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(General theorems, indeterminate forms)

(Chapter no. 3)

Before starting this chapter, we should know the following terms.

Definition ① let S be a non empty subset of real no's. An element $M \in R$ is said to be an upper bound of S if

$$x \leq M \quad \forall x \in S$$

An element $m \in R$ is said to be a lower bound of S if $m \leq x \quad \forall x \in S$

If S has an upper bound then S is said to be bounded above & if S has a lower bound then S is said to be bounded below.

A set S is said to be a bounded set if it is both bounded above as well as bounded below.

Ex ①

let $S = \{1, 2, 3, \ldots\ldots\ldots, 20\}$

then every real no. $M \geq 20$ is an upper bound of S & every real no. $m \leq 1$ is a lower bound of S.

Ex ② Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\}$

An upper bound of $S = 1$

A lower bound of $S = 0$

Def (2) Let S be a non empty subset of R (2)

(1) If S is bounded above then an upper bound of S is called a least upper bound of S or supremum of S if M is less than any other upper bound of S.

(2) If S is bounded below then a lower bound m of S is called greatest lower bound of S or infimum of S if m is greater than any other lower bound of S. In notations we write it as

$M = \sup S$ or $M = \text{lub } S$

$m = \inf S$ or $m = \text{glb } S$

In example (1)

$M = 2.0$

$m = 1$

In example (2) $M = 1$, $m = 0$

Def (3) A function f is said to be continuous in $[a,b]$ if there is no gap or breakage in the graph of f in the interval $[a,b]$.

e.g., In fig f is continuous in $[1,2]$ & discontinuous in $[4,5]$

Rolle's theorem :-

Statement: let a function f be

① Continous on closed interval $[a,b]$.

② Derivable in the open interval $]a,b[$

③ $f(a) = f(b)$

Then there exists atleast one point $c \in]a,b[$ such that $f'(c) = 0$

proof :-

Because f is continous on $[a,b]$ so it is bounded.

let $M = \sup f$

& $m = \inf f$.

Case ① If $M = m$

Then f is const. on $[a,b]$ & so $f'(x) = 0 \quad \forall x \in [a,b]$

& we have the required proof.

Case ② If $M \neq m$.

Then atleast one of M & m is diffrent from $f(a)$ & $f(b)$.

Suppose $M \neq f(a) = f(b)$ ——— ①

Since f attains its superimum on $[a,b]$, so there is a pt. $c \in [a,b]$ s.that $f(c) = M$

But from ① $M \neq f(a) = f(b)$

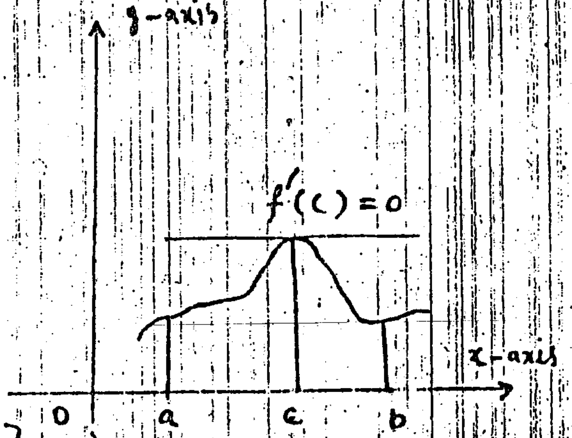
so $c \neq a$ & $c \neq b$

$\Rightarrow c \in]a,b[$

Suppose that h be a +ve real no. then

$$\left.\begin{array}{l} f(c+h) \leq f(c) \\ \& \quad f(c-h) \leq f(c) \end{array}\right\}$$

$$\left.\begin{array}{l} \dfrac{f(c+h)-f(c)}{h} \leq 0 \\ \& \\ \dfrac{f(c-h)-f(c)}{h} \leq 0 \end{array}\right\}$$



or

$$\left.\begin{array}{l} \dfrac{f(c+h)-f(c)}{h} \leq 0 \\ \& \\ \dfrac{f(c+(-h))-f(c)}{-h} \geq 0 \end{array}\right\} \quad Ⓐ$$

Taking limit as $h \to 0$

Eq. Ⓐ becomes

$$\left.\begin{array}{l} f'(c) \leq 0 \\ \& \quad f'(c) \geq 0 \end{array}\right\}$$

$$\Rightarrow \boxed{f'(c) = 0} \text{ — As required.}$$

Mean-value theoram (OR) (Lagrange's M.V.T.):-

<u>Statement</u>:-

let a function f be

① Continous on closed interval $[a,b]$

② Derivable in the open interval $]a,b[$

Then there exists a point $c \in]a,b[$
s. that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Proof:-

Define a new function

$$\phi(x) = Ax + f(x) \quad \text{——} \quad (1)$$

where A is a Constt. to be determined
s.that

$$\phi(a) = \phi(b)$$

obviously the function Ax is continous on $[a,b]$ & derivable in $]a,b[$.
Now $\phi(x)$ satisfies all the conditions of Rolle's theoram. So there is a point

$$c \in]a,b[\text{ s. that}$$

$$\phi'(c) = 0$$

$$A + f'(c) = 0$$

$$\Rightarrow f'(c) = -A \quad \text{——} \quad (2)$$

From $\phi(a) = \phi(b)$

$$\Rightarrow Aa + f(a) = Ab + f(b) \quad \text{or}$$

$$Aa - Ab = f(b) - f(a)$$

$$A(a-b) = f(b) - f(a)$$

$$\Rightarrow -A(b-a) = f(b) - f(a)$$

$$= -A = \frac{f(b)-f(a)}{b-a}$$

Put value of $-A$ in (2)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

so $$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{as desired.}$$

Note :- Another form of mean value theorem

If we take the interval as $[a, a+h]$ instead of $[a,b]$. Then mean value theorem becomes :-

$$\frac{f(a+h)-f(a)}{a+h-a} = f'(c)$$

where c is a pt s.that $a<c<a+h$

$\Rightarrow f(a+h)-f(a) = hf'(c)$ ———— (1)

If we write $c = a+\theta h$ where $0<\theta<1$

Then clearly $c = a+\theta h \in]a, a+h[$.

So (1) becomes

$$f(a+h)-f(a) = hf'(a+\theta h) \quad \text{where } 0<\theta<1$$

which is another form of M.V.T.

Cauchy's Mean Value Theorem (OR) Generalized M.V.T :-

Statement :-

If two function f & ϕ are

(1) Continous on $[a,b]$

(2) Derivable in $]a,b[$

(3) $\phi'(x) \neq 0$ for all $x \in]a,b[$ Then there

exits at least one pt $c \in]a,b[$ s. that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}$$

Proof :-

Define a new function

$$F(x) = f(x) + A\,\phi(x)$$

where A is a Constt. to be determined s. that $F(a) = F(b)$

So $f(a) + A\phi(a) = f(b) + A\phi(b)$

or $A\phi(b) - A\phi(a) = f(a) - f(b)$

$\Rightarrow A(\phi(b) - \phi(a)) = -(f(b) - f(a))$

Now

$$\phi(b) - \phi(a) \neq 0$$

Because if we Suppose that

$\phi(b) - \phi(a) = 0$ Then $\phi(a) = \phi(b)$

& then ϕ Satisfies all the Conditions of Rolle's theorem. So we must have

$\phi'(c) = 0$ for Some $c \in]a,b[$

which is a contradiction to statement

So $\phi(b) - \phi(a) \neq 0$

$$\Rightarrow A = -\frac{f(b)-f(a)}{\phi(b)-\phi(a)} \quad \text{--------} \quad (1)$$

Now the function F being sum of two derivable functions is derivable in $]a,b[$

Hence by Rolle's theorem

$$F'(c) = 0$$

$$\Rightarrow f'(c) + A\,\phi'(c) = 0$$

$$\Rightarrow A = -\frac{f'(c)}{\phi'(c)} \quad \text{——— (2)}$$

So from (1) & (2)

$$-\frac{f'(c)}{\phi'(c)} = -\frac{f(b)-f(a)}{\phi(b)-\phi(a)}$$

$\therefore$ So $\dfrac{f(b)-f(a)}{\phi(b)-\phi(a)} = \dfrac{f'(c)}{\phi'(c)}$ as desired.

Note :- Another form of Cauchy's M.V.T

If we take the interval as $[a, a+h]$ instead of $[a,b]$. Then Cauchy's M.V.T becomes :-

$$\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \quad \text{——— (1)}$$

where c is a pt s.that $a<c<a+h$

If we write $c = a+\theta h$ where $0<\theta<1$

Then clearly $a+\theta h \in]a, a+h[$

So (1) becomes

$$\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)} \quad 0<\theta<1$$

VII another form of Cauchy's M.V.T.

Increasing & decreasing functions :-

function is said to be an increasing function on $[a,b]$ if for $x_1, x_2 \in]a,b[$

$$f(x_2) > f(x_1) \text{ whenever } x_2 > x_1$$

& f is said to decreasing function on $[a,b]$ if for $x_1, x_2 \in]a,b[$

$$f(x_2) < f(x_1) \text{ whenever } x_2 > x_1$$

Theorem :- Suppose f is continous on $[a,b]$ & has derivative at each point of $]a,b[$

(1) If f' is +ve in $]a,b[$. Then f is increasing fn in $[a,b]$.

(2) If f' is -ve in $]a,b[$. Then f is decreasing function in $[a,b]$.

Proofs :-

(1) let $x_1, x_2 \in]a,b[$ s. that $x_2 > x_1$

By Lagrange's M.V.T There is pt. c b/w x_1 & x_2 s. that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{——— (1)}$$

Now $f'(c)$ is +ve & also $x_2 - x_1 > 0$

So ① $\Rightarrow f(x_2) - f(x_1) > 0$

So $f(x_2) > f(x_1)$ for $x_2 > x_1$

Hence f is an increasing function.

(2) let $x_1, x_2 \in]a,b[$ s. that $x_2 > x_1$

By Lagranges M.V.T There is a pt c ~~s. that~~

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for } c \in [x_1, x_2]$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{———— (2)}$$

Now $f'(c) < 0$ & $x_2 - x_1 > 0$

So $f(x_2) < f(x_1)$ whenever $x_2 > x_1$. Thus f is a decreasing function.

Exercise 3.1

(1) Discuss the validity of Rolle's theorem for the following functions.

(i) $f(x) = x^2 - 3x + 2$ on $[1, 2]$

Sol

let $f(x) = x^2 - 3x + 2$

$f(1) = 1 - 3 + 2 = 0$

$f(2) = 4 - 6 + 2 = 0$

Thus $f(1) = f(2)$

Clearly $f(x)$ is continous on $[1,2]$ & derivable in $]1,2[$. Since all conditions of Rolles' theorem are satisfied. Hence there must exist a pt. $c \in]1,2[$ s. that

$f'(c) = 0$

Now $f'(x) = 2x - 3$

$\Rightarrow f'(c) = 2c - 3$

So $2c - 3 = 0$

$\Rightarrow \boxed{c = \frac{3}{2}}$

Hence Rolle's theorem is valid & $c = \frac{3}{2}$

(ii) let $f(x) = \sin^2 x$ on $[0, \pi]$

Now $f(0) = \sin^2 0 = 0$

$f(\pi) = \sin^2 \pi = 0$

$f(0) = f(\pi)$

Clearly $f(x)$ is continous on $[0, \pi]$ and derivable in $]0, \pi[$. Since all conditions of Rolles' theoram are satisfied. Hence there must exist a pt $c \in]0, \pi[$ s. that

$$f'(c) = 0$$

Now $f'(x) = 2 \sin x \cos x$

So $2 \sin c \cos c = 0$

$$\Rightarrow \sin c \cos c = 0$$

$$\Rightarrow \boxed{c = 0, \frac{\pi}{2}}$$

Hence Rolles' theoram is valid at $c = \frac{\pi}{2}$

(iii) let $f(x) = 1 - x^{3/4}$ on $[-1, 1]$

now $f(-1) = 1 - (-1)^{3/4}$

$$= 1 - (-1)$$

$$f(-1) = 1+1 = 2$$

$$f(1) = 1 - (1)^{3/4}$$

$$= 1 - 1$$

$$f(1) = 0$$

Hence $f(-1) \neq f(1)$

Since one of the conditions of Rolles' theoram is not satisfied. Hence Rolles' theoram is not valid & we cannot calculate c.

Sol. By Lagrange's Mean Value Theorem

$$f(b) - f(a) = (b-a)f'(c) \quad \text{------} \quad (1)$$

where $c \in \left]-\frac{11}{7}, \frac{13}{7}\right[$

Now

$$f(a) = f\left(-\frac{11}{7}\right) = \left(-\frac{11}{7}\right)^3 - 3\left(-\frac{11}{7}\right) - 1$$

$$= -\frac{1331}{343} + \frac{33}{7} - 1$$

$$= \frac{-1331 + 1617 - 343}{343}$$

$$f\left(-\frac{11}{7}\right) = \frac{-57}{343}$$

Also

$$f'(x) = 3x^2 - 3$$

$$\Rightarrow f'(c) = 3c^2 - 3$$

putting values in (1)

$$\frac{-57}{343} - \left(-\frac{57}{343}\right) = \left[\left(\frac{13}{7}\right) - \left(-\frac{11}{7}\right)\right](3c^2 - 3)$$

$$\cancel{\frac{-57}{343}} + \cancel{\frac{57}{343}} = \left(\frac{13}{7} + \frac{11}{7}\right)(3c^2 - 3)$$

$$0 = \left(\frac{24}{7}\right)(3c^2 - 3)$$

$$\Rightarrow 3c^2 - 3 = 0$$

$$\Rightarrow c^2 - 1 = 0$$

So $\boxed{c = \pm 1}$

(ii) Here $f(x) = \sin x$ on $[a, b]$

By Lagrange's M.V theorem,

$f(b) - f(a) = (b-a) f'(c)$ ———— (i) for some $c \in]a, b[$

Now $f(a) = \sin a$

& $f(b) = \sin b$

also $f'(x) = \cos x$

$\Rightarrow f'(c) = \cos c$

So from (1)

$\sin b - \sin a = (b-a) \cos c \Rightarrow c = \cos^{-1}\left(\frac{\sin b - \sin a}{b-a}\right)$

(iii) Here $f(x) = 2x - x^3$ on $[0, 1]$

By Lagranges M.V. Theorem

$f(b) - f(a) = (b-a) f'(c)$ ———— (i) for some $c \in]a, b[$.

$f(a) = f(0) = 0 - 0 = 0$

$f(b) = f(1) = 2 - 1 = 1$

Now

$f'(x) = 2 - 3x^2$

$\Rightarrow f'(c) = 2 - 3c^2$

putting values in (i)

$1 - 0 = (1-0)(2 - 3c^2)$

$\Rightarrow 2 - 3c^2 = 1$

$\Rightarrow 3c^2 = 1$

$\Rightarrow c^2 = \frac{1}{3}$

$\boxed{c = \pm \frac{1}{\sqrt{3}}}$ or $\boxed{c = \frac{1}{\sqrt{3}}}$ $\because \frac{-1}{\sqrt{3}} \notin]0, 1[$

(iv) Let $f(x) = x^{2/3}$ on $[-1, 1]$

By Lagrange's Mean value Theoram

$$f(b) - f(a) = (b-a) f'(c) \quad \text{———} \quad (1) \quad \text{where } a<c<b$$

Now

$$f(a) = f(-1) = (-1)^{2/3} = 1$$

& $f(b) = f(1) = 1$

Now $f(x)$ is not derivable at $x=0$

i.e $f'(0)$ doesn't exist, where $0 \in]-1,1[$

Hence mean value theoram is not applicable here.

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(3) Use M.V. Theoram to show that

(i) $|\sin x - \sin y| \leq |x-y|$ for any real nos x & y

Sol

Let $f(t) = \sin t$

Then $f(t)$ is continous & differentiable for every real no.t. Hence Lagranges M.V. theoram canbe applied in the interval $[x, y]$ where x & y are any two real nos.

So. $$\frac{f(y) - f(x)}{y-x} = \cos z \quad \text{where } z \in]x, y[$$

but $|\cos z| \leq 1$;

$$\Rightarrow \left| \frac{\sin y - \sin x}{y-x} \right| \leq 1$$

$$\Rightarrow \frac{|\sin y - \sin x|}{|y-x|} \le 1$$

$$\Rightarrow |\sin x - \sin y| \le |x-y|$$

(ii) $$\left|\frac{\cos ax - \cos bx}{x}\right| \le |b-a| \quad \text{if } x \ne 0$$

sol let $f(t) = \cos(t)$

Then $f(t)$ is continous & differentiable for every real no. t. Hence Lagranges M.V.T Can be applied in the interval $[ax, bx]$ where $x \neq 0$

i.e $\frac{f(bx) - f(ax)}{bx - ax} = -\sin z$ for some $z \in]ax, bx[$

or $$\left|\frac{\cos bx - \cos ax}{x(b-a)}\right| = |-\sin z|$$

or $$\frac{|\cos bx - \cos ax|}{|x| \cdot |b-a|} = |\sin z|$$

or $$\left|\frac{\cos ax - \cos bx}{x}\right| \cdot \frac{1}{|b-a|} \le 1 \quad \because |\sin z| \le 1$$

$$\Rightarrow \left|\frac{\cos ax - \cos bx}{x}\right| \le (b-a)$$

(iii) $|\tan x + \tan y| \geq |x+y|$, for all $x, y \in R$ in the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$

Sol :-

let $f(t) = \tan t$

Then $f(t)$ is Continous in $[-x, y]$ & differentiable in $]-x, y[$ where $[-x, y] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$

Hence Lagrange's M.V.T canbe applied in the interval $[-x, y]$.

so $\dfrac{f(y) - f(-x)}{y + x} = \sec^2 z$ for some $z \in]-x, y[$

or $\dfrac{\tan y - \tan(-x)}{y + x} = \sec^2 z$

or $\dfrac{|\tan y + \tan x|}{|y + x|} = |\sec^2 z|$

But $|\sec^2 z| \geq 1 \quad \forall z \in]-x, y[$

So $\dfrac{|\tan y + \tan x|}{|y + x|} \geq 1$

$\Rightarrow |\tan x + \tan y| \geq |x + y| \quad \forall x, y \in R$ in interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$

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(4) Let a function f be continous on $[a, b]$ & $f'(x) = 0$ for all $x \in]a, b[$. prove that f is constt. Use this to show that $\sin^2 x + \cos^2 x = 1$.

Sol :- Given f is continous on $[a, b]$ & differentiable in $]a, b[$.

Let $x_1, x_2 \in [a,b]$ s. that $x_2 > x_1$

Applying Langrange's M.V.T on $[x_1, x_2] \subseteq [a,b]$

$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ for $c \in]x_1, x_2[$

But $f'(c) = 0$

$\Rightarrow f(x_2) - f(x_1) =$

So $f(x_2) = f(x_1)$ whenever $x_2 > x_1$

Hence f is a Constt. function.

Now Suppose that $f(x) = \sin^2 x + \cos^2 x$

$\Rightarrow f'(x) = 2 \sin x \cos x - 2 \cos x \sin x = 0$

So $f'(x) = 0$ for all real nos. x.

Hence f is Constt.

Suppose $f(x) = \sin^2 x + \cos^2 x = c$ ——— ①

where c is an arbitrary constt.

Since eq ① holds for every real no. x

So In particular $\sin^2 0 + \cos^2 0 = c$

$\Rightarrow \boxed{c = 1}$

Hence from ① $\sin^2 x + \cos^2 x = 1$

③ Show that $f(x) = x^3 - 3x^2 + 3x + 2$ is monotonically increasing on every interval.

Sol we know that if f is Continous on $[a,b]$ & has derivative at each pt of $]a,b[$. Then f is an increasing function

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if f' is +ve in $]a, b[$

Now $f(x) = x^3 - 3x^2 + 3x + 2$

$f'(x) = 3x^2 - 6x + 3$

$= 3(x^2 - 2x + 1)$

$= 3(x-1)^2$

So $f'(x) = 3(x-1)^2$ is +ve for all real no.s x Hence f is monotonically increasing on every interval.

(6) Prove that $f(x) = 2x - \tan^{-1}x - \ln(x+\sqrt{x^2+1})$ increases steadily on $[0, \infty[$.

Sol Let $f(x) = 2x - \tan^{-1}x - \ln(x+\sqrt{x^2+1})$

Diff w.r.to x

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}} \cdot \left[1 + \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x\right]$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{(x+\cancel{\sqrt{x^2+1}})} \cdot \left[\frac{(\cancel{\sqrt{x^2+1}}+x)}{\sqrt{x^2+1}}\right]$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}}$$

$$= \frac{2(1+x^2) - 1 - \sqrt{1+x^2}}{1+x^2}$$

or $$f'(x) = \frac{2+2x^2-1-\sqrt{x^2+1}}{(1+x^2)}$$

Now if $f(x)$ is steadily increasing then

$f'(x) \geq 0$

i.e. $2x^2+1-\sqrt{x^2+1} \geq 0$

or $2(x^2+1) - \sqrt{x^2+1} - 1 \geq 0$

or $2(x^2+1) - 2\sqrt{x^2+1} + \sqrt{x^2+1} - 1 \geq 0$

or $2\sqrt{x^2+1}\left(\sqrt{x^2+1}-1\right) + 1\left(\sqrt{x^2+1}-1\right) \geq 0$

or $\left(2\sqrt{x^2+1}+1\right)\left(\sqrt{x^2+1}-1\right) \geq 0$

which is true for all $x \in [0, \infty[$.

Hence

$f(x)$ increases steadily in $[0, \infty[$

⑦ Show that $\frac{\tan x}{x}$ is an increasing fn for $0 < x < \frac{\pi}{2}$.

Sol :-

Let $f(x) = \frac{\tan x}{x}$

$\Rightarrow f'(x) = \frac{x \sec^2 x - \tan x}{x^2}$ ——— ①

Let $\phi(x) = x \sec^2 x - \tan x$

$\phi'(x) = \sec^2 x + x(2 \sec x \cdot \sec x \tan x) - \sec^2 x$

$= \sec^2 x + 2x \sec^2 x \tan x - \sec^2 x$

so $\phi'(x) = 2x \sec^2 x \tan x$

now clearly $\phi'(x) > 0$ when $0 < x < \frac{\pi}{2}$

Now

obviously $\phi(0) = 0$

$\Rightarrow \phi(x) > \phi(0) = 0$ $\therefore \phi$ is increasing fn.

i.e $x \sec^2 x - \tan x > 0$

$\Rightarrow x \sec^2 x - \tan x > 0$ for $0 < x < \frac{\pi}{2}$

Hence from ① we conclude that

$f'(x) > 0$ for $0 < x < \frac{\pi}{2}$

Hence $f(x)$ is an increasing fn for $0 < x < \frac{\pi}{2}$.

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⑧ Determine the intervals in which $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

Sol :-

Let $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\Rightarrow f'(x) = 6x^2 - 30x + 36$

If $f(x)$ is an increasing function

Then $f'(x) > 0$

i.e $6x^2 - 30x + 36 > 0$

or $6(x^2 - 5x + 6) > 0$

or $(x^2 - 5x + 6) > 0$

or $(x-2)(x-3) > 0$

Now either $x - 2 > 0$ & $x - 3 > 0$

or $x - 2 < 0$ & $x - 3 < 0$

Case (i) If $x - 2 > 0$ & $x - 3 > 0$

$\Rightarrow$ $x > 2$ & $x > 3$

Hence we see that $f(x)$ is an increasing function for all $x > 3$ i.e for all $x \in]3, \infty[$

Case (ii) If $x - 2 < 0$ & $x - 3 < 0$

$\Rightarrow$ $x < 2$ & $x < 3$

So we see that $f(x)$ is an increasing fn of all $x < 2$. i.e for all $x \in]-\infty, 2[$.

Now if $f(x)$ is a decreasing function

then $f'(x) < 0$

i.e $6x^2 - 30x + 36 < 0$

or $6(x^2 - 5x + 6) < 0$

or $x^2 - 5x + 6 < 0$

or $(x-2)(x-3) < 0$

Now either $(x-2) < 0$ & $(x-3) > 0$

or $(x-2) > 0$ & $(x-3) < 0$

Case (1) If $x-2<0$ & $x-3>0$

$\Longrightarrow x<2$ & $x>3$

which is impossible for any $x \in R$.

Case (2) If $x-2>0$ & $x-3<0$

$\Longrightarrow x>2$ & $x<3$

So $x \in \,]2,3[$

Hence $f(x)$ is a decreasing function for $x \in \,]2,3[$.

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(9) If $x>0$ Then prove that $x-\ln(1+x) > \dfrac{x^2}{2(1+x)}$

Sol:- Let $f(x) = x - \ln(1+x) - \dfrac{x^2}{2(1+x)}$

$$f'(x) = 1 - \frac{1}{1+x} - \frac{1}{2}\left[\frac{(1+x)2x - x^2 \cdot 1}{(1+x)^2}\right]$$

$$= 1 - \frac{1}{1+x} - \frac{1}{2}\left[\frac{2x+2x^2-x^2}{(1+x)^2}\right]$$

$$= 1 - \frac{1}{1+x} - \frac{1}{2}\left(\frac{2x+x^2}{(1+x)^2}\right)$$

$$= 1 - \frac{1}{1+x} - \frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{2(1+x)^2 - 2(1+x) - 2x - x^2}{2(1+x)^2}$$

$$= \frac{\cancel{2}+\cancel{4x}+2x^2-\cancel{2}-\cancel{2x}-\cancel{2x}-x^2}{2(1+x)^2}$$

$$f'(x) = \frac{x^2}{2(1+x)^2}$$

obviously $f'(x) > 0$ for all $x>0$

Hence $f(x)$ is an increasing fn for $x>0$

But $f(0)=0$

$\Rightarrow f(x) > f(0)$ for $x>0$

i.e $x - \ln(1+x) - \dfrac{x^2}{2(1+x)} > 0$

i.e, $x - \ln(1+x) > \dfrac{x^2}{2(1+x)}$ for $x>0$

Solved Examples :-

① verify Rolles theorem for $f(x) = 1 - x^{2/3}$ on $[-1,1]$

Sol :- Let $f(x) = 1 - x^{2/3}$

$f(-1) = 1 - (-1)^{2/3}$

$= 1 - 1 = 0$

Hence $f(-1) = f(1)$

Clearly $f(x)$ is Continous on $[-1,1]$ & not derivable in $]-1,1[$.

Because $f'(x) = -\frac{2}{3} x^{-1/3}$

$\Rightarrow f'(0) = -\dfrac{2}{3x^{1/3}} = \dfrac{2}{0}$

$\therefore$ Hence $f(x)$ is not derivable at $x=0 \in]-1,1[$

$\therefore$ Rolle's Theoram is not applicable.

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② If $f(x) = x(x-1)(x-2)$, $a=0$, $b=\frac{1}{2}$ find c of M.V.T.

Sol:- Here $f(x) = x(x-1)(x-2)$

By Langrange's M.V.T.

$f(b) - f(a) = (b-a) f'(c)$ —— ① for Some $c \in]a,b[$

Now

$$f(a) = f(0) = 0$$

$$f(b) = f\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)$$

$$= \frac{3}{8}$$

Also $\quad f(x) = x^3 - 3x^2 + 2x$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

put value in (1)

$$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)\left(3c^2 - 6c + 2\right)$$

$$\Rightarrow \frac{6}{8} = 3c^2 - 6c + 2$$

or $\quad 3c^2 - 6c + 2 = \dfrac{3}{4}$

$$12c^2 - 24c + 8 = 3$$

$$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{(24)^2 - 4(12)\cdot 8}}{2.12}$$

or

$$c = \frac{24 \pm \sqrt{336}}{24}$$

$$= \frac{24 \pm 4\sqrt{21}}{24}$$

$$c = \frac{6 \pm \sqrt{21}}{6}$$

$$= 1 \pm \frac{\sqrt{21}}{6}$$

$$= 1 \pm \sqrt{\frac{21}{36}}$$

$$c = 1 \pm \sqrt{\frac{7}{12}}$$

Now $1+\sqrt{\frac{7}{12}} \notin \left[0, \frac{1}{2}\right]$

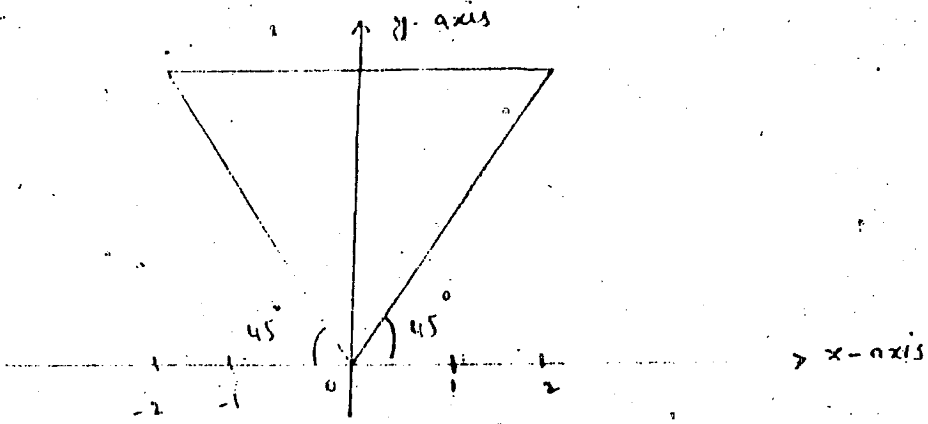
$\Rightarrow$ $\boxed{c = 1-\sqrt{\frac{7}{12}}}$ as required.

---

(3). For the function $f(x) = |x|$ check whether M.V.T holds on the interval $[-2, 2]$.

Sol:-

The graph of the function $f(x) = |x|$ is as :-



Now obviously $f(x)$ is continuous on $[-2, 2]$.

The slope of line through pts $A(-2, 2)$ & $B(2, 2)$ is $\frac{2-2}{2+2} = 0$

Now the function $f(x)$ doesn't have a derivative at $x = 0$ because

~~$f'(x) \neq 0$ for any~~ $x \in]-2, 2[$.

So one of the conditions of M.V.T is not satisfied. Hence M.V.T doesnot hold in $[-2, 2]$

---

(4) If a function f satisfies the hypothesis of M.V.T on $[a, b]$ & $|f'(x)| \leq M$ for

all $x \in]a,b[$. Then prove that

$$|f(b)-f(a)| \leq M(b-a).$$

Sol:- Given that f satisfies the hypothesis of M.V.T on $[a,b]$.

So by M.V.T

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for Some } c \in]a,b[.$$

Taking modulus on both sides

$$\left|\frac{f(b)-f(a)}{b-a}\right| = |f'(c)|$$

$$\Rightarrow \left|\frac{f(b)-f(a)}{b-a}\right| \leq M \qquad \because |f'(c)| \leq M \text{ where } a<c<b$$

$$\Rightarrow \frac{|f(b)-f(a)|}{|b-a|} \leq M$$

$$\Rightarrow |f(b)-f(a)| \leq M(b-a)$$ ——— Ans.

(5) Prove that for $x > 0$ $\frac{x}{x+1} < \ln(x+1) < x$

Sol :- Let $f(x) = \ln(x+1) - \frac{x}{x+1}$

$$f'(x) = \frac{1}{x+1} - \left[\frac{(x+1).1 - x}{(x+1)^2}\right]$$

$$= \frac{1}{x+1} - \frac{\cancel{x}+1-\cancel{x}}{(x+1)^2}$$

$$= \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

$$= \frac{x+1-1}{(x+1)^2}$$

$$f'(x) = \frac{x}{(x+1)^2}$$

So $f'(x) > 0$ for all $x > 0$

Hence $f(x)$ is an increasing function.

Now $f(0) = 0$

$\Rightarrow f(x) > f(0) = 0$ for all $x > 0$

$\Rightarrow \ln(x+1) - \frac{x}{x+1} > 0$

$\Rightarrow \ln(x+1) > \frac{x}{x+1}$

Hence

$\frac{x}{x+1} < \ln(x+1)$ ———— (1) for $x > 0$

Again suppose $\phi(x) = x - \ln(x+1)$

$$\phi'(x) = 1 - \frac{1}{x+1}$$

or $$= \frac{x+1-1}{x+1} = \frac{x}{x+1}$$

so $\phi'(x) > 0$ for $x > 0$

Now $\phi(0) = 0$

So $\phi(x) > \phi(0)$

$\Rightarrow x - \ln(x+1) > 0$

or $x > \ln(x+1)$

or $\ln(x+1) < x$ ——— ② if $x > 0$.

Combining ① & ②, we have

$$\frac{x}{x+1} < \ln(x+1) < x \quad \text{if } x > 0$$

---

(6). prove that $f(x) = \frac{\ln(x+1)}{x}$ decreases in $]0, \infty[$.

Sol:-

Here $f(x) = \frac{\ln(x+1)}{x}$

Diff w.r.t. x

$$f'(x) = \frac{x \cdot \frac{1}{x+1} - \ln(x+1)}{x^2}$$

$$= \frac{\frac{x}{x+1} - \ln(x+1)}{x^2}$$

But we know that

$$\frac{x}{x+1} < \ln(x+1) \quad \text{for } x > 0$$

So $$\frac{x}{x+1} - \ln(x+1) < 0$$

Hence $f'(x) = \frac{-ve}{+ve} = -ve$ for $x > 0$

Hence $f(x)$ is decreasing function $x>0$
i.e, $f(x)$ decreases for $]0,\infty[$.

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(7) Let $f(x)=x^2$ & $\phi(x)=x^3$. Verify Cauchy M.V.T in $[1,2]$. Also find c.

Sol :-

Given $\left.\begin{array}{l} f(x)=x^2 \\ \phi(x)=x^3 \end{array}\right]$

Obviously $f(x)$ & $\phi(x)$ are continous in $[1,2]$ & derivable in $]1,2[$.

Hence by Cauchy's M.V.T

$$\frac{f(2)-f(1)}{\phi(2)-\phi(1)}=\frac{f'(c)}{\phi'(c)} \quad \text{for some } c\in\,]1,2[$$

$$\Rightarrow \quad \frac{2^2-1^2}{2^3-1^3}=\frac{2c}{3c^2}$$

$$\Rightarrow \quad \frac{4-1}{8-1}=\frac{2}{3c}$$

$$\frac{3}{7}=\frac{2}{3c} \quad \Rightarrow \quad 9c=14$$

$$\Rightarrow \boxed{c=\frac{14}{9}}$$

Hence Cauchy's M.V.T holds for $c=\frac{14}{9}\in\,]1,2[$
