \frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\{x\cos x + \sin x\} - \sin x + (1+x)^{-2}}{2}$$

$$\boxed{l = \frac{1}{2}}$$

---

(13) $\lim_{x \to 0} \frac{\sin x - \ln(e^x \cos x)}{x \sin x}$

Sol Let l be the req. limit. Then

$$l = \lim_{x \to 0} \frac{\sin x - \ln(e^x \cos x)}{x \sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x - \frac{1}{e^x \cos x}\left\{-e^x \sin x + \cos x \, e^x\right\}}{x\cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x + \tan x - 1}{x\cos x + \sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to} \frac{-\sin x + \sec^2 x}{-x\sin x + \cos x + \cos x}$$

$$\Rightarrow \boxed{l = \frac{1}{2}}$$

---

(14) $\lim_{x \to 0} \frac{1 + \sin x - \cos x + \ln(1-x)}{x \tan^2 x}$

Sol:- Let l be the required limit
Then

$$l = \lim_{x \to 0} \frac{1 + \sin x - \cos x + \ln(1-x)}{x \tan^2 x}$$

$$l = \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{2x \tan x \sec^2 x + \tan^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{2x \tan x (1 + \tan^2 x) + \tan^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{2x \tan x + 2x \tan^3 x + \tan^2 x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{2x \sec^2 x + 2 \tan x + 6x \tan^2 x \sec^2 x + 2 \tan^3 x + 2 \tan x \sec^2 x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - \frac{2}{(1-x)^3}}{2 \sec^2 x + 4x \sec^2 x \tan x + 2 \sec^2 x + 6 \tan^2 x \sec^2 x + 12 \sec^2 x \tan^3 x + 12x \tan x \sec^4 x + 6 \tan^2 x \sec^2 x + 4 \sec^2 x \tan^2 x + 2 \sec^4 x}$$

$$= \frac{-3}{6} \implies \boxed{l = -\frac{1}{2}}$$

---

(15) If $\lim_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ be finite find the value of a & limit.

Sol. Let l be req. limit. Then

$$l = \lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$l = \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} \text{ ——— } (1)$$

Now as $3x^2 \to 0$ as $x \to 0$

But $l \to$ a finite number (given)

So we conclude that $2 \cos 2x + a \cos x \to 0$ as $x \to 0$

$\implies 2 \cos 2(0) + a \cos(0)^8 = 0$

$2 + a = 0$

$\implies \boxed{a = -2}$

So Given limit becomes

$$l = \lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$\therefore \; l = -\frac{8+2}{6} = -1 \implies \boxed{l = -1}$$

(16) $\lim_{x \to a} \dfrac{x\cos x - \ln(1+x)}{x^2}$

Sol:- let l be the req limit. Then

$$l = \lim_{x \to a} \frac{x\cos x - \ln(1+x)}{x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{-x\sin x + \cos x - \frac{1}{1+x}}{2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{-x\sin x - x\cos x - \sin x + \frac{1}{(1+x)^2}}{2}$$

$$\implies \boxed{l = \frac{1}{2}}$$

(17) $\lim_{x \to 0} \dfrac{(1+x)^{1/x} - e}{x}$

Sol: let l be the req limit, Then

$$l = \frac{(1+x)^{1/x} - e}{x} \quad \text{——— (1)}$$

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Since we know that

$\lim_{x\to 0} (1+x)^{1/x} = e$ (Page 26 calculus)

so we have

$$l = \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} \quad \left(\frac{0}{0}\right)$$

Now we find derivative of $(1+x)^{1/x}$

Let $y = (1+x)^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\ln(1+x) + \frac{1}{x(1+x)}$$

$$\frac{dy}{dx} = y\left[\frac{-1}{x^2}\ln(1+x) + \frac{1}{x(1+x)}\right]$$

or $\frac{dy}{dx} = (1+x)^{1/x}\left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right]$

so from ① using hospital's rule; we have

$$l = \lim_{x\to 0} \frac{(1+x)^{1/x}\left[\frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2}\right]}{1}$$

$$= \lim_{x\to 0} (1+x)^{1/x}\left[\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right]$$

$$= \lim_{x\to 0} (1+x)^{1/x} \cdot \lim_{x\to 0}\left[\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right]$$

$$= e \cdot \lim_{x\to 0}\left[\frac{x-(1+x)\ln(1+x)}{x^2(1+x)}\right] \quad \left(e \cdot \frac{0}{0}\right)$$

$$= e \lim_{x\to 0} \frac{1-\left[1\cdot\ln(1+x) + (1+x)\cdot\frac{1}{1+x}\right]}{x^2\cdot 1 + (1+x)\cdot 2x}$$

$$= e \lim_{x\to 0} \frac{\cancel{1} - \ln(1+x) - \cancel{1}}{2x+3x^2} \quad \left(e \cdot \frac{0}{0}\right)$$

$$= e \cdot \lim_{x\to 0} \frac{\frac{-1}{1+x}}{2+6x} = e \cdot \frac{-1}{2} \Rightarrow \boxed{l = -\frac{e}{2}}$$

The form $\frac{\infty}{\infty}$:-

If $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} \phi(x)$

Then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$$ (i.e. L. Hospital rule is also applicable in this form)

Evaluate the following limits

① $\lim_{x \to 0} \frac{\ln(\sin 3x)}{\ln(\sin x)}$

Sol :: let l be the req limit. Then

$$l = \lim_{x \to 0} \frac{\ln(\sin 3x)}{\ln(\sin x)} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{3\cos 3x}{\sin 3x}}{\frac{\cos x}{\sin x}}$$

$$= \lim_{x \to 0} \frac{3\cos 3x \sin x}{\cos x \sin 3x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{3\left[-3\sin 3x \sin x + \cos 3x \cos x\right]}{-\sin x \sin 3x + \cos x \cdot (3\cos 3x)}$$

$$= \lim_{x \to 0} \frac{-9\sin 3x \sin x + 3\cos 3x \cos x}{-\sin x \sin 3x + 3\cos x \cos 3x}$$

$$= \frac{3}{3} \Rightarrow \boxed{l = 1}$$

② $\lim_{x \to 0} \frac{\ln x^2}{\cot^2 x}$

Sol :- let l be the req limit, Then

$$l = \lim_{x \to 0} \frac{\ln x^2}{\cot^2 x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{x^2}(2x)}{2\cot x(-\operatorname{cosec}^2 x)}$$

$$= \lim_{x \to 0} \frac{\frac{2}{x}}{-2\cot x \operatorname{cosec}^2 x}$$

$$= \lim_{x \to 0} \frac{\sin x \sin^2 x}{-x \cos x}$$

$$= \lim_{x \to 0} -\frac{\sin^3 x}{x \cos x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} -\frac{3\sin^2 x \cos x}{-x\sin x + \cos x}$$

$$= -\frac{0}{0+1} = 0 \implies \boxed{l = 0}$$

---

(3) $\lim_{x \to a} \dfrac{\ln(x-a)}{\ln(e^x - e^a)}$

Sol:- Let l be the req limit, Then

$$l = \lim_{x \to a} \frac{\ln(x-a)}{\ln(e^x - e^a)} \qquad \left(\frac{\infty}{\infty}\right)$$

$$l = \lim_{x \to a} \frac{\frac{1}{x-a}}{\frac{e^x}{e^x - e^a}}$$

$$= \lim_{x \to a} \frac{e^x - e^a}{e^x(x-a)} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{e^x}{e^x + (x-a)e^x}$$

$$= \lim_{x \to a} \frac{1}{1+(x-a)}$$

$$= \frac{1}{1+0} = 1 \implies \boxed{l = 1}$$

(4) $\lim\limits_{x\to 0} \dfrac{\ln \tan x}{\ln x}$

Sol:- Let l be the req limit, so

$$l = \lim_{x\to 0} \frac{\ln \tan x}{\ln x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos^2 x}\cdot\frac{\cos x}{\sin x}}{\frac{1}{x}}$$

or $$l = \lim_{x\to 0} \frac{x}{\sin x \cos x}$$

$$= \lim_{x\to 0} \frac{2x}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}\cos 2x}$$

$$= \frac{1}{\cos 0} \Rightarrow \boxed{l = 1}$$

(5) $\lim\limits_{x\to 0} \log_{\tan x}(\tan 2x)$

Sol:- Let l be req limit, Then

$$l = \lim_{x\to 0} \log_{\tan x} \tan 2x$$

$$= \lim_{x\to 0} \frac{\ln \tan 2x}{\ln \tan x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{\sec^2 x}{\tan x}}$$

$$= \lim_{x\to 0} \frac{\frac{2}{\cos^2 2x}\cdot\frac{\cos 2x}{\sin 2x}}{\frac{1}{\cos^2 x}\cdot\frac{\cos x}{\sin x}}$$

$$= \lim_{x \to 0} \frac{\frac{2}{\sin 2x \cos 2x}}{\frac{1}{\sin x \cos x}}$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin 2x \cos 2x}$$

$$= \lim_{x \to 0} \frac{2 \sin 2x}{\sin 4x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \cos 2x . 2}{4 \cos 4x}$$

$$= \lim_{x \to 0} \frac{\cos 2x}{\cos 4x}$$

$$= \frac{\cos 0}{\cos 0} \implies \boxed{l = 1}$$

(b) $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

sol:- let l be the req limit, so

$$l = \lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{3 \sec^2 3x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 3x}{3 \cos^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2 \cos 3x (-\sin 3x) . 3}{6 \cos x . (-\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-3 (2 \sin 3x \cos 3x)}{-3 (2 \sin x \cos x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin 6x}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{6\cos 6x}{2\cos 2x}$$

$$= \lim_{x \to \frac{\pi}{2}} \left(\frac{3\cos 6x}{\cos 2x}\right)$$

$$= \frac{3\cos(3\pi)}{\cos(\pi)} = \frac{3(-1)}{(-1)}$$

$$\boxed{l = 3}$$

---

(7) $\lim_{x \to 0} \frac{\ln x^2}{\cot x^2}$

Sol:- Let l be the req limit, so

$$l = \lim_{x \to 0} \frac{\ln x^2}{\cot x^2} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{2x}{x^2}}{-\operatorname{cosec}^2(x)^2 \cdot 2x}$$

$$= \lim_{x \to 0} -\frac{\frac{\cancel{2}}{x}}{\cancel{2}x\operatorname{cosec}^2 x^2}$$

$$= \lim_{x \to 0} -\frac{\sin^2 x^2}{x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} -\frac{2\sin x^2 \cos x^2 \cdot 2\cancel{x}}{2\cancel{x}}$$

$$= \lim_{x \to 0} -\frac{2\sin x^2 \cos x^2}{1}$$

$$\Rightarrow \boxed{l = 0}$$

---

(8) $\lim_{x \to \infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\ln\left(1-\frac{1}{x}\right)}$

Sol:- Let l be req limit Then

$$l = \lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\ln(1-\frac{1}{x})} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{1}{x}}\right)\cdot\left(-\frac{1}{x^2}\right)}{\left(\frac{1}{1-\frac{1}{x}}\right)\cdot\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} - \frac{\frac{1}{1+\frac{1}{x}}}{\frac{1}{1-\frac{1}{x}}} = \lim_{x\to\infty} - \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$

$$= \lim_{x\to\infty} - \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{1-x}{1+x}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}-1}{\frac{1}{x}+1} = -1$$

$$\boxed{l = -1}$$

The Form $(0\times\infty)$ or $(\infty\times 0)$:-

To evaluate $\lim_{x\to a} [f(x)\cdot\phi(x)]$

where $\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} \phi(x) = \infty$

Consider $f(x)\cdot\phi(x) = \dfrac{f(x)}{\frac{1}{\phi(x)}} \quad \left(\frac{0}{0}\right)$

or $\dfrac{\phi(x)}{\frac{1}{f(x)}} \quad \left(\frac{\infty}{\infty}\right)$

so that limits of these forms canbe calculated by previous method.

Evaluate the following Limits

(1) $\lim_{x\to a} (x-a)\operatorname{Cosec}\left(\frac{\pi x}{a}\right)$

Sol:- Let $l = \lim_{x\to a} (x-a)\operatorname{Cosec}\left(\frac{\pi x}{a}\right)$

$$= \lim_{x \to a} \frac{x-a}{\sin\left(\frac{\pi x}{a}\right)} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{1}{\frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)} \qquad \text{(Using L.H. Rule)}$$

so $$l = \frac{1}{\frac{\pi}{a}(-1)} = -\frac{a}{\pi}$$

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(2) $\lim_{x \to 1} (1-x)\tan\left(\frac{\pi x}{2}\right)$

sol:-

Let $$l = \lim_{x \to 1} (1-x)\tan\left(\frac{\pi x}{2}\right) \qquad (0 \times \infty)$$

$$= \lim_{x \to 1} \frac{1-x}{\frac{1}{\tan\left(\frac{\pi x}{2}\right)}}$$

$$= \lim_{x \to 1} \frac{1-x}{\cot\left(\frac{\pi x}{2}\right)} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1}{-\operatorname{cosec}^2\left(\frac{\pi x}{2}\right)\cdot\left(\frac{\pi}{2}\right)}$$

$$= \lim_{x \to 1} \frac{\sin^2\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}}$$

$$= \frac{1}{\frac{\pi}{2}}$$

$$\boxed{l = \frac{2}{\pi}}$$

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(3) $\lim_{x \to 0} x \ln \tan(x)$

sol:-

let $$l = \lim_{x \to 0} x \ln \tan x \qquad (0 \times \infty)$$

$$= \lim_{x \to 0} \frac{\ln \tan x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty}\right)$$

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$$= \lim_{x \to 0} \frac{\frac{1}{\tan(x)} \sec^2(x)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{\frac{\cos x}{\sin x} \frac{1}{\cos^2 x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} -\frac{x^2}{\sin x \cos x}$$

$$= \lim_{x \to 0} \frac{-2x^2}{\sin 2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} -\frac{4x}{2\cos 2x}$$

$$= \lim_{x \to 0} -\frac{2x}{\cos 2x}$$

$$\Rightarrow \boxed{l = 0}$$

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(4) $\lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right)$

Sol:- Let $l = \lim_{x \to 0} x \tan\left(\frac{\pi}{2} - x\right) \quad (0 \times \infty)$

$$\lim_{x \to 0} \frac{\tan\left(\frac{\pi}{2} - x\right)}{\frac{1}{x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{-\sec^2(\pi/2 - x)}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{x^2}{\cos^2\left(\frac{\pi}{2} - x\right)}$$

$$= \lim_{x \to 0} \frac{x^2}{\sin^2 x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2x}{2\sin x \cos x}$$

$$= \lim_{x \to 0} \frac{2x}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{2\cos 2x}$$

$$= \lim_{x \to 0} \frac{1}{\cos 2x}$$

$$\Rightarrow \boxed{l = 1}$$

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(5) $\lim_{x \to \frac{\pi}{2}} \tan x \cdot \ln \sin x$

Sol :- Let $l = \lim_{x \to \frac{\pi}{2}} \tan x \cdot \ln \sin x \quad (\infty \times 0)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\ln \sin x}{\cot x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} -\frac{\cos x}{\sin x} \times \sin^2 x$$

$$= \lim_{x \to \frac{\pi}{2}} -\sin x \cos x$$

or $\quad l = \lim_{x \to \frac{\pi}{2}} -\frac{1}{2} \sin 2x$

$$\Rightarrow \boxed{l = 0}$$

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(6) $\lim_{x \to \infty} x\left(a^{1/x} - 1\right)$

Sol :- let $l = \lim_{x \to \infty} x\left[a^{1/x} - 1\right] \quad (\infty \times 0)$

$$= \lim_{x \to \infty} \frac{a^{1/x} - 1}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

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$$l = \lim_{x \to \infty} \frac{a^{\frac{1}{x}} \ln(a) \cdot \left(\cancel{\frac{-1}{x^2}}\right)}{\left(\cancel{\frac{-1}{x^2}}\right)}$$

$$= \lim_{x \to \infty} \left(a^{1/x} \ln(a)\right)$$

$$= 1 \cdot \ln a \qquad (\because a^0 = 1)$$

$$\Rightarrow \boxed{l = \ln a}$$

The form $(\infty - \infty)$:-

To evaluate $\lim_{x \to a} \left[f(x) - \phi(x) \right]$

where $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} \phi(x)$

To evaluate above limit, we write

$$f(x) - \phi(x) = \frac{\frac{1}{\phi(x)} - \frac{1}{f(x)}}{\frac{1}{\phi(x) f(x)}} \quad \text{viz } \left(\frac{0}{0}\right) \text{ form}$$

& canbe evaluated by previous method

Evaluate the following limits

① $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

Sol:- let $l = \lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{e^x - 1} \right] \qquad (\infty - \infty)$

$$= \lim_{x \to 0} \frac{e^x - 1 - x}{x(e^x - 1)}$$

$$= \lim_{x \to 0} \frac{e^x - 1 - x}{x e^x - x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x - 1}{x e^x + e^x + e^x}$$

$$= \lim_{x \to 0} \frac{e^x}{x e^x + 2 e^x}$$

$$= \lim_{x \to 0} \frac{1}{x+2}$$

$$= \boxed{l = \frac{1}{2}}$$

② $\lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right) \right]$

Sol:- let $l = \lim_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right) \right]$ $(\infty - \infty)$

$$= \lim_{x \to 0} \frac{\frac{1}{\cot(\frac{x}{a})} - \frac{1}{a/x}}{\frac{1}{\cot(\frac{x}{a}) \cdot \frac{a}{x}}}$$

$$= \lim_{x \to 0} \frac{\tan\left(\frac{x}{a}\right) - \frac{x}{a}}{\frac{x}{a} \tan\left(\frac{x}{a}\right)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a} - \frac{1}{a}}{\frac{x}{a} \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a} + \tan\left(\frac{x}{a}\right) \cdot \frac{1}{a}}$$

$$= \lim_{x \to 0} \frac{\sec^2\left(\frac{x}{a}\right) - 1}{\frac{x}{a} \sec^2\left(\frac{x}{a}\right) + \tan\left(\frac{x}{a}\right)}$$

$$= \lim_{x \to 0} \frac{\tan^2\left(\frac{x}{a}\right)}{\frac{x}{a} \sec^2\left(\frac{x}{a}\right) + \tan\left(\frac{x}{a}\right)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2\tan\left(\frac{x}{a}\right) \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a}}{\frac{1}{a} \sec^2 \frac{x}{a} + \frac{x}{a} \cdot 2 \sec\frac{x}{a} \cdot \sec\frac{x}{a} \tan\frac{x}{a} \cdot \frac{1}{a} + \sec^2\frac{x}{a} \cdot \frac{1}{a}}$$

$$= \frac{0}{\frac{1}{a} + \frac{1}{a}} \qquad \boxed{l = 0}$$

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(3) $\lim\limits_{x \to 1} \left[\dfrac{x}{[illegible]} - \dfrac{1}{\ln x} \right]$

sol :- let $l = \lim\limits_{x \to 1} \left[\dfrac{x}{x-1} - \dfrac{1}{\ln x} \right]$

$$= \lim_{x \to 1} \left[\frac{x \ln x - x + 1}{(x-1) \ln x} \right] \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{x \cdot \frac{1}{x} + \ln x - 1}{(x-1) \cdot \frac{1}{x} + \ln x}$$

$$= \lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}}$$

$$= \frac{1}{1+1} \implies \boxed{l = \frac{1}{2}}$$

(4) $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

sol :- $l = \lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x) \quad (\infty - \infty)$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{1 - \sin x}{\cos x} \right] \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{-\cos x}{-\sin x} \right]$$

$$\implies \boxed{l = 0}$$

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(5) $\lim_{x \to 1} \left[\frac{2}{x^2-1} - \frac{1}{x-1} \right]$

Sol :- let $l = \lim_{x \to 1} \left[\frac{2}{x^2-1} - \frac{1}{x-1} \right]$ $(\infty - \infty)$

$$= \lim_{x \to 1} \left[\frac{2-(x+1)}{x^2-1} \right]$$

$$= \lim_{x \to 1} \frac{1-x}{x^2-1} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1}{2x}$$

$$\Rightarrow \boxed{l = -\frac{1}{2}}$$

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(6) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x \right)$

Sol :- Let $l = \lim_{x \to 0} \left(\frac{1}{x} - \frac{\cos x}{\sin x} \right)$

$$= \lim_{x \to 0} \left(\frac{\sin x - x\cos x}{x \sin x} \right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left(\frac{\cancel{\cos x} + x\sin x - \cancel{\cos x}}{x\cos x + \sin x} \right)$$

$$= \lim_{x \to 0} \left(\frac{x \sin x}{x\cos x + \sin x} \right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left[\frac{x\cos x + \sin x}{-x\sin x + \cos x + \cos x} \right]$$

$$= \frac{0}{0+1+1} = 0$$

$$\Rightarrow \boxed{l = 0}$$

The forms 0^0, 1^∞, ∞^0

To Evaluate $\lim_{x\to a} \{f(x)\}^{\phi(x)}$

when

① $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} \phi(x)$. (0^0)

② $\lim_{x\to a} f(x) = 1$ & $\lim_{x\to a} \phi(x) = \infty$ (1^∞)

③ $\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} \phi(x) = 0$

Method for Evaluation :-

suppose $y = \{f(x)\}^{\phi(x)}$

Taking ln on both sides

$$\ln y = \phi(x) \ln f(x) \quad \text{—— (A)}$$

If ① $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} \phi(x)$

The (A) is of the form $0 \times \infty$

(2) If $\lim_{x\to a} f(x) = 1$ & $\lim_{x\to a} \phi(x) = \infty$

The one is again of the form $\infty \times 0$

③ If $\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} \phi(x) = 0$

: Then (A) is of form $0 \times \infty$.

Q Evaluate the following limits

① $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$

Sol :- Let $y = (1+x)^{1/x}$

$$\ln y = \frac{1}{x} \ln(1+x)$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \left[\frac{\ln(1+x)}{x} \right] \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x}}{1}$$

$$= \lim_{x\to 0} \frac{1}{1+x}$$

so $\lim_{x\to 0} \ln y = 1$

$$\Rightarrow \lim_{x\to 0} y = e^1 \quad \Rightarrow \quad \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

(2) $\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$

Sol: $y = \left(\frac{1}{x}\right)^{\tan x}$ $\quad (\infty^0)$

$$\ln y = \tan x \cdot \ln\left(\frac{1}{x}\right)$$
$$= \tan x \left[\ln 1 - \ln x\right]$$
$$= -\tan x \ln x$$

$$\ln y = -\frac{\ln x}{\cot x}$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{-\ln(x)}{\cot x} \quad \left(\frac{-\infty}{\infty}\right)$$

$$= -\lim_{x\to 0} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2\sin x \cos x}{1}$$

$$= \frac{0}{1}$$

$\therefore \lim_{x\to 0} \ln y = 0$

$$\Rightarrow \lim_{x\to 0} y = e^0$$

or

$$\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x} = 1$$

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(3) $\underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\frac{\pi}{2} - x}$

Sol :- 1^0

Let $y = (\cos x)^{\pi/2 - x}$

$\ln y = (\frac{\pi}{2} - x) \ln \cos x$

$\underset{x \to \frac{\pi}{2}}{Lt} \ln y = \underset{x \to \frac{\pi}{2}}{Lt} \left[(\frac{\pi}{2} - x) \ln \cos x \right]$ $\quad 0 \times \infty$

$= \underset{x \to \frac{\pi}{2}}{Lt} \left[\dfrac{\ln \cos x}{\frac{1}{\frac{\pi}{2} - x}} \right]$ $\quad \left(\frac{\infty}{\infty}\right)$

$= \underset{x \to \frac{\pi}{2}}{Lt} \dfrac{-\frac{\sin x}{\cos x}}{(\frac{\pi}{2} - x)^{-2}}$

$= \underset{x \to \frac{\pi}{2}}{Lt} \dfrac{-\tan x}{(\frac{\pi}{2} - x)^{-2}}$ $\quad \left(\frac{\infty}{\infty}\right)$

$= \underset{x \to \frac{\pi}{2}}{Lt} - \dfrac{(\frac{\pi}{2} - x)^2}{\cot x}$ $\quad \left(\frac{0}{0}\right)$

$= \underset{x \to \frac{\pi}{2}}{Lt} - \dfrac{2(\frac{\pi}{2} - x).(-1)}{-\operatorname{cosec}^2 x}$

$= \underset{x \to \frac{\pi}{2}}{Lt} \dfrac{-2(\frac{\pi}{2} - x)}{\operatorname{cosec}^2 x}$

$= \dfrac{0}{1}$

$\therefore \underset{x \to \frac{\pi}{2}}{Lt} \ln y = 0$

$\Rightarrow \underset{x \to \frac{\pi}{2}}{Lt} y = e^0$

or $\underset{x \to \frac{\pi}{2}}{Lt} (\cos x)^{\pi/2 - x} = 1$

(4) $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

sol:-

Let $y = (\sin x)^{\tan x}$ $\quad (0^0)$

$\ln y = \tan x \ln \sin x$

$\lim_{x \to \frac{\pi}{2}} \ln y = \lim_{x \to \frac{\pi}{2}} \left[\tan x \ln \sin x\right] \quad \infty \times 0$

$= \lim_{x \to \frac{\pi}{2}} \frac{\ln \sin x}{\cot x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\cot x}{-\operatorname{cosec}^2 x}$

$= \lim_{x \to \frac{\pi}{2}} \left(-\frac{\cos x}{\sin x} \times \sin^2 x\right)$

$= \lim_{x \to \frac{\pi}{2}} (-\sin x \cos x)$

$= 0$

so $\lim_{x \to \frac{\pi}{2}} \ln y = 0$

$\Rightarrow \lim_{x \to \frac{\pi}{2}} y = e^0 = 1$

or

$\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$

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(5) $\lim_{x \to 0} \left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}}$

sol:- Let $y = \left(\frac{\sinh x}{x}\right)^{1/x^2}$

$\ln y = \frac{1}{x^2} \ln\left(\frac{\sinh x}{x}\right)$

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$$\lim_{x\to 0} \ln y = \lim_{x\to 0}\left[\frac{1}{x^2}\ln\left(\frac{\sinh x}{x}\right)\right] \quad \infty\times 0$$

$$= \lim_{x\to 0}\left[\frac{\ln\left(\frac{\sinh x}{x}\right)}{x^2}\right] \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left(\frac{\ln \sinh x - \ln x}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sinh x}(\cosh x) - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{\frac{\cosh x}{\sinh x} - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{x\cosh x - \sinh x}{2x^2 \sinh x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{x \sinh x + \cancel{\cosh x} - \cancel{\cosh x}}{2x^2\cosh x + 4x\sinh x}$$

$$= \lim_{x\to 0} \frac{\sinh x}{2x\cosh x + 4\sinh x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cosh x}{2x\sinh x + 2\cosh x + 4\cosh x}$$

$$= \lim_{x\to 0} \frac{\cosh x}{2x\sinh x + 6\cosh x}$$

$$= \frac{1}{2(0)+6(1)} = \frac{1}{6}$$

so $\lim_{x\to 0} \ln y = \frac{1}{6}$

or

$$\lim_{x\to 0} y = e^{1/6}$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{\frac{1}{x^2}} = e^{1/6}$$

(6) $\lim_{x\to 0} (\tan x)^{\sin 2x}$ (0^0)

Sol:-

let $y = (\tan x)^{\sin 2x}$

$\ln y = \sin 2x \ln \tan x$

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \left[\sin 2x \ln \tan x\right]$ $0\times\infty$

$= \lim_{x\to 0} \left[\dfrac{\ln \tan x}{\operatorname{cosec} 2x}\right]$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x\to 0} \dfrac{\sin 2x}{(\ln \tan x)^{-1}}$

$= \lim_{x\to 0} \dfrac{\sin 2x}{-\ln \tan x}$

$= \lim_{x\to 0} - \dfrac{2\cos^2 2x}{\dfrac{1}{\tan x}\cdot \sec^2 x}$

$= \lim_{x\to 0} \dfrac{-2\cos 2x}{\dfrac{\cos x}{\sin x}\cdot\dfrac{1}{\cos^2 x}}$

$\lim_{x\to 0} \dfrac{-2\cos 2x}{\dfrac{1}{\sin x \cos x}}$

$= \lim_{x\to 0} -2\cos 2x \cdot \sin x \cos x$

$= \lim_{x\to 0} -\sin 2x \cos 2x$

$\lim_{x\to 0} \ln y = 0 \implies \lim_{x\to 0} y = e^0 = 1$

$\implies \lim_{x\to 0} (\tan x)^{\sin 2x} = 1$

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(7) $\underset{x\to 0}{Lt} (1+\sin x)^{\cot x}$ $\quad 1^{\infty}$

Sol :- Let $y = (1+\sin x)^{\cot x}$

$\ln y = \cot x \ln(1+\sin x)$

$\underset{x\to 0}{Lt} \ln y = \underset{x\to 0}{Lt} \left[\cot x \ln(1+\sin x)\right] \quad \infty \times 0$

$= \underset{x\to 0}{Lt} \dfrac{\ln(1+\sin x)}{\tan x} \quad \left(\frac{0}{0}\right)$

$= \underset{x\to 0}{Lt} \dfrac{\frac{\cos x}{1+\sin x}}{\sec^2 x}$

$= \underset{x\to 0}{Lt} \dfrac{\cos^3 x}{1+\sin x}$

$= \dfrac{1}{1+0} = 1$

$\therefore \underset{x\to 0}{Lt} \ln y = 1$

$\Rightarrow \underset{x\to 0}{Lt} y = e^1$

$\Rightarrow \underset{x\to 0}{Lt} (1+\sin x)^{\cot x} = e$

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(8) $\underset{x\to \frac{\pi}{2}}{Lt} (\sec x)^{\cot x}$

Sol :-

Let $y = (\sec x)^{\cot x}$

$\ln y = \cot x \ln(\sec x)$

$\underset{x\to \frac{\pi}{2}}{Lt} \ln y = \underset{x\to \frac{\pi}{2}}{Lt} \left[\cot \ln(\sec x)\right] \quad (0\times\infty)$

$= \underset{x\to \frac{\pi}{2}}{Lt} \dfrac{\ln \sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right)$

$= \underset{x\to \frac{\pi}{2}}{Lt} \dfrac{\frac{1}{\sec x}\cdot \sec x \tan x}{\sec^2 x} = \underset{x\to \frac{\pi}{2}}{Lt} \dfrac{\tan x}{\sec^2 x} \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{2\sec x \cdot \sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{2\tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{2}(\cot x) = 0$$

$$\lim_{x \to \frac{\pi}{2}} \ln y = 0$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} y = e^0 = 1$$

$$\Rightarrow \lim_{x \to \frac{\pi}{2}} (\sec x)^{\cot x} = 1$$

(9) $\lim_{x \to 1} (1-x^2)^{\frac{1}{\ln(1-x)}}$

Sol :- let $y = (1-x^2)^{\frac{1}{\ln(1-x)}}$

$$\ln y = \frac{1}{\ln(1-x)} \ln(1-x^2)$$

$$\lim_{x \to 1} \ln y = \lim_{x \to 1} \left[\frac{1}{\ln(1-x)} \ln(1-x^2) \right]$$

$$= \lim_{x \to 1} \left[\frac{\ln(1-x^2)}{\ln(1-x)} \right] \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 1} \frac{\frac{-2x}{1-x^2}}{\frac{-1}{1-x}}$$

$$= \lim_{x \to 1} \frac{2x}{1+x}$$

$$= \frac{2}{1+1} = 1$$

$$= \lim_{x \to 1} \ln y = 1 \Rightarrow \lim_{x \to 1} y = e^1$$

$$\Rightarrow \underset{x\to 1}{Lt} (1-x^2)^{\frac{1}{\ln(1-x)}} = e$$

(10) $$\underset{x\to 1}{Lt} \left[\tan\left(\frac{\pi x}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$$

sol:- let $y = \left[\tan\left(\frac{\pi x}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$ (1^{∞})

$$\ln y = \tan\left(\frac{\pi x}{2}\right) \ln\left[\tan\left(\frac{\pi x}{4}\right)\right]$$

$$\underset{x\to 1}{Lt} \ln y = \underset{x\to 1}{Lt}\left[\tan\left(\frac{\pi x}{2}\right) \ln \tan\left(\frac{\pi x}{4}\right)\right] \quad (\infty \times 0)$$

$$= \underset{x\to 1}{Lt} \frac{\ln \tan\left(\frac{\pi x}{4}\right)}{\cot\left(\frac{\pi x}{2}\right)} \quad \left(\frac{0}{0}\right)$$

$$= \underset{x\to 1}{Lt} \frac{\frac{1}{\tan\left(\frac{\pi x}{4}\right)} \cdot \sec^2\left(\frac{\pi x}{4}\right) \cdot \frac{\cancel{\pi}}{\cancel{4}\,2}}{-\operatorname{cosec}^2\left(\frac{\pi x}{2}\right) \cdot \frac{\cancel{\pi}}{\cancel{2}}}$$

$$= \underset{x\to 1}{Lt} - \frac{\frac{\cos\left(\cancel{\frac{\pi x}{4}}\right)}{\sin\left(\frac{\pi x}{4}\right)} \cdot \frac{1}{\cos^{\cancel{2}}\left(\frac{\pi x}{4}\right)}}{2 \cdot \frac{1}{\sin^2\left(\frac{\pi x}{2}\right)}}$$

$$= \underset{x\to 1}{Lt} \frac{-\sin^2\left(\frac{\pi x}{2}\right)}{2\sin\left(\frac{\pi x}{4}\right)\cos\left(\frac{\pi x}{4}\right)}$$

$$= \underset{x\to 1}{Lt} \frac{-\sin^{\cancel{2}} \frac{\pi x}{2}}{\cancel{\sin}\frac{\pi x}{2}} = \underset{x\to 1}{Lt}\left(-\sin\frac{\pi x}{2}\right)$$

$$= -1$$

so $\underset{x\to 1}{Lt} \ln y = -1 \Rightarrow \underset{x\to 1}{Lt}\, y = e^{-1} = \frac{1}{e}$

so $\underset{x\to 1}{Lt} \left[\tan\left(\frac{\pi x}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)} = \frac{1}{e}$

(ii) $\lim_{x \to \frac{\pi}{2}} (1-\sin x)^{\cos x}$ 0^0

Sol :- Let $y = (1-\sin x)^{\cos x}$

$$\ln y = \cos x \ln(1-\sin x)$$

$$\lim_{x \to \frac{\pi}{2}} \ln y = \lim_{x \to \frac{\pi}{2}} \left[\cos x \ln(1-\sin x) \right] \quad 0 \times \infty$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\ln(1-\sin x)}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{-\cos x}{1-\sin x}}{\sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{(1-\sin x)\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\cos^3 x}{\sin x(1-\sin x)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{3\cos^2 x \sin x}{\sin x(-\cos x) + \cos x(1-\sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{3\sin x \cos x}{-\sin x + 1 - \sin x}$$

$$= \frac{0}{-1+1-1} = 0$$

$$\lim_{x \to \frac{\pi}{2}} \ln y = 0 \implies \lim_{x \to \frac{\pi}{2}} y = e^0 = 1$$

So $\lim_{x \to \frac{\pi}{2}} (1-\sin x)^{\cos x} = 1$

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(12) V. Imp $\quad \lim_{x \to 0} (\cot x)^{\sin 2x} \qquad (\infty^0)$

Sol :

$$\text{let } y = (\cot x)^{\sin 2x}$$

$$\ln y = \sin 2x \ln(\cot x)$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \left[\sin 2x \ln(\cot x)\right] \qquad (0 \times \infty)$$

$$= \lim_{x \to 0} \left[\frac{\ln \cot x}{\operatorname{cosec} 2x} \right] \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{-\operatorname{cosec}^2 x}{\cot x}}{-\operatorname{cosec} 2x \cot 2x . 2}$$

$$= \lim_{x \to 0} \frac{\operatorname{cosec}^2 x}{2 \cot x \operatorname{cosec} 2x \cot 2x}$$

$$= \lim_{x \to 0} \frac{\tan x \sin 2x \tan 2x}{\sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sec^2 x \sin 2x \tan 2x + \tan x \tan 2x . 2\cos 2x + \tan x \sin 2x . \sec^2 2x . 2}{2 \sin x . \cos x}$$

Dividing num. & den. by $\sin x$ $\quad 2\cos 2x$

$$= \lim_{x \to 0} \frac{\sec^2 x (2\cos x)\tan 2x + \sec x . \tan 2x . 2\cos 2x + \sec x . \sin 2x . 2\sec^2 2x}{2\cos x}$$

$$\lim_{x \to 0} \ln(y) = \frac{0+0+0}{2(1)} = 0 \Rightarrow \lim_{x \to 0} y = e^0 = 1$$

$$\lim_{x \to 0} (\cot x)^{\sin 2x} = 1$$

---

(13) $\quad \lim_{x \to 0} x^x$

Sol :

$$\text{Let } y = x^x$$

$$\ln y = x \ln x$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} x \ln x \qquad (0 \times \infty)$$

$$= \lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} (-x)$$

$$= 0$$

so $\lim_{x \to 0} \ln y = 0 \implies \lim_{x \to 0} y = e^0$

or $\lim_{x \to 0} x^x = 1$

---

(14) $\lim_{x \to 1} x^{\frac{1}{x-1}}$

sol :-

Let $y = x^{\frac{1}{x-1}}$

$$\ln y = \frac{1}{x-1} \ln x$$

$$\lim_{x \to 1} \ln y = \lim_{x \to 1} \left(\frac{\ln x}{x-1} \right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x \to 1} \frac{1}{x} = 1$$

so $\lim_{x \to 1} \ln y = 1$

$\implies \lim_{x \to 1} y = e^1$

$\implies \lim_{x \to 1} x^{\frac{1}{x-1}} = e$

---

(15) $\lim_{x \to 0} (\cot x)^{\sin 2x}$

sol: let $y = (\cot x)^{\sin 2x}$

$$\ln y = \sin 2x \ln(\cot x)$$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \left[\sin 2x \ln(\cot x) \right] \qquad (0 \times \infty)$$

$$= \underset{x \to 0}{Lt} \frac{\ln \cot x}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \underset{x \to 0}{Lt} \frac{\frac{-\operatorname{cosec}^2 x}{\cot x}}{-2\operatorname{cosec} 2x \cot 2x}$$

$$= \underset{x \to 0}{Lt} \frac{\frac{\sin x}{\cos x} \cdot \frac{1}{\sin^2 x}}{\frac{2\cos 2x}{\sin^2 2x}}$$

$$= \underset{x \to 0}{Lt} \frac{\sin^2 2x}{2\cos 2x \sin x \cos x}$$

$$= \underset{x \to 0}{Lt} \frac{\sin^{\cancel{2}} 2x}{\cancel{\sin 2x} \cos 2x}$$

$$= \underset{x \to 0}{Lt} \frac{\sin 2x}{\cos 2x} = 0$$

So

$$\underset{x \to 0}{Lt} \ln y = 0 \implies \underset{x \to 0}{Lt} y = e^0 = 1$$

So

$$\underset{x \to 0}{Lt} (\cot x)^{\sin 2x} = 1$$

Evaluate the following limits by using expansions

① $\underset{x \to 0}{Lt} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$

Sol: Let $l = \underset{x \to 0}{Lt} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$

$$= \underset{x \to 0}{Lt} \left[\frac{1}{(\cancel{1} + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - \cancel{1}} - \frac{1}{x} \right]$$

$$= \lim_{x \to 0} \left[\frac{1}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} - \frac{1}{x} \right]$$

$$= \lim_{x \to 0} \left[\frac{1}{x\left(1 + \frac{x}{2!} + \frac{x^2}{3!} \cdots \right)} - \frac{1}{x} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots} - 1 \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right)^{-1} - 1 \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\cancel{1} - \frac{x}{2!} - \frac{x^2}{3!} - \cdots - \cancel{1} \right]$$

$$= \lim_{x \to 0} \left[-\frac{1}{2} - \frac{x}{3!} - \frac{x^2}{4!} \cdots \right]$$

$$\boxed{l = -\frac{1}{2}}$$

(1) $\displaystyle \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{\sin x}}{x^{3/2}}$

Sol.— Let $\displaystyle l = \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{\sin x}}{x^{3/2}}$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right\}^{1/2}}{x^{3/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x} \left\{ 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots \right\}^{1/2}}{x^{3/2}}$$

$$= \lim_{x \to 0} \frac{1 - \left\{ 1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots \right) \right\}^{1/2}}{[illegible]}$$

$$= \lim_{x \to 0} \frac{1 - \left\{1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)\right\}^{1/2}}{x^2}$$

$$= \lim_{x \to 0} \frac{1 - \left\{1 - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)\right\}}{x^2}$$

$$= \lim_{x \to 0} \frac{1 - 1 + \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} x^2 \left[\frac{1}{6} - \frac{x^2}{120} + \cdots\right]}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2}\left\{\frac{1}{6} - \frac{x^2}{120} + \cdots\right\}$$

$$\boxed{l = \frac{1}{12}}$$

(3) $\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$

Sol :- Let l be req limit, Then

$$l = \lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} \qquad \text{——— (1)}$$

Consider $t = (1+x)^{1/x}$

$$\ln(t) = \frac{1}{x}\ln(1+x)$$

or

$$\ln(t) = \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

$$\ln(t) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

$$\Rightarrow t = e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots\right)}$$

$$\therefore t = e^1 \cdot e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots\right)}$$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\cdots\right)+\frac{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\cdots\right)^2}{2!}\right]$$

$$= e\left[1-\frac{x}{2}+\left(\frac{x^2}{3}+\frac{x^2}{8}\right)+\left(-\frac{x^3}{4}-\frac{x^3}{18}\right)+\cdots\right]$$

$$= e\left[1-\frac{x}{2}+\left(\frac{8x^2+3x^2}{24}\right)+\left(\frac{-9x^3-2x^3}{36}\right)+\cdots\right]$$

$$(1+x)^{\frac{1}{x}} = e\left[1-\frac{x}{2}+\frac{11x^2}{24}-\frac{11x^3}{36}+\cdots\infty\right]$$

So from ①, we have

$$l = \underset{x\to 0}{Lt} \frac{e\left[1-\frac{x}{2}+\frac{11x^2}{24}-\frac{11x^3}{36}+\cdots\right]-e}{x}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{e}+e\left[-\frac{x}{2}+\frac{11x^2}{24}-\frac{11x^3}{36}\right]-\cancel{e}}{x}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x}\, e\left[-\frac{1}{2}+\frac{11x}{24}-\frac{11x^2}{36}\right]}{\cancel{x}}$$

$$= \underset{x\to 0}{Lt}\; e\left[-\frac{1}{2}+\frac{11x}{24}-\frac{11x^2}{36}\cdots\right]$$

$$\boxed{l = -\frac{e}{2}}$$

④ $$\underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}}{x^2}$$

Sol:- let $$l = \underset{x\to 0}{Lt} \frac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}}{x^2}$$

$$= \underset{x\to 0}{Lt} \frac{e\left[1-\frac{x}{2}+\frac{11x^2}{24}-\frac{11}{36}x^3+\cdots\right]-e+\frac{ex}{2}}{x^2}$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{e} + e\left[\cancel{\frac{x}{2}}\right] + e\left[\frac{11x^2}{24} - \frac{11}{36}x^3 + \cdots\right] - \cancel{e} + \cancel{\frac{ex}{2}}}{x^2}$$

$$= \underset{x \to 0}{Lt} \frac{e\cancel{x^2}\left[\frac{11}{24} - \frac{11x}{36} + \cdots\right]}{\cancel{x^2}}$$

$$= e\left(\frac{11}{24}\right) \Rightarrow \boxed{l = \frac{11}{24}e}$$

(6) $\underset{x \to 0}{Lt} \dfrac{x - \sin x}{x^3}$

sol: $\underset{x \to 0}{Lt} \dfrac{x - \sin x}{x^3}$

$$= \underset{x \to 0}{Lt} \left[\frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}{x^3}\right]$$

$$= \underset{x \to 0}{Lt} \frac{\cancel{x} - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^3}$$

$$= \underset{x \to 0}{Lt} \frac{1}{3!} - \frac{x^2}{5!} + \cdots$$

$$= \frac{1}{3!} \Rightarrow \boxed{l = \frac{1}{6}}$$

(7) $\underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\ln(1-x)}$

Sol:- Let $l = \underset{x \to 0}{Lt} \dfrac{e^x \sin x - x - x^2}{x^2 + x\ln(1-x)}$

$$= \underset{x \to 0}{Lt} \frac{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right] - x - x^2}{x^2 + x\left[-x - \frac{x^2}{2} - \frac{x^3}{3} \cdots\right]}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - x^2 - \frac{x^4}{6} + \frac{x^6}{5!} \cdots + \frac{x^2}{2} - \frac{x^5}{12} \cdots - \cancel{x}}{\cancel{x^2} - \cancel{x^2} - \frac{x^3}{2} - \frac{x^4}{3} \cdots}$$

$$= \lim_{x\to 0} \left[\frac{\frac{x^3}{3} - \frac{x^4}{6} + \cdots}{-\frac{x^3}{2} - \frac{x^4}{3} \cdots} \right]$$

$$= \lim_{x\to 0} \left[\frac{\frac{1}{3} - \frac{x}{2} + \cdots}{-\frac{1}{2} - \frac{x}{3} \cdots} \right]$$

$$= \frac{\frac{1}{3} - 0 + 0}{-\frac{1}{2} - 0 - \cdots}$$

$$= \boxed{l = -\frac{2}{3}}$$

(5).

Q: $\lim\limits_{x\to 0} \dfrac{e^{x} - e^{\sin x}}{x - \sin x}$

Sol Let l be the req. limit then

$$l = \lim_{x\to 0} \frac{e^{x} - e^{\sin x}}{x - \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^{x} - e^{\sin x}.\cos x}{1 - \cos x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^{x} - e^{\sin x}.\cos^2 x - e^{\sin x}(-\sin x)}{0 - (-\sin x)}$$

$$= \lim_{x\to 0} \frac{e^{x} - e^{\sin x}.\cos^2 x + e^{\sin x}\sin x}{\sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^{x} - \left(e^{\sin x}.\cos^3 x + e^{\sin x}.2\cos x.(-\sin x)\right) + \left(e^{\sin x}.\cos x\sin x + e^{\sin x}.\cos x\right)}{\cos x}$$

$$= \frac{1 - (1+0) + (0+1)}{1}$$

$$= \frac{1-1+1}{1}$$

$$= \frac{1}{1}$$

$$l = 1$$

Q $\lim_{x\to 1} \dfrac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2}$

Sol: Let l be the req. limit. then

$$l = \lim_{x\to 1} \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} \qquad \left(\frac{0}{0}\right)$$

Applying L.H. rule

$$= \lim_{x\to 1} \frac{n(n+1)x^n - n(n+1)x^{n-1}}{2(x-1)} \qquad \left(\frac{0}{0}\right)$$

Again by L.H. Rule

$$= \lim_{x\to 1} \frac{n^2(n+1)x^{n-1} - n(n+1)(n-1)x^{n-2}}{2}$$

$$= \frac{n^2(n+1)\cdot 1 - n(n+1)(n-1)\cdot 1}{2}$$

$$= \frac{n^3+n^2 - n(n^2-1)}{2}$$

$$= \frac{\cancel{n^3}+n^2-\cancel{n^3}+n}{2}$$

$$l = \frac{n(n+1)}{2}$$

Q $\lim_{x\to 0} \dfrac{\tan x - \sin x}{x^2\tan x}$

Sol. Let l be the req. limit then

$$l = \lim_{x\to 0} \frac{\tan x - \sin x}{x^2 \tan x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^2 \cdot \frac{\sin x}{\cos x}}$$

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$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} - 1}{\frac{x^2}{\cos x}}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{0-(-\sin x)}{2x} \qquad \text{(By L.H. rule)}$$

$$= \lim_{x\to 0} \frac{\sin x}{2x}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \frac{1}{2} \cdot \lim_{x\to 0} \frac{\cos x}{1} \qquad \text{(By L.H. rule)}$$

$$= \frac{1}{2} \cdot \frac{\cos 0}{1}$$

$$= \frac{1}{2}(1)$$

$$= \frac{1}{2}$$

Q. $\lim_{x\to 0} \left(\frac{1}{x \sin^{-1} x} - \frac{1}{x^2} \right)$

Soln. Let l be the req. limit then

$$l = \lim_{x\to 0} \left(\frac{1}{x \sin^{-1} x} - \frac{1}{x^2} \right)$$

$$= \lim_{x\to 0} \frac{x - \sin^{-1} x}{x^2 \sin^{-1} x} \qquad \left(\frac{0}{0}\right)$$

Put $\sin^{-1} x = t$

$\Rightarrow x = \sin t$

Here as $x \to 0$, $t \to 0$

So above limit becomes

$$l = \lim_{t \to 0} \frac{\sin t - t}{\sin^2 t \cdot t}$$

$$= \lim_{t \to 0} \frac{\sin t - t}{t \sin^2 t} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{t \to 0} \frac{\cos t - 1}{t(2\sin t \cos t) + \sin^2 t} \quad \text{(By L.H rule)}$$

$$= \lim_{t \to 0} \frac{\cos t - 1}{t \sin 2t + \sin^2 t} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{t \to 0} \frac{-\sin t}{t(\cos 2t \cdot 2) + \sin 2t + 2\sin t \cos t}$$

$$= \lim_{t \to 0} \frac{-\sin t}{2t\cos 2t + \sin 2t + \sin 2t}$$

$$= \lim_{t \to 0} \frac{-\sin t}{2t\cos 2t + 2\sin 2t} \quad \left(-\frac{0}{0}\right)$$

$$= \lim_{t \to 0} \frac{-\cos t}{2(t(-\sin 2t \cdot 2) + \cos 2t) + 2\cos 2t \cdot 2} \quad \text{(By L.H rule)}$$

$$= \lim_{t \to 0} \frac{-\cos t}{-4t\sin 2t + 2\cos 2t + 4\cos 2t}$$

$$= \frac{-1}{0 + 2(1) + 4(1)}$$

$$= \frac{-1}{2+4}$$

$$\therefore l = \frac{-1}{6} \quad \text{——— Ans}$$

Q $\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$

Sol. Let l be the req. limit then

$l = \lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$

Multiplying num. & den. by $\sqrt{x^2+5x}+x$

$$= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+5x}-x\right)\left(\sqrt{x^2+5x}+x\right)}{\left(\sqrt{x^2+5x}+x\right)}$$

$$= \lim_{x\to\infty}\left(\frac{\cancel{x^2}+5x-\cancel{x^2}}{\sqrt{x^2+5x}+x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{5x}{\sqrt{x^2+5x}+x}\right)$$

$$= \lim_{x\to\infty}\frac{5x}{\left(x\sqrt{1+\frac{5}{x}}+x\right)}$$

$$= \lim_{x\to\infty}\frac{5\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{5}{x}}+1\right)}$$

$$= \lim_{x\to\infty}\frac{5}{\left(\sqrt{1+\underset{0}{\frac{5}{x}}}+1\right)}$$

$$= \frac{5}{\left(\sqrt{1+0}+1\right)}$$

$$= \frac{5}{1+1}$$

$$= \frac{5}{2}$$ Ans.

Q Use L' Hospital rule to prove that,

$$\lim_{x\to\infty}\left[\frac{a^{1/x}+b^{1/x}}{2}\right]^{x} = \sqrt{ab} \quad . \; a>0,\; b>0$$

Sol:- let l be the req. limit then

$$l = \lim_{x\to\infty}\left[\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right]^{x}$$

Let $y = \left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^{x}$

taking ln on both sides

$$\ln y = \ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^{x}$$

$$\ln y = x\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x.\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right) \qquad (\infty\cdot 0)$$

$$= \lim_{x\to\infty}\frac{\ln\left(a^{\frac{1}{x}}+b^{\frac{1}{x}}\right)-\ln 2}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{1}{(a^{\frac{1}{x}}+b^{1/x})}\left\{a^{\frac{1}{x}}.\ln a.\frac{-1}{x^2}+b^{\frac{1}{x}}.\ln b.\frac{-1}{x^2}\right\}}{\frac{-1}{x^2}} \qquad (\text{By L' Hospital rule})$$

$$= \lim_{x\to\infty}\frac{-x^2\left[a^{\frac{1}{x}}.\ln a.\frac{1}{x^2}+b^{\frac{1}{x}}.\ln b\,\frac{1}{x^2}\right]}{-(a^{\frac{1}{x}}+b^{1/x})}$$

$$= \lim_{x\to\infty}\frac{\left(a^{1/x}\ln a+b^{1/x}\ln b\right)}{\left(a^{1/x}+b^{1/x}\right)}$$

$$= \frac{a^0\ln a+b^0\ln b}{a^0+b^0}$$

$$= \frac{\ln a+\ln b}{1+1}$$

$$\lim_{x\to\infty} \ln y = \frac{1}{2}\ln(ab)$$

$$\lim_{x\to\infty} \ln y = \ln(ab)^{1/2}$$

$$\lim_{x\to\infty} y = (ab)^{1/2}$$

$$l = \sqrt{ab}$$

Q If f is a thrice differentiable function, prove by using L' Hospital rule that

(a) $$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$$

(b) $$\lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$$

(c) $$\lim_{h\to 0} \frac{f(x+h)-f(x)-hf'(x)-\frac{h^2}{2}f''(x)}{h^3} = \frac{f'''(x)}{6}$$

Sol.

Consider the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f'(x+h)-f'(x-h)(-1)}{2} \qquad \text{(By L.H. rule)}$$

$$= \lim_{h\to 0} \frac{f'(x+h)+f'(x-h)}{2}$$

$$= \frac{f'(x+0)+f'(x-0)}{2}$$

$$= \frac{f'(x)+f'(x)}{2}$$

$$= \frac{2f'(x)}{2}$$

$$= f'(x) \quad \text{Ans.}$$

(b) Consider the limit

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f'(x+h) + f'(x-h)(-1)}{2h} \qquad \text{(By L.H. Rule)}$$

$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f''(x+h) - f''(x-h)(-1)}{2} \qquad \text{(By L.H. Rule)}$$

$$= \lim_{h\to 0} \frac{f''(x+h) + f''(x-h)}{2}$$

$$= \frac{f''(x+0) + f''(x-0)}{2}$$

$$= \frac{f''(x) + f''(x)}{2}$$

$$= \frac{2f''(x)}{2}$$

$$= f''(x) \quad \text{—— Ans}$$

(C). Sol.- Consider the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - hf'(x) - \frac{h^2}{2} f''(x)}{h^3} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x) - \frac{2h}{2} f''(x)}{3h^2} \qquad \text{(By L.H. Rule)}$$

(9) $\lim_{x\to 0} \frac{\tanh x - \sinh x}{x^2}$

Sol: Let l be the req. limit then

$$l = \lim_{x\to 0} \frac{\tanh x - \sinh x}{x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\operatorname{sech}^2 x - \cosh x}{2x} \quad \text{(By L.H. rule)}$$

$$= \lim_{x\to 0} \frac{2\operatorname{sech} x \cdot \operatorname{sech} x \cdot \tanh x - \sinh x}{2} \quad \text{(By L.H. rule)}$$

$$= \lim_{x\to 0} \frac{2\operatorname{sech}^2 x \tanh x - \sinh x}{2}$$

$$= \frac{2\operatorname{sech}^2(0) \cdot \tanh(0) - \sinh 0}{2}$$

$$= \frac{2(1)(0) - 0}{2}$$

$$= \frac{0}{2}$$

$$= 0$$

Q $\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$

Sol. Let l be the req. limit then

$$l = \lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$$

Let $y = \left(\frac{x+a}{x-a}\right)^x$

taking ln on both sides

$$\ln y = \ln\left(\frac{x+a}{x-a}\right)^x$$

$$\ln y = x\ln\left(\frac{x+a}{x-a}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) \qquad (\infty\times 0)$$

Put $x = \frac{1}{t}$ $\Rightarrow$ as $x\to\infty$, $t\to 0$

So $$\lim_{x\to\infty}\ln y = \lim_{t\to 0}\frac{1}{t}\ln\left(\frac{\frac{1}{t}+a}{\frac{1}{t}-a}\right)$$

$$= \lim_{t\to 0}\frac{1}{t}\ln\left(\frac{1+at}{1-at}\right) \qquad (\infty\times 0)$$

$$= \lim_{t\to 0}\frac{\ln(1+at)-\ln(1-at)}{t} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{t\to 0}\frac{\frac{a}{1+at}-\frac{-a}{1-at}}{1} \qquad \text{(By L.H. rule)}$$

$$= \lim_{t\to 0}\left(\frac{a}{1+at}+\frac{a}{1-at}\right)$$

$$= \frac{a}{1+0}+\frac{a}{1-0}$$

$$= a+a$$

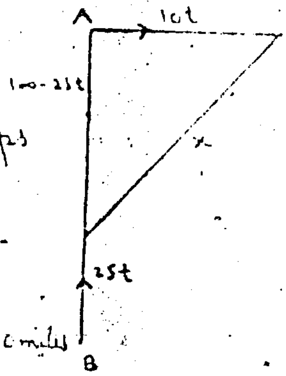
$$\lim_{x\to\infty}\ln y = 2a$$

$$\lim_{x\to\infty} y = e^{2a}$$

Empty

Q A ship sails east from port A at 10 mph. At the same time another ship leaves port B which is 100 miles due south of port A & sails north at 25 mph. For how long is the distance between the ships decreasing?

Sol. Let at $t=0$, first ship is at port A & second ship is at port B.



Let x be the distance between the ships after t hours.

Since vel. of first ship is 10 mph.

So distance travelled by first ship in ~~1 hour~~ = 10 miles

Hence distance travelled by first ship after t hours = 10t miles

Now distance travelled by second ship in 1 hour = 25 miles

So distance travelled by second ship after t hours = 25t miles

Now by Pythagoras theorem

$$x^2 = (10t)^2 + (100-25t)^2$$

$$x^2 = 100t^2 + 10000 - 5000t + 625t^2$$

$$x^2 = 725t^2 - 5000t + 10000$$

Diff. w.r.t. t

$$2x\frac{dx}{dt} = 1450t - 5000$$

$$\Rightarrow \frac{dx}{dt} = \frac{725t - 2500}{x}$$

Now $\frac{dx}{dt} < 0$ when $725t - 2500 < 0$

i.e., when $725t < 2500$

i.e., when $t < \frac{2500}{725}$

i.e., when $t < \frac{100}{29}$

So the distance b/w ships decreases for $\frac{100}{29}$ hours.

Q4. Discuss the validity of Rolle's theorem & find the value of c. 112

$$f(x) = \frac{1-x^2}{1+x^2} \quad \text{on } [-1,1]$$

Sol. Given function is

$$f(x) = \frac{1-x^2}{1+x^2} \quad \text{on } [-1,1]$$

Now $f(-1) = \frac{1-(-1)^2}{1+(-1)^2} = \frac{1-1}{1+1} = \frac{0}{2} = 0$

& $f(1) = \frac{1-(1)^2}{1+(1)^2} = \frac{1-1}{1+1} = \frac{0}{2} = 0$

So $f(-1) = f(1)$

Clearly $f(x)$ is continuous in $[-1,1]$

& $f(x)$ is derivable in $]-1,1[$

Since all the conditions of Rolle's theorem are satisfied. Hence by Rolle's theorem

$f'(c) = 0$ for some $c \in]-1,1[$

Now $f'(x) = \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$

$= \frac{-2x-\cancel{2x^3}-2x+\cancel{2x^3}}{(1+x^2)^2}$

$f'(x) = \frac{-4x}{(1+x^2)^2}$

$\Rightarrow f'(c) = \frac{-4c}{(1+c^2)^2}$

But $f'(c) = 0$

So $\frac{-4c}{(1+c^2)^2} = 0$

$\Rightarrow -4c = 0$

or $\boxed{c = 0}$

Q14. Prove that $(1+x)^a > 1+ax$ where $a>1$, $x>0$ (Bernoulli's inequality) by M.V. theorem.

Sol.

Let $f(x) = (1+x)^a - (1+ax)$

We note that $f(x)$ is continuous in $[0,x]$

$f(x)$ is derivable in $]0,x[$

Since f satisfies the conditions of M.V. th. So

by M.V. th.

$$\frac{f(x)-f(0)}{x-0} = f'(c) \qquad \text{for some } c \in]0,x[$$

or $$f(x)-f(0) = x f'(c) \text{ ——— (1)}$$

Here $f(x) = (1+x)^a - (1+ax)$

$\Rightarrow f(0) = (1+0)^a - (1+0)$

$f(0) = 0$

& $f'(x) = a(1+x)^{a-1} - a$

$f'(c) = a(1+c)^{a-1} - a$

Put values in (1)

$$(1+x)^a - (1+ax) - 0 = x\left[a(1+c)^{a-1} - a\right]$$

or $(1+x)^a - (1+ax) = ax\left((1+c)^{a-1} - 1\right)$

Since $a>1$, $x>0$ & $(1+c)^{a-1} - 1 > 0$

So $(1+x)^a - (1+ax) > 0$

or $(1+x)^a > 1+ax$

Q115 use M.V. theorem to show that

$$\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$$

Also approximate $\sqrt{168}$ by using M.V. theorem.

Sol:- Consider a function f defined as

$$f(x) = \sqrt{x} \quad \text{on interval } [25,27].$$

Clearly $f(x)$ is continuous on $[25,27]$

& $f(x)$ is derivable on $]25,27[$

So $f(x)$ satisfies conditions of M.V. th on $[25,27]$

Hence by M.V. th.

$$\frac{f(27)-f(25)}{27-25} = f'(c) \quad \text{for some } c \in]25,27[$$

$$\frac{\sqrt{27}-\sqrt{25}}{2} = \frac{1}{2\sqrt{c}} \quad \text{for some } c \in]25,27[.$$

$$\text{or } \sqrt{27} - 5 = \frac{1}{\sqrt{c}} \quad \text{——} (1) \text{ for some } c \in]25,27[.$$

Since $c \in]25,27[$

So $25 < c < 27$

or $\sqrt{25} < \sqrt{c} < \sqrt{27}$

or $\dfrac{1}{\sqrt{25}} > \dfrac{1}{\sqrt{c}} > \dfrac{1}{\sqrt{27}}$

or $\dfrac{1}{\sqrt{27}} < \dfrac{1}{\sqrt{c}} < \dfrac{1}{5}$

Also $\dfrac{1}{\sqrt{36}} < \dfrac{1}{\sqrt{27}} < \dfrac{1}{\sqrt{c}}$

or $\dfrac{1}{6} < \dfrac{1}{\sqrt{c}} < \dfrac{1}{5}$

3.35   E.)

or $\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$.   from eq. ①

is the req. proof!

Again Consider the function

$$f(x) = \sqrt{x} \quad \text{on} \quad [168, 169]$$

Clearly $f(x)$ is Continuous on $[168,169]$

& $f(x)$ is derivable on $]168,169[$

~~Hence $f(x)$ Satisfies the Conditions of M.V.Th.~~

So by M.V. Th.

$$\frac{f(169) - f(168)}{169 - 168} = f'(c) \quad \text{for some } c \in]168,169[$$

$$\text{or} \quad \sqrt{169} - \sqrt{168} = \frac{1}{2\sqrt{c}} \quad \text{for some } 168 < c < 169$$

Suppose c is nearly equal to 169

$$\text{So} \quad \sqrt{169} - \sqrt{168} = \frac{1}{2\sqrt{169}}$$

$$13 - \sqrt{168} = \frac{1}{2 \times 13}$$

$$13 - \sqrt{168} = \frac{1}{26}$$

$$-\sqrt{168} = \frac{1}{26} - 13$$

$$\sqrt{168} = 13 - \frac{1}{26}$$

$$= 13 - 0.0385$$

$$= 12.9615$$

C.3

$$= \lim_{h\to 0} \frac{f'(x+h) - f'(x) - hf''(x)}{3h^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f''(x+h) - f''(x).1}{6h} \qquad (\text{By L.H. Rule})$$

$$= \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{6h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0} \frac{f'''(x+h)}{6}$$

$$= \frac{f'''(x+0)}{6}$$

$$= \frac{f'''(x)}{6} \text{ ——— Ans.}$$

End of Ch-3

(Thank God)