Chapter 03
Calculus with analytic geometry

Ex. Let \( S = \{1, 2, 3\} \)

An element \( x \in S \) is said to be bounded above if there is a number \( M \geq x \) for every \( x \in S \). The least of such numbers is called the upper bound of \( S \).

An element \( x \in S \) is said to be bounded below if there is a number \( m \leq x \) for every \( x \in S \). The greatest of such numbers is called the lower bound of \( S \).

Ex. Let \( S = \{1, 2, 3, 4, 5\} \)

An element \( x \in S \) is said to be bounded above by \( 5 \) and bounded below by \( 1 \).

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An element \( x \in S \) is said to be bounded above by \( 5 \) and bounded below by \( 1 \).
Let $S$ be a non-empty subset of $\mathbb{R}$.  

1. If $S$ is bounded above, then an upper bound of $S$ is called a least upper bound of $S$, or supremum of $S$. If $M$ is less than any other upper bound of $S$.  
2. If $S$ is bounded below, then a lower bound $m$ of $S$ is called greatest lower bound of $S$, or infimum of $S$. If $m$ is greater than any other lower bound of $S$. In notation we write $m = \sup S$ or $M = \sup S$.  

In example 1, 

$M = 20$ 
$m = 1$  

In example 2, 

$M = 30$, $m = 0$.  

A function $f$ is said to be continuous in $[a, b]$ if there is no gap in the graph of $f$ in the interval $[a, b]$.  

For the given example, $f$ is continuous in $[a, b]$ and discontinuous in $[c, d]$.  

[Graph of a function with discontinuity at $x = c$]
**Rolle's Theorem:**

**Statement:**
Let a function \( f \) be

1. Continuous on a closed interval \([a, b]\).
2. Differentiable in the open interval \((a, b)\).
3. \( f(a) = f(b) \)

Then there exists at least one point \( c \in [a, b] \) such that

\[ f'(c) = 0. \]

**Proof:**

Because \( f \) is continuous on \([a, b]\),

\[ f \]

is bounded.

Let \( M = \sup f \) and \( m = \inf f \).

**Case 1:** \( M = m \)

Then \( f \) is constant on \((a, b)\) and so,

\[ f(x) = 0 \quad \forall x \in [a, b] \]

we have the required proof.

**Case 2:** \( M \neq m \)

Then at least one of \( M \) and \( m \) is different from \( f(a) \) and \( f(b) \).

Suppose \( M \neq f(a) = f(b) \)

Since \( f \) attains its supremum on \([a, b]\), so there is a pt. \( c \in [a, b] \) such that \( f(c) = M \).

But from (i), \( M \neq f(a) = f(b) \)

so \( c \neq a \) and \( c \neq b \)

\[ \Rightarrow c \in (a, b) \]
Suppose that \( h \) be a real number, then

\[
\frac{f(c+h) - f(c)}{h} \leq 0
\]

or

\[
\frac{f(c+h) - f(c)}{h} \geq 0
\]

Taking limit as \( h \to 0 \)

Eq. (a) becomes

\[
\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0
\]

\[
f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0
\]

\[
f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0
\]

\[
\Rightarrow f'(c) = 0
\]

---

Mean-Value Theorem (or) (Lagrange's M.V.T).

Statement:

Let a function \( f \) be

1. Continuous on closed interval \([a, b]\)
2. Differentiable in the open interval \((a, b)\)
Then there exists a point \( c \in [a, b] \) such that
\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

Proof:

Define a new function
\[
\phi(x) = Ax + f(x)
\]
where \( A \) is a constant to be determined.

\( \phi(a) = \phi(b) \)

Obviously, the function \( Ax \) is continuous on \([a, b]\) and \( f(x) \) is derivable in \([a, b]\).

Now \( \phi(x) \) satisfies all the conditions of Rolle's theorem. So there is a point \( c \in [a, b] \) such that
\[
\phi'(c) = 0
\]
\[
A + f'(c) = 0
\]
\[
\Rightarrow f'(c) = -A
\]

From \( \phi(a) = \phi(b) \)
\[
\Rightarrow Aa + f(a) = Ab + f(b)
\]
\[
A(a - b) = f(b) - f(a)
\]
\[
A(b - a) = f(b) - f(a)
\]
\[
\Rightarrow -A(b - a) = f(b) - f(a)
\]
\[
= -A = \frac{f(b) - f(a)}{b - a}
\]
but value of $-a$ in (2)
\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]
so \[ \frac{f(b) - f(a)}{b - a} = f'(c) \] as desired.

**Note:** Another form of mean value theorem.

If we take the interval as \([a, a+h]\) instead of \([a, b]\). Then mean value theorem becomes:
\[ f(a+h) - f(a) = f'(c) \]
where \(c\) is a point such that \(a < c < a+h\).

Then clearly \(c = a + \theta h\) where \(0 < \theta < 1\).

So (1) becomes:
\[ f(a+h) - f(a) = hf'(a+\theta h) \]
which is another form of M.V.T.

**Cauchy's Mean Value Theorem** (OR)

**Generalized M.V.T:**

**Statement:**

9 if two functions $f$ and $g$ are

1. Continuous on \([a, b]\)
2. Derivable in \(]a, b[\)
3. $g(x) \neq 0$ for all $x \in ]a, b[$ Then there
exists: at least one \( c \in [a, b] \) such that

\[
\frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f(c)}{\phi(c)}
\]

**Proof:**

Define a new function

\[ F(x) = f(x) + A \phi(x) \]

where \( A \) is a constant to be determined so that \( F(a) = F(b) \).

So

\[ f(a) + A \phi(a) = f(b) + A \phi(b) \]

or

\[ A \phi(b) - A \phi(a) = f(a) - f(b) \]

\[ \Rightarrow A (\phi(b) - \phi(a)) = - (f(b) - f(a)) \]

Now

\[ \phi(b) - \phi(a) \neq 0 \]

Because if we suppose that

\[ \phi(b) - \phi(a) = 0 \]

then \( \phi(a) = \phi(b) \)

and then \( \phi \) satisfies all the conditions of Rolle's theorem. So we must have

\[ \phi(c) = 0 \]

for some \( c \in [a, b] \)

which is a contradiction to statement.

So

\[ \phi(b) - \phi(a) \neq 0 \]

\[ \Rightarrow A = - \frac{f(b) - f(a)}{\phi(b) - \phi(a)} \]

Now the function \( F \), being sum of two derivable functions is derivable in \([a, b]\).
Hence by Rolle's theorem
\[ F'(c) = 0 \]
\[ \rightarrow f'(c) + A \phi'(c) = 0 \]
\[ \Rightarrow A = \frac{f'(c)}{\phi'(c)} \]  \hspace{1cm} (2)

So from (1) and (2)
\[ \frac{-f'(c)}{\phi'(c)} = \frac{f(b)-f(a)}{\phi(b)-\phi(a)} \]

So \[ \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \] as desired.

Note: Another form of Cauchy's M.V.T.
If we take the interval as \([a, a+h]\) instead of \([a, b]\). Then Cauchy's M.V.T. becomes:
\[ \frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(c)}{\phi'(c)} \] \hspace{1cm} (i)

where \( c \) is a pt s.t. \( a < c < a+h \)
If we write \( c = a + \theta h \) where \( 0 < \theta < 1 \)
Then clearly \( a + \theta h \in [a, a+h] \)
So, \( (i) \) becomes
\[ \frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)} \quad 0 < \theta < 1 \]

viz. another form of Cauchy's M.V.T.
Increasing & decreasing functions:

A function $f$ is said to be an increasing function on $[a, b]$ if for $x_1, x_2 \in [a, b]$,

$$f(x_2) > f(x_1) \text{ whenever } x_2 > x_1$$

A function $f$ is said to be a decreasing function on $[a, b]$ if for $x_1, x_2 \in [a, b]$,

$$f(x_2) < f(x_1) \text{ whenever } x_2 > x_1$$

Theorem:

Suppose $f$ is continuous on $[a, b]$ and has derivative at each point in $]a, b[$.

1. If $f'$ is non-decreasing on $]a, b[$, then $f$ is increasing on $[a, b]$.
2. If $f'$ is non-increasing on $]a, b[$, then $f$ is decreasing on $[a, b]$.

Proof:

Let $x_1, x_2 \in ]a, b[$ such that $x_2 > x_1$.

By Lagrange's MVT, there is a point $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{(i)}$$
Now \( f'(c) \) is +ve a\_alio \( x_2 - x_1 > 0 \)

So \( D \Rightarrow f(x_2) - f(x_1) > 0 \)

So \( f(x_2) > f(x_1) \) for \( x_2 > x_1 \).

Hence \( f \) is an increasing function.

(2) let \( x_1, x_2 \in ]a, b[ \) s. that \( x_2 > x_1 \).

By Lagrange’s M.V.T. There is a pt \( c \) s. that

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for } c \in \{x_1, x_2\}
\]

\[
\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c) \quad (2)
\]

Now \( f'(c) < 0 \) a \( x_2 - x_1 > 0 \)

So \( f(x_2) < f(x_1) \) whenever \( x_2 > x_1 \). Thus \( f \) is a decreasing function.
Exercise 3.1

1. Discuss the validity of Rolle's theorem for the following functions.

(i) \( f(x) = x^2 - 3x + 2 \) on \([1, 2]\)

\[ \text{Let } f(x) = x^2 - 3x + 2 \]
\[ f(1) = 1 - 3 + 2 = 0 \]
\[ f(2) = 4 - 6 + 2 = 0 \]

Thus, \( f(1) = f(2) \)

Clearly, \( f(x) \) is continuous on \([1, 2]\) and derivable in \((1, 2)\). Since all conditions of Rolle's theorem are satisfied, there must exist a point \( c \in (1, 2) \) such that
\[ f'(c) = 0 \]

Now \( f'(x) = 2x - 3 \)
\[ \Rightarrow f'(c) = 2c - 3 \]
So \( 2c - 3 = 0 \)
\[ \Rightarrow c = \frac{3}{2} \]

Hence, Rolle's theorem is valid with \( c = \frac{3}{2} \)

(ii) \( \text{Let } f(x) = \sin x \) on \([0, \pi]\)

Now \( f(0) = \sin 0 = 0 \)
\[ f(\pi) = \sin \pi = 0 \]
\[ f'(c) = 0 \]

Now \( f'(x) = 2 \sin x \cos x \)

so \( 2 \sin c \cos c = 0 \)

\[ \Rightarrow \sin c \cos c = 0 \]

\[ \Rightarrow c = 0, \frac{\pi}{2} \]

Hence, Rolle's theorem is valid if \( c = \frac{\pi}{2} \).

(iii) Let \( f(x) = 1 - x^{3/4} \) on \([-1, 1]\)

Now \( f(-1) = 1 - (-1)^{3/4} \)

\[ = 1 - (-1) \]

\[ f(-1) = 1 + 1 = 2 \]

\( f(1) = 1 - (1)^{3/4} \)

\[ = 1 - 1 \]

\[ f(1) = 0 \]

Hence, \( f(-1) \neq f(1) \)

Since one of the conditions of Rolle's theorem is not satisfied, hence Rolle's theorem is not valid and we cannot calculate \( c \).
By Lagrange's Mean Value Theorem,
\[ f(b) - f(a) = (b - a) f'(c) \quad (1) \]
where \( c \in \left[ -\frac{11}{7}, \frac{13}{7} \right] \).

Now
\[ f'(c) = f\left( -\frac{11}{7} \right) = \left( -\frac{11}{7} \right)^3 - 3 \left( -\frac{11}{7} \right) - 1 \]
\[ = -\frac{1331}{343} + \frac{33}{7} - 1 \]
\[ = -\frac{1331 + 1617 - 343}{343} \]
\[ f\left( -\frac{11}{7} \right) = -\frac{57}{343} \]

Also,
\[ f'(x) = \frac{3}{2} x^2 - 3 \]
\[ \Rightarrow f'(c) = 3c^2 - 3 \]
putting values in (1)
\[ -\frac{57}{343} \left( -\frac{57}{343} \right) = \left( \frac{13}{7} - \left( -\frac{11}{7} \right) \right) \left( 3c^2 - 3 \right) \]
\[ -\frac{57}{\sqrt{13}} + \frac{57}{343} = \left( \frac{13}{7} + \frac{11}{7} \right) \left( 3c^2 - 3 \right) \]
\[ 0 = \left( \frac{24}{7} \right) \left( 3c^2 - 3 \right) \]
\[ \Rightarrow 3c^2 - 3 = 0 \]
\[ \Rightarrow c^2 - 1 = 0 \]
\[ \Rightarrow c = \pm 1 \]

(ii) Hence \( f(x) = \sin x \) on \( [a, b] \).
By Lagrange's M.V Theorem,
\[ f(b) - f(a) = (b - a) f'(c) \quad (i) \]
for some \( c \in [a, b] \).

Now
\[ f(a) = \sin a \]
and \( f(b) = \sin b \)

also \( f'(x) = \cos x \)
\[ \Rightarrow f'(c) = \cos c \]

So from (i)
\[ \sin b - \sin a = (b - a) \cos c \]

\[ \Rightarrow c = \cos \left( \frac{b - a}{b - a} \right) \]

(iii): Here \( f(x) = 2x - x^3 \) on \([0, 1]\).

By Lagrange's M.V. Theorem
\[ f(b) - f(a) = (b - a) f'(c) \quad (i) \]
for some \( c \in [a, b] \).

Now
\[ f(a) = f(0) = 0 \cdot 0 = 0 \]
and \( f(b) = f(1) = 2 - 1 = 1 \)

Putting values in (i)
\[ 1 - 0 = (1 - 0) (2 - 3c^2) \]
\[ \Rightarrow 2 - 3c^2 = 1 \]
\[ \Rightarrow 3c^2 = 1 \]
\[ \Rightarrow c^2 = \frac{1}{3} \]
\[ \Rightarrow c = \pm \frac{1}{\sqrt{3}} \]

\[ \int_{-1}^{1} f(x) \, dx \]
(iv) Let \( f(x) = x^{2/3} \) on \([-1, 1]\).

By Lagrange's Mean Value Theorem,
\[
f(b) - f(a) = (b-a) f'(c)
\]
where \( a < c < b \).

Now,
\[
f(a) = f(-1) = (-1)^{2/3} = 1
\]
and
\[
f(b) = f(1) = 1
\]
Now, \( f(x) \) is not derivable at \( x = 0 \),

i.e. \( f'(0) \) doesn't exist, where \( 0 \in ]-1, 1[ \).

Hence Mean Value theorem is not applicable here.

(3) Use M.V. Theorem to show that

(i) \[
|\sin x - \sin y| \leq |x - y| \text{ for any real nos. } \ x \neq y
\]

Sol. Let \( f(t) = \sin t \),

Then \( f(t) \) is continuous & differentiable for

every real no. Hence Lagrange's M.V. theorem can be applied in the interval \([x, y]\)

where \( x \neq y \) are any two real nos.

So,
\[
\frac{f(y) - f(x)}{y - x} = \cos z \quad \text{where } z \in ]x, y[.
\]

But, \( |\cos z| \leq 1 \),

\[
\Rightarrow \left| \frac{\sin y - \sin x}{y - x} \right| \leq 1
\]
\[
\begin{align*}
\cos bx - \cos ax &= \frac{\sin x}{x} \quad \text{if } x \neq 0 \\
&= \sin z \\
&\text{for some } z \in [ax, bx] \\
\cos ax - \cos bx &= \frac{\sin x}{x} (b - a) \\
\Rightarrow \quad \left| \frac{\cos bx - \cos ax}{x} \right| &= \left| \frac{\sin z}{x} \right| \\
\Rightarrow \quad \left| \frac{\cos ax - \cos bx}{x} \right| &= \frac{1}{|b - a|} |\sin z| \\
\Rightarrow \quad \left| \frac{\cos ax - \cos bx}{x} \right| &\leq (b - a) \\
\tan x + \tan y &\geq |x + y| \quad \text{for all } x, y \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]
\end{align*}
\]
Let \( f(t) = \tan t \)

Then \( f(t) \) is continuous in \([-\pi, \pi]\) and differentiable in \([-\pi, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]\).

Hence Lagrange's M.V.T can be applied in the interval \([-\pi, \pi]\).

So  
\[
\frac{f(y)-f(-x)}{y-x} = \sec^2 z \quad \text{for some } z \in (-\pi, \pi]  
\]

or  
\[
\tan y - \tan (-x) = \sec^2 z 
\]

or  
\[
|\tan y + \tan x| = |\sec^2 z| 
\]

But  
\[
|\sec^2 z| = 1 \quad \forall z \in (-\pi, \pi]  
\]

So  
\[
\frac{|\tan y + \tan x|}{|y+x|} > 1 
\]

\[
\Rightarrow |\tan x + \tan y| > |x+y| \quad \forall x, y \in \text{intervals } [-\frac{\pi}{2}, \frac{\pi}{2}]  
\]

Q1) Let a function \( f \) be continuous on \((a, b)\) and \( f(x) = 0 \) for all \( x \in ]a, b[\). Prove that \( f \) is constant. Use this to show that  
\[
\sin^2 x + \cos^2 x = 1. 
\]

Q2) Given \( f \) is continuous on \([a, b]\) and differentiable in \([a, b[\).
Let \( x_1, x_2 \in [a, b] \) s.t. \( x_2 > x_1 \).

Applying L'Hopital's M.V.T on \( \{ x_1, x_2 \} \subseteq [a, b] \):

\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad \text{for } c \in ]x_1, x_2[
\]

But \( f'(c) = 0 \)

\[
\Rightarrow f(x_2) - f(x_1) = \]

So \( f(x_2) = f(x_1) \) whenever \( x_2 > x_1 \).

Hence \( f \) is a constant function.

Now suppose that \( f(x) = \sin x + \cos x \)

\[
\Rightarrow f'(x) = 2 \sin x \cos x - 2 \cos x \sin x = 0
\]

So \( f'(x) = 0 \) for all real nos. \( x \).

Hence \( f \) is constant.

Suppose \( f(x) = \sin x + \cos x = c \) \( \quad \text{(i)} \)

where \( c \) is an arbitrary constant.

Since \( f \) is constant, \( f \) holds for every real no. \( x \).

So in particular \( \sin 0 + \cos 0 = c \)

\[
\Rightarrow [c = 1]
\]

Hence from \( (i) \) \( \sin x + \cos x = 1 \)

\( \square \)

(ii) Show that \( f(x) = x^3 - 3x^2 + 3x + 2 \) is monotonically increasing on every interval.

So if we know that if \( f \) is continuous on \( [a, b] \) a has derivative at each pt \( \text{on } ]a, b[ \). Then \( f \) is an increasing function.
If \( f' \text{ is } +ve \text{ in } ]a, b[ \),

Now \( f(x) = x^3 - 3x^2 + 3x + 1 \).

\[
\begin{align*}
   f'(x) &= 3x^2 - 6x + 3 \\
   &= 3(x^2 - 2x + 1) \\
   &= 3(x-1)^2 \\
\end{align*}
\]

so \( f'(x) = 3(x-1)^2 \text{ is } +ve \text{ for all real nos. } x \). Hence \( f \) is monotonically increasing on every interval.

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(c) prove that \( f(x) = 2x - \tan^{-1}x - \ln(x + \sqrt{x^2 + 1}) \)

increases steadily on \([0, \infty]\)

**sol.** Let \( f(x) = 2x - \tan^{-1}x - \ln(x + \sqrt{x^2 + 1}) \)

\[
diff \ \text{w. r. to } x \\
f'(x) = 2 - \frac{1}{1 + x^2} - \frac{1}{x + \sqrt{x^2 + 1}} \left[ \frac{1 + x^2}{\sqrt{x^2 + 1}} \right] \\
= 2 - \frac{1}{1 + x^2} - \frac{1}{(x + \sqrt{x^2 + 1})} \left[ \frac{(x^2 + 1 + x)}{\sqrt{x^2 + 1}} \right] \\
= 2 - \frac{1}{1 + x^2} - \frac{1}{\sqrt{x^2 + 1}} \\
= \frac{2(1 + x^2) - 1 - \sqrt{1 + x^2}}{1 + x^2} \\
= \frac{2 + 2x^2 - 1 - \sqrt{x^2 + 1}}{(1 + x^2)} \\
f'(x) = \frac{2 + 2x^2 - 1 - \sqrt{x^2 + 1}}{(1 + x^2)}
\]

Now if \( f(x) \) is steadily increasing then \( f'(x) > 0 \)

i.e. \( 2x^2 + 1 - \sqrt{x^2 + 1} > 0 \)

or \( 2(x^2 + 1) - \sqrt{x^2 + 1} - 1 > 0 \)

or \( 2(x^2 + 1) - 2\sqrt{x^2 + 1} + \sqrt{x^2 + 1} - 1 > 0 \)

or \( 2\sqrt{x^2 + 1} (\sqrt{x^2 + 1} - 1) + 1 (\sqrt{x^2 + 1} - 1) > 0 \)

or \( (2\sqrt{x^2 + 1} + 1)(\sqrt{x^2 + 1} - 1) > 0 \)

which is true for all \( x \in [0, \infty) \)

Hence

\( f(x) \) increases steadily in \([0, \infty)\)
(7) Show that \( \frac{\tan x}{x} \) is an increasing fn \( \implies 0 < x < \frac{\pi}{2} \).

\[ f(x) = \frac{\tan x}{x} \]
\[ f'(x) = \frac{x \sec^2 x - \tan x}{x^2} \]

\[ \phi(x) = x \sec^2 x - \tan x \]
\[ \phi'(x) = \sec^4 x + x (2 \sec x \sec x \tan x) - \sec^2 x \]
\[ = \sec^4 x + 2x \sec^3 x \tan x - \sec^2 x \]
\[ \phi'(x) = 2x \sec^3 x \tan x \]

Now clearly \( \phi'(x) > 0 \) when \( 0 < x < \frac{\pi}{2} \).

Now obviously \( \phi(0) = 0 \)
\[ \implies \phi(x) > \phi(0) = 0 \implies \phi \text{ is increasing fn.} \]
\[ \text{i.e. } x \sec^2 x - \tan x > 0 \]
\[ \implies x \sec^2 x - \tan x > 0 \text{ for } 0 < x < \frac{\pi}{2} \]

Hence from (1) we conclude that

\[ f'(x) > 0 \]
\[ \text{for } 0 < x < \frac{\pi}{2} \]

 Hence \( f(x) \) is an increasing fn for
\[ 0 < x < \frac{\pi}{2} \]

(8) Determine the intervals in which
\[ f(x) = 2x^3 - 15x^2 + 36x + 1 \]
is increasing or decreasing.

\[ f'(x) = 6x^2 - 30x + 36 \]

\[ f'(x) = 0 \]
\[ \text{for } x = 3, 2 \]

\[ \text{Hence } f(x) \text{ is decreasing for } 0 < x < 2 \text{ or } x > 3 \]
\[ f'(x) = 6x^2 - 30x + 36 \]

If \( f(x) \) is an increasing function

Then \( f'(x) > 0 \)

i.e. \( 6x^2 - 30x + 36 > 0 \)

or \( 6(x^2 - 5x + 6) > 0 \)

or \( (x^2 - 5x + 6) > 0 \)

or \( (x-2)(x-3) > 0 \)

Now either \( x-2 > 0 \) or \( x-3 > 0 \)

or \( x-2 < 0 \) or \( x-3 < 0 \)

\text{Case (1)}: \( x - 2 > 0 \) or \( x - 3 > 0 \)

\[ \Rightarrow x > 2 \quad \text{or} \quad x > 3 \]

Hence we see that \( f(x) \) is an increasing function for all \( x > 3 \), i.e. for all \( x \in ]3,\infty[ \)

\text{Case (2)}: \( x - 2 < 0 \) or \( x - 3 < 0 \)

\[ \Rightarrow x < 2 \quad \text{or} \quad x < 3 \]

So we see that \( f(x) \) is an increasing function for all \( x < 2 \), i.e. for all \( x \in ]-\infty,2[ \)

Now either \( f(x) < 0 \)

\[ \text{Then} \quad f'(x) < 0 \]

i.e. \( 6x^2 - 30x + 36 < 0 \)

or \( 6(x^2 - 5x + 6) < 0 \)

or \( (x^2 - 5x + 6) < 0 \)

or \( (x-2)(x-3) < 0 \)

Now either \( (x-2) < 0 \) or \( (x-3) > 0 \)

or \( (x-2) > 0 \) or \( (x-3) < 0 \)
Case (i) \( x - 2 < 0 \) and \( x - 3 > 0 \)
\[ \Rightarrow x < 2 \quad \text{and} \quad x > 3 \]
which is impossible for any \( x \in \mathbb{R} \).

Case (ii) \( x - 2 > 0 \) and \( x - 3 < 0 \)
\[ \Rightarrow x > 2 \quad \text{and} \quad x < 3 \]
so \( x \in \{2, 3\} \)
Hence \( f(x) \) is a decreasing function for
\[ x \in \{2, 3\} \]

(i) \( \forall x > 0 \). Then prove that \( x - \ln(1 + x) \geq \frac{x^2}{2(1 + x)} \)

Scl: Let \( f(x) = x - \ln(1 + x) - \frac{x^2}{2(1 + x)} \)
\[
f'(x) = 1 - \frac{1}{1 + x} - \frac{1}{2} \left( \frac{(1 + x)2x - x^2}{(1 + x)^2} \right)
\]
\[
= 1 - \frac{1}{1 + x} - \frac{1}{2} \left( \frac{2x + x^2}{(1 + x)^2} \right)
\]
\[
= 1 - \frac{1}{1 + x} - \frac{2x + x^2}{2(1 + x)^2}
\]
\[
= \frac{2(1 + x)^2 - 2(1 + x) - 2x - x^2}{2(1 + x)^2}
\]
\[
= \frac{x^2}{2(1 + x)^2}
\]
\[
f'(x) = \frac{x^2}{2(1 + x)^2}
\]

obviously \( f(x) > 0 \) for all \( x > 0 \).
Hence $f(x)$ is an increasing fn for $x > 0$.

But $f(0) = 0$

$\Rightarrow f(x) > f(0)$ for $x > 0$

$\therefore x - \ln(1 + x) > \frac{x^2}{2(1 + x)} > 0$

i.e. $x - \ln(1 + x) > \frac{x^2}{2(1 + x)}$ for $x > 0$.

Solved Examples:

1) Verify Rolle's theorem for $f(x) = 1 - \frac{x^2}{2}$ on $[-1, 1]$.

Solution:

Let $f(x) = 1 - \frac{x^2}{2}$

$f(-1) = 1 - (-1)^2 = 0$

Hence $f(-1) = f(1)$

Clearly $f(x)$ is continuous on $[-1, 1]$ and not derivable in $[-1, 1]$.

Because $f'(x) = -\frac{2}{3} x^{1/3}$

$\Rightarrow f'(0) = -\frac{2}{3} 0^{1/3} = \frac{2}{0}$

Hence $f(x)$ is not derivable at $x = 0 \in ]-1, 1[$.

Rolle's Theorem is not applicable.

2) If $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$,

find $c$ of M.V.T.

Solution:

Here $f(x) = x(x-1)(x-2)$.

By Lagrange's M.V.T.

$f(b) - f(a) = (b-a) f'(c)$ for some $c \in ]a, b[$.
Now
\[ f(a) = f(0) = 0 \]
\[ f(b) = f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{8} \]

Also
\[ f(x) = x^3 - 3x^2 + 2x \]
\[ f'(x) = 3x^2 - 6x + 2 \]
\[ f'(c) = 3c^2 - 6c + 2 \]

Put value in (i)
\[ \frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right) \left(3c^2 - 6c + 2\right) \]
\[ \Rightarrow \frac{6}{8} = 3c^2 - 6c + 2 \]
\[ 3c^2 - 6c + 2 = \frac{3}{4} \]
\[ 12c^2 - 24c + 8 = 3 \]
\[ 12c^2 - 24c + 5 = 0 \]
\[ c = 2.4 \pm \sqrt{(2.4)^2 - 4(12)(0.8)} \]
\[ = 2.4 \pm \sqrt{3.12} \]
\[ c = \frac{2.4 \pm \sqrt{3.12}}{2.4} \]
\[ = 9 2.4 \pm 4\sqrt{21} \]
\[ c = \frac{6 \pm \sqrt{21}}{6} \]
\[ = 1 \frac{1}{6} \pm \sqrt{21} \]
\[ = 1 \pm \sqrt{21} \]
\[ c = 1 \pm \sqrt{21} / 36 \]
\[ N(x) = 1 + \frac{\pi^2}{12} e^{\left[0, \frac{1}{2}\right]} \]
\[ C = 1 - \frac{\pi^2}{12} \]

\[
\Rightarrow
\]

(3) For the function \( f(x) = |x| \), check whether, \( M.V.T \) holds on the interval \([-2, 2]\).

The graph of the function \( f(x) = |x| \) is

Now obviously \( f(x) \) is continuous on \([-2, 2]\).

The slope of the line through points \( A(-2, 2) \) and \( B(2, 2) \) is \( \frac{2 - 2}{2 + 2} = 0 \).

Now the function \( f(x) \) doesn't have a derivative at \( x = 0 \) because

\[ f'(x) \neq 0 \quad \text{for any} \quad x \in [-2, 2] \]

So one of the conditions of \( M.V.T \) is not satisfied. Hence \( M.V.T \) doesn't hold in \([-2, 2]\).

(4) If a function \( f \) satisfies the hypothesis of \( M.V.T \) on \([a, b]\) and \( |f(x)| \leq M \) for
If \( x \in \mathbb{R} \cap \{a, b\} \), then prove that
\[
\left| \frac{f(b) - f(a)}{b - a} \right| \leq M(b - a).
\]

**Solution:**
Given that \( f \) satisfies the hypothesis of M.V.T. on \([a, b]\).
So, by M.V.T,
\[
\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some } c \in (a, b).
\]
Taking modulus on both sides
\[
\left| \frac{f(b) - f(a)}{b - a} \right| = \left| f'(c) \right|
\]
\[
\Rightarrow \left| \frac{f(b) - f(a)}{b - a} \right| \leq M \quad \text{where } a < c < b
\]
\[
\Rightarrow \frac{|f(b) - f(a)|}{|b - a|} \leq M
\]
\[
\Rightarrow |f(b) - f(a)| \leq M|b - a|
\]
\[
\Rightarrow |f(b) - f(a)| \leq M(b - a) \quad \text{Ans.}
\]
(2) Prove that for \( x > 0 \), \( \frac{x}{x+1} < \ln(x+1) < x \).

Solution: Let \( f(x) = \ln(x+1) - \frac{x}{x+1} \).

\[
f'(x) = \frac{1}{x+1} - \left[ \frac{(x+1) - x}{(x+1)^2} \right]
\]

\[
= \frac{1}{x+1} - \frac{x+1-x}{(x+1)^2}
\]

\[
= \frac{1}{x+1} - \frac{1}{(x+1)^2}
\]

\[
= \frac{x+1-1}{(x+1)^2}
\]

\[
f'(x) = \frac{x}{(x+1)^2}
\]

So \( f'(x) > 0 \) for all \( x > 0 \).

Hence \( f(x) \) is an increasing function.

Now \( f(0) = 0 \).

\( \Rightarrow f(x) > f(0) = 0 \) for all \( x > 0 \).

\( \Rightarrow \ln(x+1) - \frac{x}{x+1} > 0 \)

\( \Rightarrow \ln(x+1) > \frac{x}{x+1} \)

Hence

\( \frac{x}{x+1} < \ln(x+1) \quad \text{(1) for } x > 0 \)

Again, suppose \( g(x) = x - \ln(x+1) \).
\[
\phi'(x) = 1 - \frac{1}{x+1}
\]

\[
\omega = \frac{x+1-1}{x+1} = \frac{x}{x+1}
\]

so \( \phi'(x) > 0 \) for \( x > 0 \)

Now \( \phi(0) = 0 \)

so \( \phi(x) > \phi(0) \)

\implies x - \ln(x+1) > 0 \at\ x > \ln(x+1)

or \ln(x+1) < x

Combining (1) and (2), we have

\[
\frac{x}{x+1} < \ln(x+1) < x \quad \forall x > 0
\]

(6) prove that \( f(x) = \frac{\ln(x+1)}{x} \) decreases in \( ]0, \infty[ \)

so \( f'(x) = \frac{\ln(x+1)}{x} \)

Diff w. x.t. x

\[
f'(x) = \frac{x \cdot \frac{1}{x+1} - \ln(x+1)}{x^2}
\]

\[
= \frac{x - \ln(x+1)}{x^2}
\]

but we know that

\[
\frac{x}{x+1} < \ln(x+1) \quad \forall x > 0
\]

so \( \frac{x}{x+1} - \ln(x+1) < 0 \)

Hence \( f'(x) = \frac{-ve}{+ve} = -ve \quad \forall x > 0 \)
Hence \( f(x) \) is decreasing function \( x > 0 \) i.e., \( f(x) \) decreases for \( \{0, \infty\} \).

(1) Let \( f(x) = x^2 \) and \( \phi(x) = x^3 \). Verify Cauchy M.V.T in \([1, 2]\). Also find \( c \).

Sol.:

Given \( f(x) = x^2 \)

\( \phi(x) = x^3 \)

Obviously \( f(x) \) and \( \phi(x) \) are continuous in \([1, 2]\) and derivable in \([1, 2]\).

Hence by Cauchy's M.V.T

\[
\frac{f(2) - f(1)}{\phi(2) - \phi(1)} = \frac{f'(c)}{\phi'(c)} \text{ for some } c \in \{1, 2\}
\]

\[
\frac{2^2 - 1^2}{2^3 - 1^3} = \frac{2c}{3c^2}
\]

\[
\frac{4 - 1}{8 - 1} = \frac{2}{3c}
\]

\[
\frac{3}{7} = \frac{2}{3c} \Rightarrow 9c = 14
\]

\[
\boxed{c = \frac{14}{9}}
\]

Hence Cauchy's M.V.T holds for \( c = \frac{14}{9} \in \{1, 2\} \).

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