Exercise # 3.3

1. $\lim_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \dfrac{e^x - (e^{-x}(-1))}{\cos x}$

$= \lim_{x\to 0} \dfrac{e^x + e^{-x}}{\cos x}$

$= \dfrac{e^0 + e^0}{\cos 0} = \dfrac{1+1}{1}$

$= 2$

2. $\lim_{x\to 0} \dfrac{e^{x^2} - 1}{\cos x - 1}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \dfrac{e^{x^2}(2x) - 0}{-\sin x - 0}$

$= \lim_{x\to 0} \dfrac{2xe^{x^2}}{-\sin x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \dfrac{2(e^{x^2}(1) + x.e^{x^2}(2x))}{-\cos x}$

$= \lim_{x\to 0} \dfrac{2(e^{x^2} + 2x^2e^{x^2})}{-\cos x}$

$= \dfrac{2(e^0 + 0)}{-\cos 0} = \dfrac{2(1+0)}{-1}$

$= -2$

3. $\lim_{x\to 0} \dfrac{x - \tan x}{x - \sin x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule

$= \lim_{x\to 0} \dfrac{1 - \sec^2 x}{1 - \cos x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \dfrac{0 - 2\sec x(\sec x \tan x)}{0 + \sin x}$

$= \lim_{x\to 0} \dfrac{-2\sec^2 x \tan x}{\sin x}$ $\left(\frac{0}{0}\right)$ L-Hospital

$= \lim_{x\to 0} \dfrac{-2[\sec^2 x \sec^2 x + \tan x . 2\sec^2 x \tan x]}{\cos x}$

$= \lim_{x\to 0} \dfrac{-2[\sec^4 x + 2\sec^2 x \tan^2 x]}{\cos x}$

$= \dfrac{-2[\sec^4(0) + 2(0)]}{\cos 0}$

$= \dfrac{-2[1+0]}{1} = -2$

4. $\lim_{x\to \pi} \dfrac{\sin^2 x}{\cos 3x + 1}$ $\left(\frac{0}{0}\right)$

Sol: Applying L-Hospital Rule

$= \lim_{x\to \pi} \dfrac{2\sin x \cos x}{-3\sin 3x}$

$= \lim_{x\to \pi} \dfrac{\sin 2x}{-3\sin 3x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital.

$= \lim_{x\to \pi} \dfrac{2\cos 2x}{-3(3\cos 3x)}$

$= \dfrac{2\cos(2\pi)}{-9\cos(3\pi)}$

$= \dfrac{2(1)}{-9(-1)}$

$= \dfrac{2}{9}$

5. $\lim_{x\to 0} \dfrac{\sin x}{\sqrt{1-\cos x}}$.

$\because \sqrt{\dfrac{1-\cos x}{2}} = \sin\dfrac{x}{2} \Rightarrow \dfrac{\sqrt{1-\cos x}}{\sqrt{2}} = \sin\dfrac{x}{2}$

$\Rightarrow \sqrt{1-\cos x} = \sqrt{2}\sin(x/2)$

$\lim_{x\to 0} \dfrac{\sin x}{\sqrt{1-\cos x}} = \lim_{x\to 0} \dfrac{\sin x}{\sqrt{2}\sin(\frac{x}{2})}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital

$= \lim_{x\to 0} \dfrac{\cos x}{\sqrt{2}\cos(\frac{x}{2}).\frac{1}{2}}$

$$= \lim_{x\to 0} \frac{2\cos x}{\sqrt{2}\cos(\frac{x}{2})}$$

$$= \frac{2\cos 0}{\sqrt{2}\cos 0} = \frac{2(1)}{\sqrt{2}(1)}$$

$$= \frac{\sqrt{2}.\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$$

$$= \sqrt{2}$$

6. $$\lim_{x\to 0} \frac{\tan x - \sin x}{x^2 \tan x} \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^2 \frac{\sin x}{\cos x}}$$

$$= \lim_{x\to 0} \frac{\frac{\sin x - \sin x \cos x}{\cancel{\cos x}}}{x^2 \frac{\sin x}{\cancel{\cos x}}}$$

$$= \lim_{x\to 0} \frac{\cancel{\sin x}(1-\cos x)}{x^2 \cancel{\sin x}}$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} \left(\frac{0}{0}\right)$$

Applying L-Hospital.

$$= \lim_{x\to 0} \frac{0+\sin x}{2x} \left(\frac{0}{0}\right)$$

Applying L-Hospital

$$= \lim_{x\to 0} \frac{\cos x}{2}$$

$$= \frac{\cos 0}{2} = \frac{1}{2}$$

7. $$\lim_{x\to 1} \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \left(\frac{0}{0}\right)$$

Applying L-Hospital

$$= \lim_{x\to 1} \frac{n(n+1)x^n - (n+1)nx^{n-1} + 0}{2(x-1)}$$

$$= \lim_{x\to 1} \frac{n(n+1)}{2}\left[\frac{x^n - x^{n-1}}{x-1}\right]$$

$$= \frac{n(n+1)}{2} \lim_{x\to 1} \left[\frac{x^n - x^{n-1}}{x-1}\right] \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \frac{n(n+1)}{2} \lim \left[\frac{nx^{n-1} - (n-1)x^{n-2}}{1-0}\right]$$

$$= \frac{n(n+1)}{2} \left[\frac{n(1)^{n-1} - (n-1)(1)^{n-2}}{1}\right]$$

$$= \frac{n(n+1)}{2} (n-(n-1))$$

$$= \frac{n(n+1)}{2} (\cancel{n} - \cancel{n} + 1)$$

$$= \frac{n(n+1)}{2}$$

8. $$\lim_{x\to 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0} \frac{e^x + 2\sin x - e^{-x}}{x\cos x + \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital

$$= \lim_{x\to 0} \frac{e^x + 2\cos x + e^{-x}}{\cos x - x\sin x + \cos x}$$

$$= \frac{e^0 + 2\cos 0 + e^0}{\cos 0 - 0 + \cos 0} = \frac{1+2+1}{1+1}$$

$$= \frac{4}{2}$$

$$= 2$$

9. $$\lim_{x\to 0} \frac{\ln(1-x^2)}{\ln(\cos x)} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x\to 0} \frac{\frac{-2x}{1-x^2}}{\frac{1}{\cos x}(-\sin x)}$$

$$= \lim_{x\to 0} \frac{\cancel{-}2x\cos x}{\cancel{-}\sin x(1-x^2)}$$

$$= \lim_{x\to 0} \frac{2x\cos x}{\sin x(1-x^2)} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x\to 0} \frac{2(\cos x - x\sin x)}{\cos x(1-x^2) + \sin x(-2x)}$$

$$= \frac{2(\cos 0 - 0)}{\cos 0(1-0)+0}$$

$$= \frac{2(1-0)}{1(1)+0} = \frac{2}{1}$$

$$= 2$$

10. $$\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital.

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{x \cos x + \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{\cos x - x \sin x + \cos x}$$

$$= \frac{\cosh(0) + \cos(0)}{\cos 0 - 0 + \cos 0} = \frac{1+1}{1+1}$$

$$= \frac{2}{2}$$

$$= 1.$$

11. $$\lim_{x \to 1} \frac{1 - x + \ln x}{1 - \sqrt{2x - x^2}} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x \to 1} \frac{0 - 1 + \frac{1}{x}}{0 - \frac{1}{2}(2x - x^2)^{-1/2}(2-2x)}$$

$$= \lim_{x \to 1} \frac{\left(\frac{-x+1}{x}\right)(2x - x^2)^{1/2}}{-\frac{1}{\cancel{2}} \cdot \cancel{2}(1-x)}$$

$$= \lim_{x \to 0} \frac{\cancel{(1-x)}(2x - x^2)^{1/2}}{-x\cancel{(1-x)}}$$

$$= \lim_{x \to 0} \frac{(2x - x^2)^{1/2}}{-x}$$

$$= \frac{(2(1) - 1^2)^{1/2}}{-1}$$

$$= \frac{(2-1)^{1/2}}{-1} = \frac{1^{1/2}}{-1}$$

$$= -1$$

12. $$\lim_{x \to 0} \frac{x \cos x - \ln(1+x)}{x^2} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x \to 0} \frac{-x \sin x + \cos x - \frac{1}{1+x}}{2x} \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-x \cos x - \sin x - \sin x - \left(-\frac{1}{(1+x)^2}\right)}{2}$$

$$= \lim_{x \to 0} \frac{-x \cos x - 2 \sin x + \frac{1}{(1+x)^2}}{2}$$

$$= \frac{0 - 0 + \frac{1}{(1+0)^2}}{2} = \frac{0 + \frac{1}{1}}{2} = \frac{1}{2}$$

13. $$\lim_{x \to 0} \frac{\sin x - \ln(e^x \cos x)}{x \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital

$$= \lim_{x \to 0} \frac{\cos x - \frac{1}{e^x \cos x}(e^x \cos x - e^x \sin x)}{x \cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x - \frac{e^x \cos x}{e^x \cos x} + \frac{e^x \sin x}{e^x \cos x}}{x \cos x + \sin x}.$$

$$= \lim_{x \to 0} \frac{\cos x - 1 + \tan x}{x \cos x + \sin x} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x \to 0} \frac{-\sin x - 0 + \sec^2 x}{-x \sin x + \cos x + \cos x}$$

$$= \frac{-\sin 0 + \sec^2 0}{-0 + \cos 0 + \cos 0}$$

$$= \frac{0+1}{1+1}$$

$$= \frac{1}{2}$$

14. $\lim_{x \to 0} \dfrac{1 + \sin x - \cos x + \ln(1-x)}{x\tan^2 x}$ (4)

Applying L-Hospital Rule,

$$= \lim_{x \to 0} \frac{0 + \cos x + \sin x + \frac{1}{1-x}(-1)}{\tan^2 x + x(2\tan x \sec^2 x)}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{\tan^2 x + 2x\tan x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{\tan^2 x + 2x\tan x(1+\tan^2 x)} = \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{\tan^2 x + 2x\tan x + 2x\tan^3 x}$$

Applying L-Hospital Rule. $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - \left(\frac{-1}{(1-x)^2}(-1)\right)}{2\tan x \sec^2 x + 2x\sec^2 x + 2\tan x + 2x(3\tan^2 x \sec^2 x) + 2\tan^3 x}$$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{6\tan^2 x \sec^2 x + 2\tan x \sec^2 x + 2x\sec^2 x + 2\tan x + 2\tan^3 x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - \left(\frac{-2}{(1-x)^2}(-1)\right)}{\begin{gathered}6(2\tan x \sec^2 x)\sec^2 x + 6\tan^2 x(2\sec x . \sec x \tan x) + 2(\sec^2 x)\sec^2 x \\ + 2\tan x(2\sec x . \sec x \tan x) + 2(1)\sec^2 x + 2x(2\sec x . \sec x \tan x) + 2\sec^2 x \\ + 2(3\tan^2 x . \sec^2 x)\end{gathered}}$$

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - \frac{2}{(1-x)^2}}{\begin{gathered}12\tan x \sec^4 x + 12\tan^3 x \sec^2 x + 2\sec^4 x + 4\sec^2 x \tan^2 x \\ + 2\sec^2 x + 4x\tan x \sec^2 x + 2\sec^2 x + 6\tan^2 x \sec^2 x\end{gathered}}$$

$$= \frac{-\cos 0 - 0 - \frac{2}{(1-0)^2}}{0 + 0 + 2\sec^4 0 + 0 + 2\sec^2 0 + 0 + 2\sec^2 0 + 0}$$

$$= \frac{-1-2}{2+2+2}$$

$$= -\frac{\cancel{3}}{\cancel{6}\,2}$$

$$= -\frac{1}{2}$$

15. If $\lim\limits_{x\to 0} \frac{\sin 2x + a\sin x}{x^3}$ exists. find value of 'a' and the Limit.

Sol:

$$\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} \longrightarrow (1)$$

Applying L-Hospital

$$= \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2}$$

As $3x^2 \to 0$ when $x\to 0$ and given that

$$\lim_{x\to 0} \frac{\sin 2x + a\sin x}{3x^2} \text{ exist}$$

So we conclude

$2\cos 2x + a\cos x \longrightarrow 0$ when $x\to 0$

i.e $2\cos 2(0) + a\cos(0) = 0$

$\Rightarrow 2(1) + a(1) = 0$

$\Rightarrow 2 + a = 0$

$\Rightarrow \boxed{a = -2}$

put in Limit (1)

$$\lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^3} \rightarrow \left(\frac{0}{0}\right)$$

Applying L-Hospital

$$= \lim_{x\to 0} \frac{2\cos 2x - 2\cos x}{3x^2} \left(\frac{0}{0}\right)$$

again L-Hospital Rule,

$$= \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} \left(\frac{0}{0}\right)$$

again L-Hospital Rule,

$$= \lim_{x\to 0} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8\cos(0) + 2\cos(0)}{6}$$

$$= \frac{-8(1)+2}{6} = \frac{\cancel{-6}}{\cancel{6}}$$

$$= -1$$

16. (5)

$$\lim_{x\to 0} \frac{\ln(\sin 3x)}{\ln(\sin x)} \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule,

$$= \lim_{x\to 0} \frac{\frac{1}{\sin 3x}(3\cos 3x)}{\frac{1}{\sin x}(\cos x)}$$

$$= \lim_{x\to 0} \frac{3\cos 3x \sin x}{\sin 3x \cos x} \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0} \frac{3\cos 3x\cos x + 3(-3\sin 3x)\sin x}{3\cos 3x\cos x + \sin 3x(-\sin x)}$$

$$= \lim_{x\to 0} \frac{3\cos 3x\cos x - 9\sin 3x\sin x}{3\cos 3x\cos x - \sin 3x\sin x}$$

$$= \frac{3\cos(0)\cos(0) - 9(0)}{3\cos(0)\cos(0) - 0} = \frac{3(1)-0}{3(1)-0}$$

$$= \frac{\cancel{3}}{\cancel{3}}$$

$$= 1$$

17. $\lim\limits_{x\to 0} \left(\frac{1}{x\sin^{-1}x} - \frac{1}{x^2}\right) \rightarrow (1)$

Let $\sin^{-1}x = y$

$\Rightarrow x = \sin y$

when $x\to 0$ Then $y\to 0$

$$(1) \Rightarrow \lim_{y\to 0} \left(\frac{1}{(\sin y)y} - \frac{1}{\sin^2 y}\right)$$

$$= \lim_{y\to 0} \left(\frac{\sin y - y}{y\sin^2 y}\right) \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{y\to 0} \frac{\cos y - 1}{\sin^2 y + y\cdot 2\sin y\cos y}$$

$$= \lim_{y\to 0} \frac{\cos y - 1}{\sin^2 y + y\sin 2y} \left(\frac{0}{0}\right)$$

L-Hospital

$$= \lim_{y\to 0} \frac{-\sin y - 0}{2\sin y\cos y + \sin 2y + 2\cos 2y \cdot y}$$

$$= \lim_{y\to 0} \frac{-\sin y}{\sin 2y + \sin 2y + 2y\cos 2y}$$

$$= \lim_{y\to 0} \frac{-\sin y}{2\sin 2y + 2y\cos 2y} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{y\to 0} \frac{-\cos y}{4\cos 2y + 2\cos 2y - 4y\sin 2y}$$

$$= \frac{-\cos 0}{4\cos 0 + 2\cos 0 - 4(0)}$$

$$= \frac{-1}{4(1)+2(1)-0} = \frac{-1}{6}$$

18. $\lim_{x\to a} \frac{\ln(x-a)}{\ln(e^x-a^x)} \left(\frac{\infty}{\infty}\right)$

Applying L-Hospital Rule.

$$= \lim_{x\to a} \frac{\frac{1}{x-a}(1)}{\frac{1}{e^x-a^x}\, e^x}$$

$$= \lim_{x\to a} \frac{e^x - a^x}{e^x(x-a)} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x\to a} \frac{e^x}{e^x(1) + e^x(x-a)}$$

$$= \frac{e^a}{e^a + e^a\cancel{(a-a)}} = \frac{e^a}{e^a+0}$$

$$= \frac{\cancel{e^a}}{\cancel{e^a}}$$

$$= 1$$

19. $\lim_{x\to 0} \frac{\ln(\tan x)}{\ln x} \left(\frac{\infty}{\infty}\right)$

Applying L-Hospital.

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{\cos x}}{\sin x} \times \frac{1}{\cos^{\cancel{2}} x}}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{x}{\sin x\cos x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0} \frac{1}{\sin x(-\sin x) + \cos x\cos x}$$

$$= \lim_{x\to 0} \frac{1}{-\sin^2 x + \cos^2 x}$$

$$= \frac{1}{-\sin^2(0) + \cos^2(0)} = \frac{1}{0+1}$$

$$= 1$$

20. $\lim_{x\to 0} \log_{\tan x}(\tan 2x)$

$$= \lim_{x\to 0} \frac{\ln(\tan 2x)}{\ln(\tan x)} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x}\cdot(2\sec^2 2x)}{\frac{1}{\tan x}\cdot \sec^2 x}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{\cos 2x}}{\sin 2x}\cdot\frac{2}{\cos^{\cancel{2}} 2x}}{\frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cos^{\cancel{2}} x}}$$

$$= \lim_{x\to 0} \frac{2\sin x\cos x}{\sin 2x\cos 2x}$$

$$= \lim_{x\to 0} \frac{\cancel{\sin 2x}}{\cancel{\sin 2x}\cos 2x}$$

$$= \lim_{x\to 0} \frac{1}{\cos 2x} = \frac{1}{\cos 0}$$

$$= \frac{1}{1} = 1$$

21. $\lim_{x\to a} (x-a)\operatorname{cosec}\left(\frac{\pi x}{a}\right)$ $(0\times\infty)$

$$= \lim_{x\to a} \frac{x-a}{\sin\left(\frac{\pi x}{a}\right)} \quad \left(\frac{0}{0}\right)$$

L-Hospital

$$= \lim_{x\to 0} \frac{1}{\cos\left(\frac{\pi x}{a}\right)\cdot\left(\frac{\pi}{a}\right)}$$

$$= \frac{1}{\frac{\pi}{a}\cos(a\pi/a)} = \frac{a}{\pi(-1)}$$

$$= -\frac{a}{\pi}$$

22. $\lim_{x\to 1} (1-x)\tan\left(\frac{\pi x}{2}\right)$ $(0\times\infty)$

$= \lim_{x\to 1} \frac{1-x}{\cot\left(\frac{\pi x}{2}\right)}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule

$= \lim_{x\to 1} \frac{\cancel{-}1}{\cancel{-}\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}$

$= \lim_{x\to 1} \frac{+\sin^2\left(\frac{\pi x}{2}\right)\cdot 2}{\pi}$

$= \frac{+\sin^2\left(\frac{\pi}{2}\right).2}{\pi}$

$= \frac{(1).2}{\pi} = \frac{2}{\pi}$

23. $\lim_{x\to 0} x\ln(\tan x)$ $0\times\infty$

$= \lim_{x\to 0} \frac{\ln(\tan x)}{1/x}$ $\left(\frac{\infty}{\infty}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\cdot\sec^2 x}{-1/x^2}$

$= \lim_{x\to 0} \frac{\frac{\cancel{\cos x}}{\sin x}\cdot\frac{1}{\cos^{\cancel{2}} x}}{-1/x^2}$

$= \lim_{x\to 0} -\frac{x^2}{\sin x\cos x}$

$= \lim_{x\to 0} -\frac{2x^2}{2\sin x\cos x}$

$= \lim_{x\to 0} -\frac{2x^2}{\sin 2x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital

$= \lim_{x\to 0} -\frac{{}^{2}\cancel{4}x}{\cos 2x\cancel{(2)}}$

$= \lim_{x\to 0} \frac{-2x}{\cos 2x}$

$= \frac{0}{\cos 0} = \frac{0}{1}$

$= 0$

24. $\lim_{x\to 0} x\tan\left(\frac{\pi}{2}-x\right)$ $(0\times\infty)$

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$= \lim_{x\to 0} x\cot x$ $(0\times\infty)$

$= \lim_{x\to 0} \frac{\cot x}{1/x}$ $\left(\frac{\infty}{\infty}\right)$

Applying L-Hospital Rule.

$= \lim_{x\to 0} \frac{\cancel{-}\csc^2 x}{\cancel{-}1/x^2}$

$= \lim_{x\to 0} x^2\csc^2 x$

$= \lim_{x\to 0} \frac{x^2}{\sin^2 x}$

$= \lim_{x\to 0} \frac{1}{\sin^2 x/x^2}$

$= \lim_{x\to 0} \frac{1}{\left(\frac{\sin x}{x}\right)^2}$

$= \frac{1}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2}$

$= \frac{1}{(1)^2} = 1.$

25. $\lim_{x\to\pi/2} \tan x\ln(\sin x)$ $(\infty\times 0)$

$= \lim_{x\to\pi/2} \frac{\ln(\sin x)}{\cot x}$ $\left(\frac{0}{0}\right)$

L-Hospital.

$= \lim_{x\to\pi/2} \frac{\frac{1}{\sin x}(\cos x)}{-\csc^2 x}$

$= \lim_{x\to\pi/2} \frac{\cos x}{-\sin x\cdot\frac{1}{\sin^2 x}}$

$= \lim_{x\to\pi/2} -\sin x\cos x$

$= -\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$

$= -(1)(0)$

$= 0$

26. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$ (P) $(\infty-\infty)$

$= \lim\limits_{x\to 0}\left(\frac{e^x-1-x}{x(e^x-1)}\right)$

$= \lim\limits_{x\to 0}\frac{e^x-1-x}{xe^x-x}$ $\left(\frac{0}{0}\right)$

L-Hospital;

$= \lim\limits_{x\to 0}\frac{e^x-0-1}{e^x+xe^x-1}$

$= \lim\limits_{x\to 0}\frac{e^x-1}{xe^x+e^x-1}$ $\left(\frac{0}{0}\right)$

L-Hospital

$= \lim\limits_{x\to 0}\frac{e^x-0}{xe^x+e^x+e^x-0}$

$= \lim\limits_{x\to 0}\frac{\cancel{e^x}}{\cancel{e^x}(x+1+1)}$

$= \lim\limits_{x\to 0}\frac{1}{x+2}$

$= \frac{1}{0+2}$

$= \frac{1}{2}$

27. $\lim\limits_{x\to 0}\left[\frac{a}{x}-\cot\left(\frac{x}{2}\right)\right]$ $(\infty-\infty)$

$= \lim\limits_{x\to 0}\left[\frac{a}{x}-\frac{\quad}{\sin(x/a)}\right]$

$= \lim\limits_{x\to 0}\left[\frac{a\sin(x/a)-x\cos(x/a)}{x\sin(x/a)}\right]$

L-Hospital Rule;

$= \lim\limits_{x\to 0}\frac{a\cos(x/a)\cdot\frac{1}{a}-\left(\cos\frac{x}{a}-\frac{1}{a}x\sin\left(\frac{x}{a}\right)\right)}{x\cos\left(\frac{x}{a}\right)\cdot\frac{1}{a}+\sin\left(\frac{x}{a}\right)}$

$= \lim\limits_{x\to a}\frac{\cancel{\cos\left(\frac{x}{a}\right)}-\cancel{\cos\left(\frac{x}{a}\right)}+\frac{x}{a}\sin\left(\frac{x}{a}\right)}{\frac{x}{a}\cos\left(\frac{x}{a}\right)+\sin\left(\frac{x}{a}\right)}$

$= \lim\limits_{x\to a}\frac{\frac{x}{a}\sin\left(\frac{x}{a}\right)}{\frac{x}{a}\cos\left(\frac{x}{a}\right)+\sin\left(\frac{x}{a}\right)}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim\limits_{x\to 0}\frac{\frac{x}{a}\cos\left(\frac{x}{a}\right)\cdot\frac{1}{a}+\frac{1}{a}\sin\left(\frac{x}{a}\right)}{-\frac{x}{a}\sin\left(\frac{x}{a}\right)\cdot\frac{1}{a}+\frac{1}{a}\cos\left(\frac{x}{a}\right)+\frac{1}{a}\cos\frac{x}{a}}$

$= \lim\limits_{x\to 0}\frac{\frac{x}{a^2}\cos\left(\frac{x}{a}\right)+\frac{1}{a}\sin\left(\frac{x}{a}\right)}{-\frac{x}{a^2}\sin\left(\frac{x}{a}\right)+\frac{2}{a}\cos\left(\frac{x}{a}\right)}$

$= \frac{\frac{(0)}{a^2}(1)+0}{0+\frac{2}{a}(1)} = \frac{0}{2/a} = 0$

$= 0$

28. $\lim\limits_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$ $(\infty-\infty)$

$= \lim\limits_{x\to 1}\left(\frac{x\ln x-(x-1)}{\ln x(x-1)}\right)$

$= \lim\limits_{x\to 1}\left(\frac{x\ln x-x+1}{\ln x(x-1)}\right)$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule,

$= \lim\limits_{x\to 1}\frac{\ln x+\cancel{x}\cdot\frac{1}{\cancel{x}}-1+0}{\frac{1}{x}(x-1)+\ln(x)(1-0)}$

$= \lim\limits_{x\to 1}\frac{\ln x+\cancel{1}-\cancel{1}}{1-\frac{1}{x}+\ln x}$

$= \lim\limits_{x\to 1}\frac{\ln x}{1-\frac{1}{x}+\ln x}$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim\limits_{x\to 1}\frac{1/x}{0+\frac{1}{x^2}+\frac{1}{x}}$

$= \frac{1/1}{1/1^2+1/1}$

$= \frac{1}{1+1}$

$= \frac{1}{2}$

29. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ $(\infty - \infty)$

$= \lim\limits_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$

$= \lim\limits_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x}\right)$ $\left(\frac{0}{0}\right)$

Applying L-Hospital Rule.

$= \lim\limits_{x \to \pi/2} \frac{0 - \cos x}{-\sin x}$

$= \lim\limits_{x \to \pi/2} \frac{-\cos x}{-\sin x}$

$= \frac{\cos \pi/2}{\sin \pi/2} = \frac{0}{1}$

$= 0$

30. $\lim\limits_{x \to 1} \left(\frac{2}{x^2-1} - \frac{1}{x-1}\right)$

$= \lim\limits_{x \to 1} \left[\frac{2 - (x+1)}{x^2-1}\right]$

$= \lim\limits_{x \to 1} \left[\frac{2 - x - 1}{x^2-1}\right]$

$= \lim\limits_{x \to 1} \left[\frac{1 - x}{x^2-1}\right]$ $\left(\frac{0}{0}\right)$

L-Hospital Rule.

$= \lim\limits_{x \to 1} \frac{-1}{2x}$

$= \frac{-1}{2(1)} = -\frac{1}{2}$

31. $\lim\limits_{x \to \infty} \left(\sqrt{x^2+5x} - x\right)$ $(\infty - \infty)$

$= \lim\limits_{x \to \infty} \frac{(\sqrt{x^2+5x} - x)}{\sqrt{x^2+5x}+x} \times (\sqrt{x^2+5x}+x)$

$= \lim\limits_{x \to \infty} \frac{(\sqrt{x^2+5x})^2 - x^2}{\sqrt{x^2+5x}+x}$

$= \lim\limits_{x \to \infty} \frac{\cancel{x^2} + 5x - \cancel{x^2}}{\sqrt{x^2(1+\frac{5}{x^2})} + x}$

$= \lim\limits_{x \to \infty} \frac{5x}{x\sqrt{1+\frac{5}{x^2}} + x}$

$= \lim\limits_{x \to \infty} \frac{5\cancel{x}}{\cancel{x}\left[\sqrt{1+\frac{5}{x^2}} + 1\right]}$

$= \lim\limits_{x \to \infty} \frac{5}{\sqrt{1+\frac{5}{x^2}} + 1}$

$= \frac{5}{\sqrt{1+\frac{5}{\infty}} + 1} = \frac{5}{\sqrt{1+0}+1} = \frac{5}{1+1}$

$= \frac{5}{2}$

32. $\lim\limits_{x \to \infty} (e^x + e^{-x})^{2/x}$ $(\infty)^0$

Let $y = (e^x + e^{-x})^{2/x}$

$\ln y = \ln(e^x + e^{-x})^{2/x}$

$\ln y = \frac{2}{x} \ln(e^x + e^{-x})$

$\ln y = \frac{2\ln(e^x + e^{-x})}{x}$

$\lim\limits_{x \to \infty} \ln y = \lim\limits_{x \to \infty} \frac{2\ln(e^x + e^{-x})}{x}$ $\left(\frac{\infty}{\infty}\right)$

Applying L-Hospital Rule,

$\lim\limits_{x \to \infty} \ln y = \lim\limits_{x \to \infty} \frac{2 \cdot \frac{1}{e^x + e^{-x}}(e^x - e^{-x})}{1}$

$= \lim\limits_{x \to \infty} \frac{2(e^x - e^{-x})}{e^x + e^{-x}}$

$= \lim\limits_{x \to \infty} \frac{2\cancel{e^x}(1 - e^{-2x})}{\cancel{e^x}(1 + e^{-2x})}$

$= \lim\limits_{x \to \infty} \frac{2(1 - e^{-2x})}{1 + e^{-2x}}$

$= \frac{2(1 - e^{-\infty})}{1 + e^{-\infty}}$

$= \frac{2(1-0)}{1+0} = \frac{2(1)}{1}$

$\lim\limits_{x \to \infty} \ln y = 2$

$\lim\limits_{x \to \infty} y = e^2$

33. $\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$ $(\infty)^0$

Let

$$y = \left(\frac{1}{x}\right)^{\tan x}$$

$$\ln y = \ln\left(\frac{1}{x^1}\right)^{\tan x}$$

$$\ln y = \tan x \ln(x^{-1})$$

$$\ln y = \tan x \ln(x)^{-1}$$

$$\ln y = -\tan x \ln x$$

$$\ln y = -\frac{\ln x}{\cot x}$$

Applying Limit

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{-\ln x}{\cot x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\cancel{-}1/x}{\cancel{-}\operatorname{cosec}^2 x}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0} \frac{2\sin x\cos x}{1}$$

$$= 2\sin(0)\cos(0) = 2(0)(1)$$

$$\lim_{x\to 0} \ln y = 0$$

$$\lim_{x\to 0} y = e^0 = 1$$

34. $\lim_{x\to \pi/2} (\cos x)^{-x+\frac{\pi}{2}}$ $(0)^0$

Let

$$y = (\cos x)^{-x+\frac{\pi}{2}}$$

$$\ln y = \ln(\cos x)^{-x+\frac{\pi}{2}}$$

$$\ln y = \left(-x+\frac{\pi}{2}\right)\ln(\cos x)$$

$$\ln y = \frac{\ln(\cos x)}{\left(-x+\frac{\pi}{2}\right)^{-1}}$$

$$\lim_{x\to\frac{\pi}{2}} \ln y = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\cos x)}{\left(-x+\frac{\pi}{2}\right)^{-1}} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital

$$= \lim_{x\to\pi/2} \frac{\frac{1}{\cos x}(-\sin x)}{-1\left(-x+\frac{\pi}{2}\right)^{-2}(-1)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\left(-x+\frac{\pi}{2}\right)^{-2}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\left(-x+\frac{\pi}{2}\right)^2}{\cot x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x\to\pi/2} \frac{2\left(-x+\frac{\pi}{2}\right)(-1)}{-\operatorname{cosec}^2 x}$$

$$= \frac{2\left(-\frac{\pi}{2}+\frac{\pi}{2}\right)}{\operatorname{cosec}^2(\pi/2)} = \frac{0}{1}$$

$$= 0$$

$$\lim_{x\to\pi/2} \ln y = 0$$

$$\lim_{x\to\pi/2} y = e^0 = 1$$

35. $\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x$ $(\infty)^\infty$

Let

$$y = \left(\frac{x+a}{x-a}\right)^x$$

$$\ln y = \ln\left(\frac{x+a}{x-a}\right)^x$$

$$\ln y = x\ln\left(\frac{x+a}{x-a}\right)$$

$$\ln y = \frac{\ln(x+a)-\ln(x-a)}{1/x} \quad \left(\frac{\infty}{\infty}\right)$$

Applying Limit

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(x+a)-\ln(x-a)}{1/x}$$

Applying L-Hospital Rule,

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\frac{1}{x+a}-\frac{1}{x-a}}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{\frac{\cancel{x}-a-\cancel{x}-a}{(x+a)(x-a)}}{-1/x^2}$$

$$= \lim_{x\to\infty} \frac{+2ax^2}{x^2-a^2}$$

$$= \lim_{x\to\infty} \frac{2a\cancel{x^2}}{\cancel{x^2}\left(1-\frac{a^2}{x^2}\right)}$$

$$= \frac{2a}{1-\frac{a}{\infty}} = \frac{2a}{1-0} = 2a$$

$$\lim_{x\to\infty} \ln y = 2a$$

$$\lim_{x\to\infty} y = e^{2a}$$

36. $\lim_{x\to 0}\left(\frac{\sinh x}{x}\right)^{1/x^2}$ $(\infty)^{\infty}$

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Let $y = \left(\frac{\sinh x}{x}\right)^{1/x^2}$

$$\ln y = \ln\left(\frac{\sinh x}{x}\right)^{1/x^2}$$

$$\ln y = \frac{1}{x^2}\ln\left(\frac{\sinh x}{x}\right)$$

$$\ln y = \frac{\ln \sinh x - \ln x}{x^2}$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln \sinh x - \ln x}{x^2}$$

Applying L-Hospital Rule,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\frac{1}{\sinh x}\cosh x - \frac{1}{x}}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{x\cosh x - \sinh x}{x\sinh x}}{2x}$$

$$= \lim_{x\to 0}\frac{x\cosh x - \sinh x}{2x^2\sinh(x)} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule.

$$= \lim_{x\to 0}\frac{\cancel{\cosh x} + x\sinh x - \cancel{\cosh x}}{2x^2\cosh x + 4x\sinh x}$$

$$= \lim_{x\to 0}\frac{\cancel{x}\sinh x}{\cancel{x}(2x\cosh x + 4\sinh x)}$$

$$= \lim_{x\to 0}\frac{\sinh x}{2x\cosh x + 4\sinh x} \quad \left(\frac{0}{0}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to 0}\frac{\cosh x}{2\cosh x + 2x\sinh x + 4\cosh x}$$

$$= \frac{\cosh(0)}{2\cosh(0) + 0 + 4\cosh(0)}$$

$$= \frac{1}{2+4} = \frac{1}{6}$$

$$\lim_{x\to 0}\ln y = \frac{1}{6}$$

$$\lim_{x\to 0} y = e^{1/6}$$

37. $\lim_{x\to 0}(\tan x)^{\sin 2x}$ $(0)^0$

Let $y = (\tan x)^{\sin 2x}$

$$\ln y = \ln(\tan x)^{\sin 2x}$$

$$\ln y = \sin 2x \ln(\tan x) = \frac{\ln(\tan x)}{\operatorname{cosec} 2x}$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(\tan x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\frac{1}{\tan x}\sec^2 x}{-2\operatorname{cosec} 2x\cot 2x}$$

$$= \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}}{-2\cdot\frac{1}{\sin 2x}\cdot\frac{\cos 2x}{\sin 2x}}$$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{-2\cos 2x\sin x\cos x}$$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{-\cos 2x(2\sin x\cos x)}$$

$$= \lim_{x\to 0}\frac{\sin^2 2x}{-\cos 2x\cdot\cancel{\sin 2x}}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{-\cos 2x}$$

$$= \frac{\sin(0)}{-\cos(0)} = \frac{0}{1} = 0$$

$$\lim_{x\to 0}\ln y = 0$$

$$\lim_{x\to 0} y = e^0 = 1.$$

38. $\lim_{x\to 0}(1+\sin x)^{\cot x}$

Let $y = (1+\sin x)^{\cot x}$

$$\ln y = \ln(1+\sin x)^{\cot x}$$

$$\ln y = \cot x\ln(1+\sin x)$$

$$\ln y = \frac{\ln(1+\sin x)}{\tan x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{\tan x}$$

Applying L-Hospital Rule,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\frac{1}{1+\sin x}\cos x}{\sec^2 x}$$

$$= \lim_{x\to 0} \frac{\cos x}{(1+\sin x)\frac{1}{\cos^2 x}}$$

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$$= \lim_{x\to 0} \frac{\cos^3 x}{1+\sin x}$$

$$= \frac{\cos^3(0)}{1+\sin 0} = \frac{1^3}{1+0}$$

$$= 1$$

$$\lim_{x\to 0} \ln y = 1$$

$$\lim_{x\to 0} y = e^1$$

39. $\lim_{x\to\frac{\pi}{2}} (\sec x)^{\cot x}$ $(\infty)^0$

Let

$$y = (\sec x)^{\cot x}$$

$$\ln y = \ln(\sec x)^{\cot x}$$

$$\ln y = \cot x \ln(\sec x)$$

$$\ln y = \frac{\ln(\sec x)}{\tan x}$$

$$\lim_{x\to\frac{\pi}{2}} \ln y = \lim_{x\to\frac{\pi}{2}} \frac{\ln(\sec x)}{\tan x} \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule,

$$\lim_{x\to\frac{\pi}{2}} \ln y = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sec x}\cdot(\sec x\tan x)}{\sec^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec^2 x} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule,

$$= \lim_{x\to\pi/2} \frac{\cancel{\sec^2 x}}{2\cancel{\sec x}.\cancel{\sec x}\tan x}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{2\tan x}$$

$$= \frac{1}{2\tan(\frac{\pi}{2})}$$

$$= \frac{1}{\infty}$$

$$= 0$$

$$\lim_{x\to\frac{\pi}{2}} \ln y = 0$$

$$\lim_{x\to\frac{\pi}{2}} y = e^0 = 1$$

40. $\lim_{x\to 1} (1-x^2)^{1/\ln(1-x)}$ $(0)^0$

Let $y = (1-x^2)^{1/\ln(1-x)}$

$$\ln y = \ln(1-x^2)^{1/\ln(1-x)}$$

$$\ln y = \frac{1}{\ln(1-x)} \ln(1-x^2)$$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln(1-x^2)}{\ln(1-x)} \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule,

$$= \lim_{x\to 1} \frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)}$$

$$= \lim_{x\to 1} \frac{2x(1-x)}{1-x^2}$$

$$= \lim_{x\to 1} \frac{2x\cancel{(1-x)}}{\cancel{(1-x)}(1+x)}$$

$$= \lim_{x\to 1} \frac{2x}{1+x}$$

$$= \frac{2(1)}{1+1} = \frac{\cancel{2}}{\cancel{2}}$$

$$\lim_{x\to 1} \ln y = 1$$

$$\lim_{x\to 1} y = e^1$$

41. $\lim_{x\to 1} \left[\tan\left(x\frac{\pi}{4}\right)\right]^{\tan\left(x\frac{\pi}{2}\right)}$

Let $y = \left[\tan\left(x\frac{\pi}{4}\right)\right]^{\tan\left(\frac{x\pi}{2}\right)}$

$$\ln y = \ln\left[\tan\left(x\frac{\pi}{4}\right)\right]^{\tan\left(\frac{x\pi}{2}\right)}$$

$$\ln y = \tan\left(\frac{x\pi}{2}\right)\ln\left(\tan\left(\frac{x\pi}{4}\right)\right)$$

$$\ln y = \frac{\ln\left(\tan\left(\frac{x\pi}{4}\right)\right)}{\cot\left(\frac{x\pi}{2}\right)}$$

$$\lim_{x\to 1} \ln y = \lim_{x\to 1} \frac{\ln\left(\tan\left(\frac{x\pi}{4}\right)\right)}{\cot\left(\frac{x\pi}{2}\right)}$$

$$= \lim_{x\to 1} \frac{\frac{1}{\tan\left(x\frac{\pi}{4}\right)}\sec^2\left(\frac{x\pi}{4}\right)\cdot\frac{\pi}{4}}{-\operatorname{cosec}^2\left(\frac{x\pi}{2}\right)\cdot\frac{\pi}{2}}$$

$$= \lim_{x\to 1} \frac{\frac{\cancel{\cos\left(\frac{x\pi}{4}\right)}}{\sin\left(\frac{x\pi}{4}\right)}\cdot\frac{1}{\cos^{\cancel{2}}\left(\frac{x\pi}{4}\right)}\cdot\frac{\cancel{\left(\frac{\pi}{4}\right)}}{2}}{-\frac{1}{\sin^2\left(\frac{x\pi}{2}\right)}\cdot\cancel{\frac{\pi}{2}}}$$

$$= \lim_{x\to 1} \frac{-\sin^2\left(\frac{\pi x}{2}\right)}{2\sin\left(\frac{x\pi}{4}\right)\cos\left(\frac{x\pi}{4}\right)}$$

$$= \lim_{x\to 1} \frac{-\sin^2\left(\frac{\pi x}{2}\right)}{\sin \cancel{2}\left(\frac{x\pi}{\cancel{4}\,2}\right)}$$

$$= \lim_{x\to 1} \frac{-\sin^{\cancel{2}}\left(\frac{\pi x}{2}\right)}{\cancel{\sin\left(\frac{\pi x}{2}\right)}}$$

$$= \lim_{x\to 1} -\sin\left(\frac{\pi x}{2}\right)$$

$$\lim_{x\to 1} \ln y = -\sin\left(\frac{\pi}{2}\right) = -1$$

$$\lim_{x\to 1} y = e^{-1} = \frac{1}{e}$$

42. $\lim_{x\to \frac{\pi}{2}} (1-\sin x)^{\cos x}$

Let

$$y = (1-\sin x)^{\cos x}$$

$$\ln y = \ln(1-\sin x)^{\cos x}$$

$$\ln y = \cos x \ln(1-\sin x)$$

$$\ln y = \frac{\ln(1-\sin x)}{\sec x}$$

$$\lim_{x\to \pi/2} \ln y = \lim_{x\to \frac{\pi}{2}} \frac{\ln(1-\sin x)}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule

$$= \lim_{x\to \frac{\pi}{2}} \frac{\frac{1}{1-\sin x}(-\cos x)}{\sec x \tan x}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{-\cos x}{(1-\sin x)\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{-\cos x\cdot\cos^2 x}{\sin x(1-\sin x)}$$

$$= \lim_{x\to \frac{\pi}{2}} -\frac{\cos x(1-\sin^2 x)}{1-\sin x}$$

$$= \lim_{x\to \frac{\pi}{2}} \frac{-\cos x\cancel{(1-\sin x)}(1+\sin x)}{\cancel{1-\sin x}}$$

$$= \lim_{x\to \pi/2} -\cos x(1+\sin x)$$

$$= -\cos\left(\frac{\pi}{2}\right)\left(1+\sin\frac{\pi}{2}\right) = -0(2)$$

$$\lim_{x\to \pi/2} \ln y = 0$$

$$\lim_{x\to \pi/2} y = e^0 = 1$$

43. $\lim_{x\to 0} (\cot x)^{\sin 2x} \quad (\infty^0)$

Let

$$y = (\cot x)^{\sin 2x}$$

$$\ln y = \ln(\cot x)^{\sin 2x}$$

$$\ln y = \sin 2x \ln \cot x$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(\cot x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty}\right)$$

Applying L-Hospital Rule.

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\frac{1}{\cot x}(\cancel{-}\operatorname{cosec}^2 x)}{\cancel{-}2\operatorname{cosec} 2x \cot 2x}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{\sin x}}{\cos x}\cdot\frac{1}{\sin^{\cancel{2}} x}}{2\frac{1}{\sin 2x}\cdot\frac{\cos 2x}{\sin 2x}}$$

$$= \lim_{x\to 0} \frac{\sin^2 2x}{2\cos 2x(\cos x\sin x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 2x}{\cos 2x(2\sin x\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^{\cancel{2}} 2x}{\cos 2x\cdot\cancel{\sin 2x}}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{\cos 2x} = \frac{\sin 0}{\cos 0} = \frac{0}{1}$$

$$\lim_{x\to 0} \ln y = 0$$

$$\lim_{x\to 0} y = e^0 = 1$$

44. $\lim_{x\to 0} \frac{\tanh x - \sinh x}{x^2} \quad \left(\frac{0}{0}\right)$

L-Hospital

$$\lim_{x\to 0} \frac{\operatorname{sech}^2 x - \cosh x}{2x}$$

L-Hospital again

$$= \lim_{x\to 0} \frac{2\operatorname{sech} x(\operatorname{sech} x\tanh x) - \sinh x}{2}$$

$$= \lim_{x\to 0} \frac{2\operatorname{sech}^2 x\tanh x - \sinh x}{2}$$

$$= \frac{2\operatorname{sech}^2(0)\tanh(0) - \sinh(0)}{2}$$

$$= \frac{0-0}{2}$$

$$= 0$$

45. $y = \lim\limits_{x\to 0} \dfrac{\sqrt{x} - \sqrt{\sin x}}{x^{5/2}} = \lim\limits_{x\to 0} \dfrac{\sqrt{x} - (\sin x)^{1/2}}{x^{5/2}}$

$$y = \lim_{x\to 0} \frac{\sqrt{x} - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots\right]^{1/2}}{x^{5/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x} - \left[x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \cdots\cdots\right)\right]^{1/2}}{x^{5/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x} - x^{1/2}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\cdots\right)^{1/2}}{x^{5/2}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x} - \sqrt{x}\left(1 - \left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \cdots\cdots\right)\right)^{1/2}}{x^{5/2}}$$

(Binomial Series)

$$= \lim_{x\to 0} \frac{\sqrt{x} - \sqrt{x}\left[1 - \frac{1}{2}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots\right) - \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!}\left(\frac{x^2}{3!} - \frac{x^4}{5!} + \cdots\right)^2 + \cdots\right.}{x^{5/2}}$$

$$= \lim_{x\to 0} \frac{x^{1/2}\left[1 - 1\left[1 - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \cdots\right) - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x^2}{6} - \frac{x^4}{12} + \cdots\right)^2 + \cdots\right.\right.}{x^{5/2}}$$

$\left\{\frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2\right.$

$$= \lim_{x\to 0} \frac{1 - 1\left[1 - \frac{1}{2}\cdot\frac{x^2}{6} + \text{heigher powers of } x\right]}{x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{1} - \cancel{1} + \frac{x^2}{12} + \text{heigher powers of } x}{x^2}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{12} + \text{heigher powers of } x}{x^2}$$

$$= \frac{1}{12} + 0$$

$$= \frac{1}{12}$$

46. $\lim\limits_{x\to 0} \dfrac{\sinh x - \sin x}{\sin^3 x}$ $\left(\frac{0}{0}\right)$ L-Hospital

$= \lim\limits_{x\to 0} \dfrac{\cosh x - \cos x}{3\sin^2 x \cos x}$ $\left(\frac{0}{0}\right)$ L-Hospital

$= \lim\limits_{x\to 0} \dfrac{\sinh x + \sin x}{6\sin x \cos^2 x + 3\sin^2 x(\sin x)}$

$= \lim\limits_{x\to 0} \dfrac{\sinh(x) + \sin x}{6\sin x \cos^2 x - 3\sin^3 x}$ $\left(\frac{0}{0}\right)$

Again Applying L-Hospital Rule.

$$= \lim_{x\to 0} \frac{\cosh x + \cos x}{6\cos x\cos^2 x + 12\sin x\cos x(-\sin x) - 9\sin^2 x\cos x}$$ 15

$$= \lim_{x\to 0} \frac{\cosh x + \cos x}{6\cos^3 x - 12\sin^2 x\cos x - 9\sin^2 x\cos x}$$

$$= \frac{\cosh(0) + \cos 0}{6\cos 0 + 0 - 0} = \frac{1+1}{6(1)} = \frac{2}{6}$$

$$= \frac{1}{3}$$

47. $$Z = \lim_{x\to 0} \frac{(1+x)^{1/x} - e + e\frac{x}{2}}{x^2} \rightarrow (1)$$

Let $y = (1+x)^{1/x}$

$$\ln y = \ln(1+x)^{1/x}$$

$$\ln y = \frac{1}{x}\ln(1+x)$$

$$\ln y = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\right)$$

$$\ln y = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots\dots$$

$$\Rightarrow y = e^{\left[1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right]}$$ $\therefore e^x = 1 + x + \frac{x^2}{2!} + \dots$

$$\Rightarrow y = e^1 \cdot e^{\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)}$$

$$\Rightarrow y = e\cdot\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} + \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} + \dots\right)^2 + \dots\right]$$

$$\Rightarrow y = e\left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{2}\left(\frac{x^2}{4}\right) - \dots\right]$$

$$\Rightarrow y = e - \frac{ex}{2} + \frac{ex^2}{3} + \frac{ex^2}{8} - \dots\dots$$

$$\Rightarrow y = e - \frac{ex}{2} + ex^2\left[\frac{1}{3} + \frac{1}{8}\right] - \dots\dots$$

$$\Rightarrow y = e - \frac{ex}{2} + ex^2\left(\frac{11}{24}\right) - \dots$$

put value of y in (1)

$$\Rightarrow Z = \lim_{x\to 0} \frac{\cancel{e} - \cancel{\frac{ex}{2}} + ex^2\left(\frac{11}{24}\right) - \dots\dots \cancel{-e} + \cancel{\frac{ex}{2}}}{x^2}$$

$$\mathcal{Z} = \lim_{x\to 0} \frac{\frac{11e}{24}x^2 + \text{heigher powers of } x}{x^2}$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}\left[\frac{11e}{24} + \text{heigher powers of } x\right]}{\cancel{x^2}}$$

$$= \lim_{x\to 0} \frac{11e}{24} + \text{heigher powers of } x$$

$$= \frac{11e}{24}$$

48. $\lim_{x\to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ [Series expansion]

$$= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\right)}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{1} - \cancel{1} + \cancel{x} - \cancel{x} + \cancel{\frac{x^2}{2}} - \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} - \frac{x^4}{8} + \frac{x^4}{4!} \cdots\cdots}{\cancel{x} - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\left[\frac{1}{3!} - \frac{x}{8} + \frac{x}{8!} + \cdots\right]}{\cancel{x^3}\left[\frac{1}{3!} - \frac{x^2}{5!} + \cdots\right]}$$

$$= \frac{\frac{1}{3!} - 0 + 0 \cdots}{\frac{1}{3!} - 0 + 0 \cdots}$$

$$= \frac{\cancel{1/3!}}{\cancel{1/3!}}$$

$$= 1$$

49. Use L'Hospital Rule to prove that

$$\lim_{x\to\infty} \left[\frac{a^{1/x} + b^{1/x}}{2}\right]^x = \sqrt{ab}\ ,\ a>0, b>0$$

Sol Let $y = \left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x$

$$\ln y = \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)^x$$

$$\ln y = x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(a^{1/x} + b^{1/x}) - \ln 2}{1/x}$$

Applying L-Hospital Rule

$$= \lim_{x \to \infty} \frac{\frac{1}{a^{1/x}+b^{1/x}} \cdot \left[a^{1/x}\ln a\left(-\frac{1}{x^2}\right) + b^{1/x}\ln b\left(-\frac{1}{x^2}\right)\right]}{-1/x^2}$$

$$= \lim_{x \to \infty} \frac{1}{a^{1/x}+b^{1/x}} \cdot \frac{\cancel{\left(-\frac{1}{x^2}\right)}\left[a^{1/x}\ln a + b^{1/x}\ln b\right]}{\cancel{-1/x^2}}$$

$$= \lim_{x \to \infty} \frac{a^{1/x}\ln a + b^{1/x}\ln b}{a^{1/x}+b^{1/x}}$$

$$= \frac{a^{1/\infty}\ln a + b^{1/\infty}\ln b}{a^{1/\infty}+b^{1/\infty}} = \frac{a^0\ln a + b^0\ln b}{a^0+b^0}$$

$$= \frac{\ln a + \ln b}{1+1} = \frac{\ln a + \ln b}{2}$$

$$= \frac{1}{2}(\ln a + \ln b) = \frac{1}{2}\ln(ab) = \ln(ab)^{1/2}$$

$$\lim_{x \to \infty} \cancel{\ln} y = \cancel{\ln}(ab)^{1/2}$$

$$\Rightarrow \lim_{x \to \infty} y = (ab)^{1/2} = \sqrt{ab}$$

Hence proved.

50. If 'f' is Thrice differentiable function then prove by L-Hospital Rule.

(i) $\lim_{h \to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$

Apply L-Hospital $\left(\frac{0}{0}\right)$

$$\lim_{h \to 0} \frac{f(x+h)-f(x-h)}{2h} = \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)(-1)}{2}$$

$$= \lim_{h \to 0} \frac{f'(x+h)+f'(x-h)}{2} = \frac{f'(x+0)+f'(x-0)}{2}$$

$$= \frac{f'(x)+f'(x)}{2}$$

$$= \frac{\cancel{2}f'(x)}{\cancel{2}}$$

$$= f'(x)$$

(ii) $\lim_{h \to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$

$$\lim_{h \to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{f'(x+h) - 0 + f'(x-h)(-1)}{2h}$$

$$= \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} \quad \left(\frac{0}{0}\right)$$

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L-Hospital Rule

$$= \lim_{h \to 0} \frac{f''(x+h) - f''(x-h)(-1)}{2}$$

$$= \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2}$$

$$= \frac{f''(x+0) + f''(x-0)}{2} = \frac{\cancel{2}f''(x)}{\cancel{2}}$$

$$= f''(x) \qquad \text{Hence Proved.}$$

(iii) $$\lim_{h \to 0} \frac{f(x+h) - f(x) - hf'(x) - \frac{h^2}{2} f''(x)}{h^3} = \frac{f'''(x)}{6}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) - hf'(x) - \frac{h^2}{2} f''(x)}{h^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{f'(x+h) - 0 - f'(x) - \frac{2h}{2} f''(x)}{3h^2}$$

$$= \lim_{h \to 0} \frac{f'(x+h) - f'(x) - hf''(x)}{3h^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{f''(x+h) - 0 - f''(x)}{6h}$$

$$= \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{6h}$$

$$= \lim_{h \to 0} \frac{f''(x+h) - f''(x)}{6h} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{f'''(x+h) - 0}{6}$$

$$= \frac{f'''(x+0)}{6}$$

$$= \frac{f'''(x)}{6} \qquad \text{Hence Proved.}$$

51. Determine a, b, c, d and e such that

$$Z = \lim_{x \to 0} \frac{\cos ax + bx^3 + cx^2 + dx + e}{x^4} = \frac{2}{3} \longrightarrow (1)$$

Sol: if limit to be of form $\left(\frac{0}{0}\right)$

then $\cos(ax) + bx^3 + cx^2 + dx + e \to 0$ when $x \to 0$

$\Rightarrow \cos(0) + b(0) + c(0) + d(0) + e = 0$

$\Rightarrow 1 + 0 + e = 0$

$\Rightarrow e = -1 \longrightarrow$ put in (1)