Rolle's Theorem: Let a function on 'f' be, ①

(i) continuous on closed interval $[a,b]$

(ii) differentiable on open interval $]a,b[$

(iii) $f(a)=f(b)$

Then there exist atleast one point $c \in [a,b]$ such that $f'(c)=0$.

Ex 3.1 (Calculus)

Discuss the validity of Rolle's Theorem. Find c (whenever possible) such that $f'(c)=0$.

1. $f(x)=x^2-3x+2$ on $[1,2]$

i) $f(x)$ is continuous on $[1,2]$

ii) $f(x)$ is differentiable on $]1,2[$

iii) $f(1) = 1^2-3(1)+2=0$

$f(2) = 2^2-3(2)+2=0$

$\Rightarrow f(1)=f(2)$

There must exist $c \in]1,2[$

$\ni f'(c)=0$

$f'(x) = 2x-3$ Then $f'(c)=2c-3$

but $f'(c)=0$

$2x-3=0 \Rightarrow 2x=3$

$x=\frac{3}{2}$

Rolle's Theorem is valid.

and $c=3/2$.

2. $f(x)=\sin^2 x$ on $[0,\pi]$

i) $f(x)$ is continuous on $[0,\pi]$

ii) $f(x)$ is differentiable on $]0,\pi[$

iii) $f(0)=\sin^2 0 = 0$

$f(\pi) = \sin^2 \pi = 0$

$\Rightarrow f(0)=f(\pi)$

So $\exists$ $c \in]0,\pi[$

$\ni f'(c)=0$

$f'(x) = 2\sin x\cos x$

Now $f'(x) = \sin 2x$

$f'(c) = \sin 2c$

Now $f'(c)=0$

$\sin 2c=0$

$2c=0, \pi$

$c=0, \frac{\pi}{2}$

Thus $c=\frac{\pi}{2}$ $\because 0 \notin]1,2[$

3. $f(x)=1-x^{3/4}$ on $[-1,1]$

i) $f(-1) = 1-(-1)^{3/4}$

$= 1-[(-1)^3]^{1/4}$

$= 1-(-1)^{1/4}$

$= 1-\left[\frac{\pm 1 \pm i}{\sqrt{2}}\right]$

$f(1) = 1-(1)^{3/4} = 1-1=0$

$f(-1) \neq f(1)$

So Rolle's Theorem is not valid.

4. $f(x) = \frac{1-x^2}{1+x^2}$ on $[-1,1]$

(i) $f(x)$ is continuous on $[-1,1]$

(ii) $f(x)$ is differentiable on $]-1,1[$

(iii) $f(-1) = \frac{1-(-1)^2}{1+(-1)^2} = \frac{1-1}{1+1} = 0$

$f(1) = \frac{1-1^2}{1+1^2} = \frac{1-1}{1+1} = 0$

So $\exists$ $c \in [-1,1]$

$\ni f'(c)=0$

$f'(x) = \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$

$$f'(x) = \frac{-2x(1+x^2+1-x^2)}{(1+x^2)^2}$$

$$f'(x) = -\frac{4x}{(1+x^2)^2}$$

Now $f'(c) = \frac{-4c}{(1+c^2)^2}$

Now $f'(c) = 0$

$$\frac{-4c}{(1+c^2)^2} = 0$$

$$-4c = 0$$

$$c = 0$$

Thus Rolle's Theorem holds and $c = 0$

5. $f(x) = x(x+3)e^{-\frac{x}{2}}$ on $[-3,0]$

Sol: (i) $f(x)$ is continuous on $[-3,0]$

(ii) $f(x)$ is differentiable on $]-3,0[$

(iii) $f(-3) = -3(-3+3)e^{-1/2(-3)} = 3(0)e^{3/2} = 0$

$f(0) = 0(0+3)e^{-1/2(0)} = 0$

$\exists\ c \in [-3,0]$

$\ni\ f'(c) = 0$

Now $f(x) = (x^2+3x)e^{-\frac{x}{2}}$

$$f'(x) = (x^2+3x)\left[-\frac{1}{2}e^{-x/2}\right] + e^{-\frac{1}{2}x}(2x+3)$$

$$f'(x) = e^{-x/2}\left[\frac{-x^2-3x}{2} + 2x+3\right]$$

$$= e^{-x/2}\left[\frac{-x^2-3x+4x+6}{2}\right]$$

$$f'(x) = \frac{e^{-x/2}}{2}(-x^2-x+6)$$

$$f'(c) = \frac{e^{-c/2}}{2}(-c^2-c+6)$$

$$f'(c) = 0$$

$$\frac{e^{-c/2}}{2}(-c^2-c+6) = 0$$

$$-c^2-c+6 = 0$$

$$c^2+c-6 = 0$$

$$c^2+3c-2c-6 = 0$$

$$c(c+3)-2(c+3) = 0$$

$$(c+3)(c-2) = 0$$

$c = 3, 2 \quad \because c \notin -3$

Rolle's theorem valid, and

$c = -2$

6. $f(x) = 2+(x-1)^{3/2}$ on $[0,2]$

Sol:

(i) $f(x)$ is continuous on $[0,2]$

(ii) $f(x)$ is differentiable on $]0,2[$

(iii) $f(0) = 2+(0-1)^{3/2} = 2+(-1)^{3/2}$

$= 2+\left[(-1)^3\right]^{1/2} = 2+(-1)^{1/2}$

$f(0) = 2+i$

$f(2) = 2+(2-1)^{3/2} = 2+(1)^{3/2}$

$f(2) = 2+1$

$f(2) = 3$

$f(0) \neq f(2)$ Rolle's Theorem is invalid.

Find 'c' (whenever possible) of the Mean Value Theorem.

7. $f(x) = x^3-3x-1 \quad \left[-\frac{11}{7}, \frac{13}{7}\right]$

$$f\left(-\frac{11}{7}\right) = \left(-\frac{11}{7}\right)^3 - 3\left(-\frac{11}{7}\right) - 1$$

$$= \frac{-1331}{343} + \frac{33}{7} - 1 = \frac{-1331+1617-343}{343}$$

$$f\left(-\frac{11}{7}\right) = -\frac{57}{343}$$

$$f\left(\frac{13}{7}\right) = \left(\frac{13}{7}\right)^3 - 3\left(\frac{13}{7}\right) - 1$$

$$= \frac{2197}{343} - \frac{39}{7} - 1 = \frac{2197-1911-343}{343}$$

$$f\left(\frac{13}{7}\right) = -\frac{57}{343}$$

Now; $f'(x) = 3x^2-3$

$f'(c) = 3c^2-3$

By M.V.T.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3c^2-3 = \frac{\left(-\frac{57}{343}\right)-\left(-\frac{57}{343}\right)}{\frac{13}{7}-\left(-\frac{11}{7}\right)}$$

$$3c^2-3 = \frac{0}{\frac{13+11}{7}} = 0$$

$3c^2-3=0 \Rightarrow 3c^2=3 \Rightarrow c^2=3/3=1$

$\Rightarrow c^2=1 \Rightarrow \boxed{c=\pm 1}$

8. $f(x) = \sqrt{x-2}$ on $[2,4]$

i) $f(x)$ is continuous on $[2,4]$

ii) $f(x)$ is differentiable on $]2,4[$

iii) $f(2) = \sqrt{2-2} = 0$

$f(4) = \sqrt{4-2} = \sqrt{2}$

$f'(x) = \frac{1}{2\sqrt{x-2}}$

$f'(c) = \frac{1}{2\sqrt{c-2}}$

$\frac{f(b)-f(a)}{b-a} = f'(c) = \frac{f(4)-f(2)}{4-2}$

$\frac{\sqrt{2}-0}{\cancel{2}} = \frac{1}{\cancel{2}\sqrt{c-2}}$

$\sqrt{2}\sqrt{c-2} = 1$ squaring

$2(c-2) = 1$

$2c-4 = 1$

$2c = 1+4$

$\boxed{c = 5/2}$

9. $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1,3]$

i) $f(x)$ is differentiable & continuous.

By M.V.T

$\frac{f(b)-f(a)}{b-a} = f'(c)$

$f(b) = f(3) = 3^3 - 5(3)^2 + 4(3) - 2$

$f(3) = 27 - 45 + 12 - 2$

$f(3) = -8$

$f(a) = f(1) = 1^3 - 5(1)^2 + 4(1) - 2$

$f(1) = -2$

$f'(x) = 3x^2 - 10x + 4$

$f'(c) = 3c^2 - 10c + 4$

then $\frac{f(3)-f(1)}{3-1} = f'(c)$

$\frac{-8-(-2)}{2} = 3c^2 - 10c + 4$

$\frac{-8+2}{2} = 3c^2 - 10c + 4$

$\frac{-6}{2} = 3c^2 - 10c + 4$

$-3 = 3c^2 - 10c + 4$

$3c^2 - 10c + 4 + 3 = 0$

$3c^2 - 10c + 7 = 0$

$3c^2 - 3c - 7c + 7 = 0$

$3c(c-1) - 7(c-1) = 0$

$(3c-7)(c-1) = 0$

$3c-7=0$ | $c-1=0$

$\Rightarrow c = 7/3$ | $c = 1$

$\because 1 \notin]1,3[$

$\Rightarrow c = 7/3$

10. $f(x) = x^{2/3}$ on $[-1,1]$

$f'(x) = \frac{2}{3x^{1/3}}$

$\because f'(0)$ is undefined.

$\Rightarrow$ function is not differentiable on $]-1,1[$

Mean Value theorem is invalid.

Use M.V.T to show that.

11. $|\sin x - \sin y| \le |x-y|$

Let $f(t) = \sin t$

$f(t)$ is continuous and differentiable $\forall \mathbb{R}$

we apply M.V.T for

$f(t) = \sin t$ on $[x,y]$

$f(x) = \sin x$

$f(y) = \sin y$

$f'(t) = \cos t$

M.V.I

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

4

$$\frac{\sin y - \sin x}{y-x} = \cos c$$

$\sin y - \sin x = (y-x)(\cos c)$

Taking modulus.

$|\sin y - \sin x| = |y-x||\cos c|$

$\because |\cos c| \le 1$

$|\sin y - \sin x| \le |y-x|(1)$

$|\sin y - \sin x| \le |y-x|$

$|\sin x - \sin y| \le |x-y|$

Proved.

12. $\left|\dfrac{\cos ax - \cos bx}{x}\right| \le |b-a|$ if $x \ne 0$.

Let $\cos t = f(t)$

$f(t)$ is continuous and differentiable for all values.

we apply M.V.T on $[ax, bx]$

$f(t) = \cos t$

$f'(t) = -\sin t$

$f(ax) = \cos ax$

$f(bx) = \cos bx$

$$\frac{\cos bx - \cos ax}{bx - ax} = -\sin c$$

$$\frac{\cos bx - \cos ax}{x(b-a)} = -\sin c$$

$$\frac{\cos bx - \cos ax}{x} = -\sin c(b-a)$$

Taking Modulus.

$\left|\dfrac{\cos bx - \cos ax}{x}\right| = |-\sin c||b-a|$

$\because |\sin c| \le 1$

$\left|\dfrac{\cos bx - \cos ax}{x}\right| \le |b-a|$

13. $|\tan x + \tan y| \geq |x+y|$ for all real Nos. from the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$

Sol: Let $f(t) = \tan t$

on interval $[-x, y] \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$

$f(-x) = \tan(-x) = -\tan x$

$f(y) = \tan y$

$f'(t) = \sec^2 t$

$f'(c) = \sec^2(c)$

M.V.T

$$\frac{\tan y - (-\tan x)}{y-(-x)} = \sec^2 c$$

$$\frac{\tan y + \tan x}{y+x} = \sec^2 c$$

Taking Modulus.

$\tan y + \tan x = \sec^2 c(y+x)$

Taking Modulus.

$|\tan y + \tan x| = |y+x||\sec^2 c|$

$\because |\cos x| < 1$

$|\cos^2 x| < 1 \Rightarrow \left|\dfrac{1}{\cos^2 x}\right| > 1$

$|\sec^2 x| > 1$

$|\tan y + \tan x| \geq |y+x|(1)$

$|\tan x + \tan y| \geq |y+x|$

14. $(1+x)^\alpha > 1 + \alpha x$ where $\alpha > 1$ and $x > 0$. [Bernoulli's inequality

Sol: Let $f(x) = (1+x)^\alpha - (1+\alpha x)$

on interval $[0, x]$

By M.V.T

$$\frac{f(x) - f(0)}{x-0} = f'(c) \rightarrow (1)$$

$f(x) = (1+x)^a - (1+ax)$

$f(0) = (1+0)^a - (1+0) = 1-1 = 0$

$f(0) = 0$

$\cancel{f'(x) = a(1+x)^{a-1} - a}$

$= a\left[(1+x)^{a-1} - 1\right]$

$f'(c) = a\left[(1+c)^{a-1} - 1\right]$

putting values in 1

$$\frac{(1+x)^a - (1+ax) - 0}{x - 0} = a\left[(1+c)^{a-1} - 1\right]$$

$$\frac{(1+x)^a - (1+ax)}{x} = a\left[(1+c)^{a-1} - 1\right]$$

$$(1+x)^a - (1+ax) = ax\left[(1+c)^{a-1} - 1\right]$$

$c+1 > 1 \quad \because c \in]0, x[$

$= (c+1)^{a-1} > 1 \quad x > 0 \quad \because a > 1 \Rightarrow a-1 > 0$

$(1+c)^{a-1} - 1 > 0$

when $a > 1$, $x > 0$

$ax\left[(1+c)^{a-1} - 1\right] > 0$

$(1+x)^a - (1+ax) = ax\left[(1+c)^{a-1} - 1\right] > 0$

$(1+x)^a - (1+ax) > 0$

$(1+x)^a > 1+ax$

Hence Proved.

15. $\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$

also find $\sqrt{168}$ by M.V.T.

Sol. Let $f(x) = \sqrt{x}$ on $[27, 25]$

By M.V.T

$$\frac{f(27) - f(25)}{27 - 25} = f'(c) \longrightarrow 1$$

$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(c) = \frac{1}{2\sqrt{c}}$

from 1

$$\frac{\sqrt{27} - \sqrt{25}}{27 - 25} = \frac{1}{2\sqrt{c}}$$

$$\frac{\sqrt{27} - 5}{2} = \frac{1}{2\sqrt{c}}$$

$$\sqrt{27} - 5 = \frac{1}{\sqrt{c}} \longrightarrow 2$$

$\because c \in]25, 27[$

$\Rightarrow 25 < c < 27$

$\Rightarrow \sqrt{25} < \sqrt{c} < \sqrt{27}$

$\Rightarrow 5 < \sqrt{c} < \sqrt{27}$

$\because \sqrt{27} < \sqrt{36} = 6$

$\Rightarrow 5 < \sqrt{c} < \sqrt{27} < 6$

$\Rightarrow 5 < \sqrt{c} < 6$

$\Rightarrow \frac{1}{5} < \frac{1}{\sqrt{c}} < \frac{1}{6}.$

from 2

$$\frac{1}{\sqrt{5}} < \sqrt{27} - 5 < \frac{1}{6}$$

(ii) Let $f(x) = \sqrt{x}$ on $[168, 169]$

By M.V.T

$$\frac{f(169) - f(168)}{169 - 168} = f'(c) \qquad \because f'(x) = \frac{1}{2\sqrt{x}}$$

$$\frac{\sqrt{169} - \sqrt{168}}{1} = \frac{1}{2\sqrt{c}}$$

$$13 - \sqrt{168} = \frac{1}{2\Gamma}$$

The exact value of c is not know but it is near 169 i.e $\sqrt{c} \approx 13$

$$13 - \sqrt{168} \approx \frac{1}{2(13)} = \frac{1}{26}$$

$$13 - \frac{1}{26} \approx \sqrt{168}$$

$$\sqrt{168} \approx 12.9615$$

16. Let a function 'f' be continuous on $[a,b]$ and $f'(x)=0 \ \forall \ x \in]a,b[$. Prove that f is constant on $[a,b]$. Use this to show that $\sin^2 x + \cos^2 x = 1$ $\forall$ real numbers x

Sol. Let $f(x)$ be function continuous on $[a,b]$ and differentiable on $]a,b[$.

Let $x_1, x_2 \in]a,b[\ \exists \ x_2 > x_1$ then

$f(x)$ is also continuous on $[x_1, x_2]$ and differentiable on $]x_1, x_2[$

By M.V.T.

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0 \quad \because f'(x)=0$$

$$\Rightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1} = 0$$

$$\Rightarrow f(x_2)-f(x_1)=0$$

$$\Rightarrow f(x_2)=f(x_1)$$

$\Rightarrow$ f has same value at two points in $[a,b]$

$\Rightarrow$ $f(x)$ is constant.

(ii) Let $f(x) = \cos^2 x + \sin^2 x = c$

$\therefore \cos^2 x + \sin^2 x = c \rightarrow (1)$

$\because$ (1) hold $\forall \ \mathbb{R}$

Put $x=0$ (it'll also true for $x=0$)

$$\cos^2(0) + \sin^2(0) = c$$

$$1 + 0 = c$$

$$\boxed{c=1}$$

put in (1)

$$\cos^2 x + \sin^2 x = 1$$

17.

(b) $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$

Does M.V.T holds for f on $[\frac{1}{2}, 2]$

Sol.

(i) $f(x)$ is continuous on $[\frac{1}{2}, 2]$

(ii) Check differentiability

$$Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1^-} \frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1^-} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$

$$= 1+1$$

$$Lf'(1) = 2$$

$$Rf'(1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1^+} \frac{\cancel{x-1}}{\cancel{x-1}}$$

$$= 1$$

$\because Rf'(1) \neq Lf'(x)$

$\Rightarrow f(x)$ is not differentiable.

$\Rightarrow$ M.V.T does not hold.

18. Let n be positive integer. Apply Rolle's Theorem to the function
$$F(x) = \begin{vmatrix} f(x) & f(a) & f(b) \\ x^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$
to obtain a result that generalizes M.V.T. Does the result hold if $n<0$?

Sol.
$$F(x) = \begin{vmatrix} f(x) & f(a) & f(b) \\ x^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$

$$F(a) = \begin{vmatrix} f(a) & f(a) & f(b) \\ a^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$

$F(a) = 0 \quad \because C_1 = C_2$

$$F(b) = \begin{vmatrix} f(b) & f(a) & f(b) \\ b^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$

$F(b) = 0 \quad \because C_1 = C_3$

Thus $F(a) = F(b)$

By Rolle's theorem

$\exists \; c \in]a, b[\; \ni \; F'(c) = 0$

$$F(x) = \begin{vmatrix} f'(x) & f(a) & f(b) \\ nx^{n-1} & a^n & b^n \\ 0 & 1 & 1 \end{vmatrix}$$

$F'(x) = f'(x)[a^n - b^n] - f(a)(nx^{n-1} - 0) + f(b)(nx^{n-1} - 0)$

$F'(x) = f'(x)(a^n - b^n) - nx^{n-1}(f(a) - f(b))$

$F'(c) = f'(c)(a^n - b^n) - nc^{n-1}(f(a) - f(b))$

$F'(c) = 0$

$f'(c)(a^n - b^n) - nc^{n-1}(f(a) - f(b)) = 0$

$f'(c)(a^n - b^n) = nc^{n-1}(f(a) - f(b))$

$$\frac{f'(c)}{nc^{n-1}} = \frac{f(a) - f(b)}{a^n - b^n}$$

which is generalization of M.V.T.

if $n < 0$ then theorem holds if $0 \notin [a, b]$

19. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on graph of the parabolla $y = f(x) = ax^2 + bx + c$. By M.V.T, there is a point (x_3, y_3) on the curve where tangent Line is parallel to chord AB. Show $x_3 = \frac{x_1 + x_2}{2}$.

Sol: Let

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

By M.V.T.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_3)$$

$$\frac{ax_2^2 + bx_2 + \cancel{c} - ax_1^2 - bx_1 - \cancel{c}}{x_2 - x_1} = 2ax_3 + b$$

$$\frac{a(x_2^2 - x_1^2) + b(x_2 - x_1)}{x_2 - x_1} = 2ax_3 + b$$

$$\frac{a(x_1 + x_2)(x_2 - x_1) + b(x_2 - x_1)}{x_2 - x_1} = 2ax_3 + b$$

$$\cancel{(x_2 - x_1)}\left[\frac{a(x_2 + x_1) + b}{\cancel{(x_2 - x_1)}}\right] = 2ax_3 + b$$

$a(x_2 + x_1) + \cancel{b} = 2ax_3 + \cancel{b}$

$\cancel{a}(x_2 + x_1) = 2\cancel{a}x_3$

$$\frac{x_2 + x_1}{2} = x_3$$

As required.

20. Show that $f(x) = x^3 - 3x^2 + 3x + 2$ monotonically increasing on every interval.

Sol: def: A function is said to be monotonically increasing (decreasing) if and only if it is entirely increasing (decreasing).

Sol:

$f(x) = x^3 - 3x^2 + 3x + 2$

$f'(x) = 3x^2 - 6x + 3 + 0$

$f'(x) = 3(x^2 - 2x + 1)$

$f'(x) = 3(x - 1)^2$

$\because 3 > 0$ and $(x - 1)^2 > 0$

$\Rightarrow f'(x) > 0$

i.e $f'(x)$ is positive for all real no's.

$\Rightarrow f(x)$ is monotonically increasing

21. Prove that $f(x)=2x-\tan^{-1}x-\ln(x+\sqrt{x^2+1})$ is increasing on $[0,\infty[$. (8)

Sol:

$$f'(x)=2-\frac{1}{1+x^2}-\frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right]$$

$$f'(x)=2-\frac{1}{1+x^2}-\frac{1}{\cancel{x+\sqrt{x^2+1}}}\left[\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right]$$

$$=2-\frac{1}{1+x^2}-\frac{1}{\sqrt{x^2+1}}$$

$$=\frac{2(1+x^2)-1-\sqrt{1+x^2}}{(1+x^2)}$$

$$=\frac{2+2x^2-1-\sqrt{x^2+1}}{1+x^2}$$

$$f'(x)=\frac{1+2x^2-\sqrt{x^2+1}}{1+x^2}$$

$$f'(x)=\frac{(1+x^2)+x^2-\sqrt{x^2+1}}{1+x^2}$$

$$f'(x)=\frac{\sqrt{1+x^2}\left(\sqrt{1+x^2}-1\right)+x^2}{1+x^2}$$

$\Rightarrow f'(x)\geq 0$

Reason:

$\because \sqrt{1+x^2}\geq 0$, $x^2\geq 0$

$1+x^2\geq 0$ $\because$ involves square.

$1+x^2\geq 1$ $\because x\in[0,\infty[$

$\Rightarrow \sqrt{1+x^2}\geq 1$

$\Rightarrow \sqrt{1+x^2}-1\geq 0$

$\because f'(x)\geq 0$ for $x\in[0,\infty[$

$\Rightarrow f(x)$ is increasing on $[0,\infty[$.

22. Show that $\frac{\tan x}{x}$ is an increasing function for $0<x<\frac{\pi}{2}$.

Sol.

$$f(x)=\frac{\tan x}{x}$$

$$\cancel{f'(x)=\frac{x\sec^2x-\tan x(1)}{x^2}}$$

$$f'(x)=\frac{x\sec^2x-\tan x}{x^2}\rightarrow (1)$$

Let $\phi(x)=x\sec^2x-\tan x$

$$\phi'(x)=(1)\sec^2x+2x\sec^2x\tan x-\sec^2x$$

$$\phi'(x)=\cancel{\sec^2x}+2x\sec^2x\tan x\cancel{-\sec^2x}$$

$$\phi'(x)=2x\sec^2x\tan x$$

$\Rightarrow \phi'(x)$ $\forall$ $0<x<\frac{\pi}{2}$ (first quadrant)

So $\phi(x)$ is increasing

and $\phi(0)=0\sec^2(0)-\tan(0)$

$=0-0=0$

$\phi(0)=0$

and $\phi(x)$ is increasing

$\Rightarrow \phi(x)>0$ $\forall\, 0<x<\frac{\pi}{2}$

$\Rightarrow x\sec^2x-\tan x>0$

and $x^2>0$ (involves square.

$$\Rightarrow \frac{x^2\sec^2x-\tan x}{x^2}>0$$

$\Rightarrow f'(x)>0$

$\Rightarrow f(x)=\frac{\tan x}{x}$ is increasing.

23. Determine the interval on which $f(x)=2x^3-15x^2+36x+1$ is increasing or decreasing.

Sol: $f(x) = 2x^3-15x^2+36x+1$

$f'(x) = 6x^2-30x+36$

f(x) is increasing when

$f'(x) > 0$

i.e $6x^2-30x+36>0$

$\Rightarrow 6(x^2-5x+6)>0$

$\Rightarrow x^2-5x+6>0$

$\Rightarrow x^2-3x-2x+6>0$

$\Rightarrow x(x-3)-2(x-3)>0$

$\Rightarrow (x-3)(x-2)>0$

Two cases

1) $x-3>0$ and $x-2>0$

$\Rightarrow x>3$ and $x>2$

$\Rightarrow$ function is increasing $\forall\ x>3$

i.e $\forall\ x \in \,]3,\infty[$

2) $x-2<0$ $\quad x-3<0$

$\Rightarrow x<2$ $\quad x<2$

$\Rightarrow$ function is increasing $\forall\ x<2$

i.e $\forall\ x\in\,]-\infty,2[$

f(x) is decreasing when,

$f'(x)<0$

i.e $6x^2-30x+36<0$

$\Rightarrow 6(x-3)(x-2)<0$

$\Rightarrow (x-3)(x-2)<0$

Two cases.

1) $x-2>0$ and $x-3<0$

$x>2$ and $x<3$

$\Rightarrow$ for $x\in\,]2,3[$ function is decreasing.

2) $x-2<0$ and $x-3>0$

$\Rightarrow x<2$ and $x>3$

There is no such No. which is less than 2 and greater than 3.

24. if $x>0$, prove that $x-\ln(1+x) > \dfrac{x^2}{2(1+x)}$

Sol:

Let $f(x) = x-\ln(1+x)-\dfrac{x^2}{2(1+x)}$

$f(x) = x-\ln(1+x)-\dfrac{1}{2}\left[x-1+\dfrac{1}{1+x}\right]$

$f(x) = x-\ln(1+x)-\dfrac{1}{2}x+\dfrac{1}{2}-\dfrac{1}{2(1+x)}$

Taking derivative

$f'(x) = 1-\dfrac{1}{1+x}-\dfrac{1}{2}+0-\dfrac{1}{2}\left(-\dfrac{1}{(1+x)^2}\right)$

$= \dfrac{1}{2}-\dfrac{1}{1+x}+\dfrac{1}{2(1+x)^2}$

$= \dfrac{(1+x)^2-2(1+x)+1}{2(1+x)^2}$

$= \dfrac{1+x^2+\cancel{2x}-2-\cancel{2x}+1}{2(1+x)^2}$

$= \dfrac{\cancel{x}-\cancel{x}+x^2}{(1+x)^2}$

$f'(x) = \dfrac{x^2}{(1+x)^2} > 0 \ \forall\ x>0$

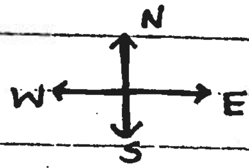
$\Rightarrow f(x)$ is increasing ftn. $\forall\ x>0$

and $f(x)=0$

$\Rightarrow x-\ln(1+x)-\dfrac{x^2}{2(1+x)}>0$

$$\Rightarrow \quad x - \ln(1+x) > \frac{x^2}{2(1+x)}$$ (10) Hence Proved.

25. A ship sails from Port A at 10 nautical miles per hour. At the same time, another ship leaves port B, which is 100 nautical miles due South of port A, and sails north at 25 nautical miles per hour. For how long is the distance b/w the ships decreasing?



Sol: Let 'S' be the distance b/w ships after 't' hours.

x = (Velocity of ship) x time

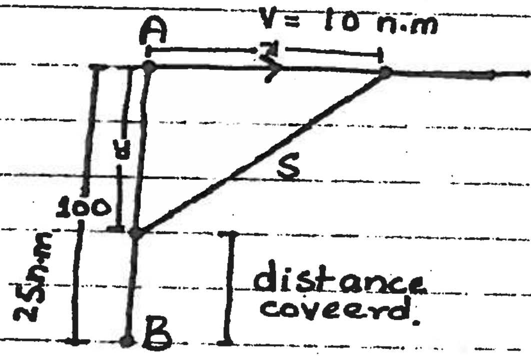
$x = 10t$

Similarly

y = AB − distance coved by ship.

$y = 100 - (\text{velocity of ship} \times \text{time})$

$y = 100 - 25t$



By pathagourus theorem

$$S^2 = x^2 + y^2$$

$$S^2 = (10t)^2 + (100-25t)^2 = 100t^2 + 10000 + 625t^2 - 5000t$$

$$S^2 = 725t^2 - 5000t + 10000$$

Differentiating w.r.t 't'

$$2S\frac{dS}{dt} = 1450t - 5000 = 2(725t - 2500)$$

$$S\frac{dS}{dt} = 725t - 2500 \Rightarrow \frac{dS}{dt} = \frac{725t - 2500}{S}$$

$\because \frac{dS}{dt} < 0$ (we find for how long distance b/w ships decreasing)

$$\frac{725t - 250}{S} < 0 \qquad \because S > 0$$

$$\Rightarrow 725t - 250 < 0 \Rightarrow 725t < 250$$

$$\Rightarrow t < \frac{250}{725}$$

$$\Rightarrow t < \frac{100}{29}$$

So $\frac{dS}{dt} < 0$ for $0 < t < \frac{100}{29}$

$\Rightarrow$ distance 's' b/w ships decreases for $\frac{100}{29}$ hours.