Differential of a function:-

Let $y = f(x)$ be a differentiable function, then its differential is defined as

$$dy = f'(x)\,dx$$

e.g.,

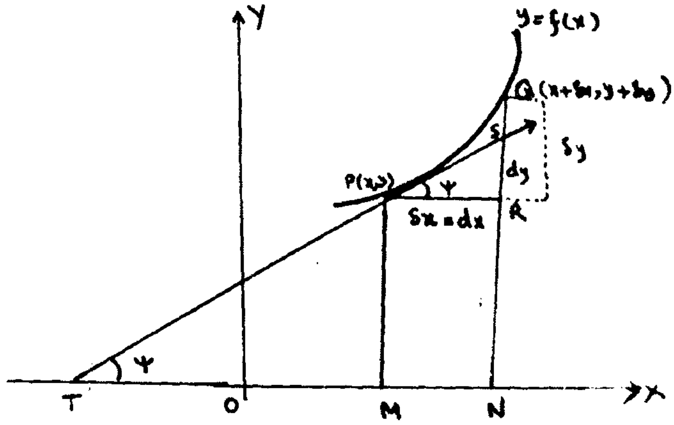
$$y = \sin^2 x$$
$$dy = 2\sin x\cos x\,dx$$

Difference b/w dy & δy:-

Let $P(x,y)$ & $Q(x+\delta x, y+\delta y)$ be any two points on $y=f(x)$.

then $\delta x = dx = PR$

$\delta y = RQ$



Now $f'(x) = \tan\psi$

$= \dfrac{RS}{PR}$

So $f'(x) = \dfrac{dy}{dx} = \dfrac{RS}{dx}$

$\Rightarrow dy = RS$

Also $\delta y = RQ$

So $\delta y \neq dy$.

Relative (average) error:- If P is the quantity to be measured & ΔP is the error in P then we define

Relative error in $P = \dfrac{\Delta P}{P}$

Percentage error in $P = \dfrac{\Delta P}{P} \times 100\ \%$

Related rate:- The rate of change of a variable with respect to time is called related rate.

EXERCISE 2.3

Find $\Delta y, dy, \Delta y - dy$ if

1. (i) $y = x^3 - 1, \quad x = 1, \quad \Delta x = -0.5$

   (ii) $y = \sqrt{3x-2}, \quad x = 2, \quad \Delta x = 0.3$

Sol.

(i) Here $y = x^3 - 1$ ——— (1)

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= [(x+\Delta x)^3 - 1] - [x^3 - 1]$$
$$= (x+\Delta x)^3 - 1 - x^3 + 1$$
$$= (x+\Delta x)^3 - x^3$$
$$= (1-0.5)^3 - (1)^3$$
$$= (0.5)^3 - 1$$
$$= 0.125 - 1$$
$$\Delta y = -0.875$$

Now from (1)

$$dy = 3x^2 dx$$
$$dy = 3(1)^2(-0.5)$$
$$= 3(-0.5)$$
$$dy = -1.5$$

Now

$$\Delta y - dy = -0.875 - (-1.5)$$
$$= -0.875 + 1.5$$
$$= 0.625 \text{ — Ans.}$$

(ii) $y = \sqrt{3x-2}, \quad x = 2, \quad \Delta x = 0.3$

Sol. Here $y = \sqrt{3x-2}$

$$\Delta y = f(x+\Delta x) - f(x)$$
$$= \sqrt{3(x+\Delta x)-2} - \sqrt{3x-2}$$
$$= \sqrt{3(2+0.3)-2} - \sqrt{3(2)-2}$$
$$= \sqrt{3(2.3)-2} - \sqrt{6-2}$$
$$= \sqrt{6.9-2} - \sqrt{4}$$
$$= \sqrt{4.9} - 2$$
$$= 2.2135 - 2$$
$$\Delta y = 0.2135$$

from (1)

$$dy = \frac{1}{2\sqrt{3x-2}} \cdot 3 \cdot dx$$
$$= \frac{3}{2\sqrt{3x-2}} dx$$
$$= \frac{3}{2\sqrt{3(2)-2}} (0.3)$$
$$= \frac{0.9}{2\sqrt{4}}$$
$$= \frac{0.9}{4}$$
$$dy = 0.2250$$

Now

$$\Delta y - dy = 0.2135 - 0.2250$$
$$= -0.0115 \text{ — Ans.}$$

2. Use differentials to approxiamte

(i) $\sqrt{26.2}$

Sol. We consider

$$y = f(x) = \sqrt{x} \quad \text{--------} \quad (1)$$

with $x = 25$ and $\Delta x = 1.2$

From (1), we have

$$dy = \frac{1}{2\sqrt{x}}\,dx \quad \text{--------} \quad (2)$$

Substituting $x = 25$, $dx = \Delta x = 1.2$ in (2), we get

$$dy = \frac{1}{2\sqrt{25}}(1.2) = \frac{1}{2\times 5}(1.2) = \frac{1.2}{10} = 0.12$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$$dy = \sqrt{x+\Delta x} - \sqrt{x}$$

$$0.12 = \sqrt{25+1.2} - \sqrt{25}$$

$$0.12 = \sqrt{26.2} - 5$$

or $\sqrt{26.2} - 5 = 0.12$

$$\sqrt{26.2} = 0.12 + 5$$

$$\sqrt{26.2} = 5.12 \text{ — Ans.}$$

(ii) $\sqrt{80.9}$

Sol. Let $y = f(x) = \sqrt{x}$

Here $x = 81$ and $\Delta x = -0.1 = dx$

Now $dy = \frac{1}{2\sqrt{x}}dx$

$$= \frac{1}{2\sqrt{81}}(-0.1) = \frac{-0.1}{2\times 9} = \frac{-0.1}{18} = -0.00555$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$$dy = \sqrt{x+\Delta x} - \sqrt{x}$$

$$-0.00555 = \sqrt{81-0.1} - \sqrt{81}$$

$$-0.00555 = \sqrt{80.9} - 9$$

$$\sqrt{80.9} - 9 = -0.00555$$

$$\sqrt{80.9} = -0.00555 + 9 = 8.99455$$

(iii) $\sqrt[3]{123}$

Sol.

Here $\sqrt[3]{123} = (123)^{1/3}$

Let $y = f(x) = x^{1/3}$

with $x = 125$ & $\Delta x = -2$

Now $dy = \frac{1}{3}x^{1/3-1}dx$

$= \frac{1}{3}x^{-2/3}dx$

$= \frac{1}{3x^{2/3}}dx$

$= \frac{1}{3(125)^{2/3}}(-2)$

$= \frac{-2}{3(5^3)^{2/3}}$

$= \frac{-2}{3(5)^2}$

$dy = -\frac{2}{75}$

$dy = -0.0266$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = (x+\Delta x)^{1/3} - x^{1/3}$

$-0.0266 = (125-2)^{1/3} - (125)^{1/3}$

$= (123)^{1/3} - (5^3)^{1/3}$

$-0.0266 = (123)^{1/3} - 5$

$(123)^{1/3} - 5 = -0.0266$

$(123)^{1/3} = -0.0266 + 5$

$\sqrt[3]{123} = 4.9734$

(iv) $\cos 61^\circ$

Sol. Let $y = f(x) = \cos x$

with $x = 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$

& $\Delta x = 1^\circ = 1 \cdot \frac{\pi}{180} = \frac{\pi}{180}$

Now $dy = -\sin x\, dx$

$= -\sin(\pi/3) \cdot \frac{\pi}{180}$

$= -\frac{\sqrt{3}}{2} \cdot \frac{\pi}{180}$

$dy = -\frac{\sqrt{3}\pi}{360}$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = \cos(60^\circ + 1^\circ) - \cos 60^\circ$

$-\frac{\sqrt{3}\pi}{360} = \cos 61^\circ - \frac{1}{2}$

$-\frac{(1.732)(3.14)}{360} = \cos 61^\circ - 0.5$

$-0.1512 = \cos 61^\circ - 0.5$

$\cos 61^\circ - 0.5 = -0.1512$

$\cos 61^\circ = -0.1512 + 0.5$

$= 0.4848$

(v) $(3.02)^4$

Sol. Let $y = f(x) = x^4$ with

$x = 3$ and $\Delta x = 0.02$

$dy = 4x^3\,dx$

$= 4 \times 3^3\,(0.02)$

$dy = 2.16$

Since $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = (x+\Delta x)^4 - x^4$

$= (3+0.02)^4 - (3)^4$

$2.16 = (3.02)^4 - 81$

$(3.02)^4 - 81 = 2.16$

$(3.02)^4 = 2.16 + 81 = 83.16$

(vi) $\tan 29^\circ$

Sol.

Let $y = f(x) = \tan x$

with $x = 30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

& $\Delta x = -1^\circ = -1 \cdot \frac{\pi}{180} = -\frac{\pi}{180}$

Now $dy = \sec^2 x\,dx$

$= \sec^2(\pi/6) \cdot \left(-\frac{\pi}{180}\right)$

$= \frac{1}{\cos^2 \pi/6} \cdot \frac{-3.14}{180}$

$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2} \cdot \frac{-3.14}{180}$

$= -\frac{4}{3} \times \frac{3.14}{180}$

$dy = -0.0233$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

$dy = \tan(x+\Delta x) - \tan x$

$-0.0233 = \tan(30^\circ - 1^\circ) - \tan 30^\circ$

$-0.0233 = \tan 29^\circ - \frac{1}{\sqrt{3}}$

$\tan 29^\circ - \frac{1}{\sqrt{3}} = -0.0233$

$\tan 29^\circ - 0.5733 = -0.0233$

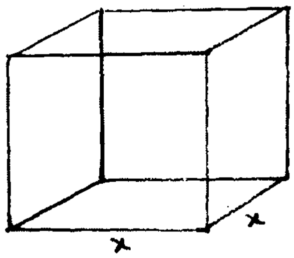
$\tan 29^\circ = -0.0233 + 0.5733$

$\tan 29^\circ = 0.55$

3. **The side of a cube is measured with a possible error of ± 2 %. Find the percentage error in the surface area of one face of the cube.**

Sol. Let x be edge of the cube

then Area A of a face is



$$A = x \cdot x = x^2$$

$$dA = 2x\,dx$$

$$\text{Relative error} = \frac{dA}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$$

But $\dfrac{dx}{x} = \pm 0.02$

Therefore,

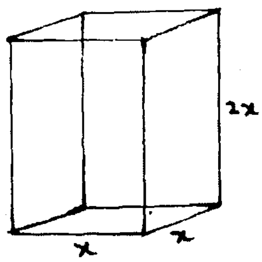
$$\frac{dA}{A} = 2(\pm 0.02)$$

$$= \pm 0.04$$

The percentage error in the surface area is $= \dfrac{dA}{A} \times 100 = \pm 0.04 \times 100 = \pm 4\%$.

4. **A box with a square base has its height twice its width. if the width of the box is 8.5 inches (in.) with a possible error of ± 0.3 in, find the possible error in the volume of the box.**

Sol. Let x in be the width of the box. Then its volume V is



$$V = L \cdot W \cdot h = x \cdot x \cdot 2x = 2x^3$$

$$dV = 6x^2\,dx$$

But $\quad dx = \pm 0.3$

Therefore change in volume is

$$dV = 6(8.5)^2(\pm 0.3)$$

$$= \pm (6)(72.25)(0.3)$$

$$= \pm 130.05 \text{ Cubic inches.}$$

5. The radius x of a circle increases from $x = 10$ cm to $x = 10.1$ cm. Find the corresponding change in the area of the circle. Also find the percentage change in the area.

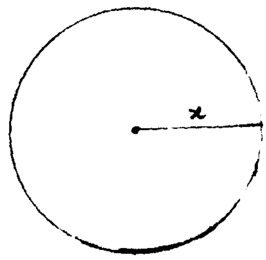
Sol. Let A be area of the circle of radius x. Then

$$A = \pi x^2$$

$$\Rightarrow \quad dA = 2\pi x\, dx$$

Now, $x = 10$ cm and $\Delta x = dx = 0.1$ cm

Change in the area of the circle is



$$dA = 2\pi(10)(0.1) = 2\pi \text{ cm}^2$$

Relative change in the area is

$$\frac{\Delta A}{A} = \frac{2\pi}{\pi(10)^2} = \frac{2}{100} = 0.02$$

Percentage change $= \dfrac{2}{100} \times 100 = 2\,\%$.

6. The diameter of a tree was 8 inches. After one year the circumference of the tree increased by 2 inches. How much did

(i) the diameter of the tree increase?

(ii) the cross-section area of the tree change?

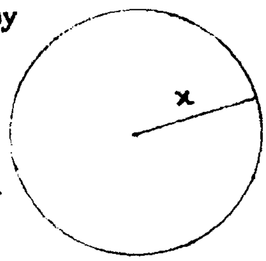
Sol. If x is the radius of the tree, then its circumference

$$C = 2\pi x$$

Therefore, $dC = 2\pi\, dx$

Change in circumference is $dC = 2$

and so the change Δx in radius is given by



$$2 = 2\pi\, dx$$

or $\quad dx = \dfrac{1}{\pi}$

Thus the diameter increased by $\dfrac{2}{\pi}$ inches.

Area A of the cross-section of tree is

$$A = \pi x^2$$

$$\Rightarrow \quad dA = 2\pi x\, dx$$

When $x = 4$, $dx = \dfrac{1}{\pi}$ and change in area is

$$dA = 2\pi\ 4 . \frac{1}{\pi} = 8 \text{ sq. inches}$$

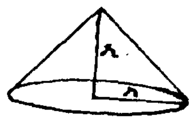
7. **Sand pouring from a chute forms a conical pile whose altitude is always equal to the radius. If the radius of the pile is 10 cm, find the approximate change in radius, when volume increases by 2 cm^3.**

Sol. The volume V of the conical pile of radius r and height r is

$$V = \frac{1}{3}\pi r^3$$

$$\Rightarrow dV = \frac{1}{3}\pi \cdot 3r^2 dr$$

$$\text{or } dV = \pi r^2 dr \quad \text{——} \ (1)$$



Now given that $\Delta V \approx dV = 2\,cm^3$ when $r = 10$

So req. change in radius $= \Delta r = dr = ?$

Hence from (1) $\quad 2 = \pi(10)^2 dr$

$$2 = 100\pi dr \quad \Rightarrow dr = \frac{2}{100\pi} = \frac{1}{50\pi}\,cm.$$

So change in radius of pile $= \frac{1}{50\pi}\,cm.$

8. **A dome is in the shape of a hemisphere with radius 60 ft. The dome is to be painted with a layer of 0.01 inch thickness. Use differentials to estimate the amount of the paint required.**

Sol. If V is volume of hemisphere with radius r, then

$$V = \frac{2\pi r^3}{3}$$

$$\Rightarrow dV = \frac{2}{3}\pi \cdot 3r^2 dr$$

$$dV = 2\pi r^2 dr \quad \text{——} \ (1)$$

we want to find dV

when $r = 60\,ft.$ & $dr = 0.01''$

$$= \frac{0.01}{12}\,ft.$$

$$= \frac{1}{12\times 100}\,ft.$$

So $dr = \frac{1}{1200}\,ft.$

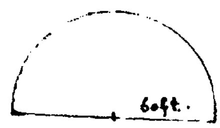
Putting values in (1)

$$dV = 2\pi(60)^2 \cdot \frac{1}{1200}$$

$$= 2\pi \cdot 3600 \cdot \frac{1}{1200}$$

$$= 2\pi \times 3$$

$$= 6\pi\,ft^3$$



9. **The side of a building is in the shape of a square surmounted by an equilateral triangle. If the length of the base is 15 m with an error of 1 %, find the percentage error in the area of the side.**

Sol. Let x m be the length of the base. Then area A of the side is given by

A = area of square + area of triangle

$$= x^2 + \frac{1}{2}\cdot x \cdot h$$

$$= x^2 + \frac{x}{2}\cdot\frac{\sqrt{3}x}{2}$$

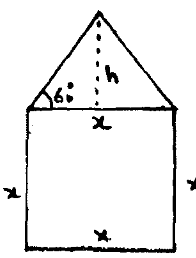
$$A = x^2 + \frac{\sqrt{3}}{4}x^2$$

$$\Rightarrow dA = \left(2x + \frac{\sqrt{3}}{4}\cdot 2x\right)dx$$

$$\frac{h}{\frac{x}{2}} = \tan 60^\circ$$

$$\frac{2h}{x} = \sqrt{3}$$

$$h = \frac{\sqrt{3}x}{2}$$



$dA = \left(2x + \frac{\sqrt{3}x}{2}\right)dx$ ——— (1)

Now given that $\frac{dx}{x} = 0.01$

Now percentage error in $A = ?$ when $x = 15$

i.e., $\frac{dA}{A} \times 100 = ?$

Now from (1)

$$\frac{dA}{A} = \frac{\left(2x + \frac{\sqrt{3}x}{2}\right)dx}{x^2 + \frac{\sqrt{3}}{4}x^2} = \left[\frac{2x + \frac{\sqrt{3}}{2}x}{x + \frac{\sqrt{3}}{4}x}\right]\frac{dx}{x}$$

$$\frac{dA}{A} = \left[\frac{2(15) + \frac{\sqrt{3}}{2}(15)}{15 + \frac{\sqrt{3}}{4}(15)}\right] \times 0.01$$

$$= \left[\frac{30 + \frac{\sqrt{3}\cdot 15}{2}}{15 + \frac{\sqrt{3}\cdot 15}{4}}\right] \times \frac{1}{100}$$

$$= \left(\frac{120 + 30\sqrt{3}}{60 + 15\sqrt{3}}\right) \times \frac{1}{100}$$

$$= 2 \times \frac{1}{100} = \frac{1}{50}$$

So % error in area

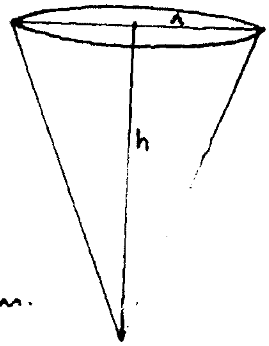
$$= \frac{dA}{A} \times 100 = \frac{1}{50} \times 100 = 2\ \%.$$

10. A boy makes a paper cup in the shape of a right circular cone with height four times its radius. If the radius is changed from 2 cm to 1.5 cm but the height remains four times the radius, find the approxiamte decrease in the capacity of the cup.

Sol. If r is the radius of the base and h is height of the cup, then its volume V is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (4r)$$

$$V = \frac{4}{3}\pi r^3$$



$$\Rightarrow dV = \frac{4}{3}\pi \cdot 3r^2 dr$$

So $dV = 4\pi r^2 dr$ ——— (1)

As it is given $r = 2\text{cm}$. & $dr = 1.5 - 2 = -0.5\text{cm}$.

So from (1) $dV = 4\pi(2)^2(-0.5)$

$$= 4\pi \cdot 4 \cdot \frac{-1}{2}$$

$$= 4\pi(-2)$$

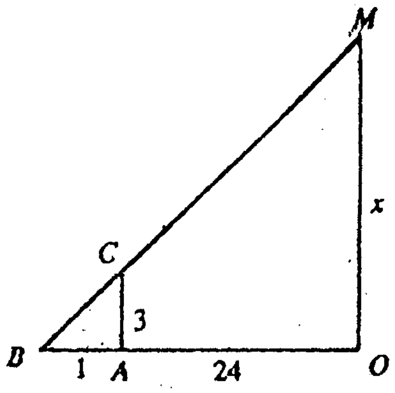
$$dV = -8\pi$$

is the req. change in Capacity of Cup.

The – ve sign shows that there is decrease in the capacity of the cup.

11. To estimate the height of Minar-i-Pakistan, the shadow of a 3 m pole placed 24 m from the Minar is measured. If the length of the shadow is 1 m with a percentage error of 1 %, find the height of the Minar. Also find the percentage error in the height so found.

Sol. Let OM be the minar & AC be the pole.



If x m is height of the Minar, then from the figure

$$\frac{x}{25} = \frac{3}{1}$$

Therefore, $x = 25 \times 3 = 75$.

Height of the minar = 75 m.

If y is the actual length of the shadow of the pole, then

$$\frac{y+24}{x} = \frac{y}{3}$$

or $\quad 3y + 72 = xy$

or $\quad 3dy = x\,dy + y\,dx$

or $\quad (3-x)\,dy = y\,dx$

or $\quad (3-x)\dfrac{dy}{y} = dx$ ——(1)

Now $\dfrac{dy}{y} = \pm\, 0.01$. When $x = 75$, relative error in the height $= \dfrac{dx}{x}$.

Now from (1) $\quad dx = (3-x) \cdot \dfrac{dy}{y}$

$= (3-75)(\pm 0.01)$

$= (-72)(\pm 0.01)$

So $\dfrac{dx}{x} = \dfrac{(-72)(\pm 0.01)}{75}$

or $\dfrac{dx}{x} = \pm 0.0096$

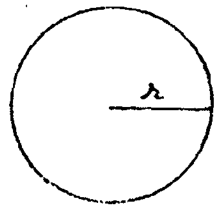
So Percentage error in height $= \dfrac{dx}{x} \times 100$

$= \pm 0.0096 \times 100$

$= \pm 0.96\%$.

12. Oil spilled from a tanker spreads in a circle whose radius increases at the rate of 2 ft/sec. How fast is the area increasing when the radius of the circle is 40 ft?

Sol. Let r be the radius of the circle at any instant t. Then area A of the circle is



$$A = \pi r^2$$

diff. w.r.t. t

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We have to find $\frac{dA}{dt}$ when $\frac{dr}{dt} = 2$ and $r = 40$. Substituting into (1), we have

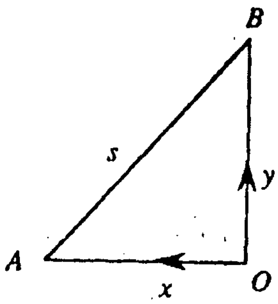
$$\frac{dA}{dt} = 2\pi \times 40 \times 2$$

$$\Rightarrow 160\pi$$

Thus area of the circle changes at the rate of 160π ft^2/sec.

13. From a point O, two cars leave at the same time. One car travels west and after t sec. its position is $x = t^2 + t$ ft. The other car travels north and it covers $y = t^2 + 3t$ ft. in t sec. At what rate is the distance between the two cars changing after 5 sec?

Sol. Let A, B be the positions of the two cars at any instant t and let s be the distance between them at this instant.



$$s^2 = x^2 + y^2 \qquad (1)$$

diff. w.r.t. t

$$2s\frac{ds}{dt} = 2x.\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow s\,dy/dt = x\frac{dx}{dt} + y\frac{dy}{dt} \qquad (2)$$

We have to find $\frac{ds}{dt}$ at the instant when $t = 5$.

We have

$$x = t^2 + t \qquad (3)$$

$$y = t^2 + 3t \qquad (4)$$

Differentating (3) and (4) w.r.t. t, we have

$$\frac{dx}{dt} = 2t + 1,$$

$$\frac{dy}{dt} = 2t + 3,$$

& at $t = 5$, $\frac{dx}{dt} = 11$ & $\frac{dy}{dt} = 13$

After 5 sec, the distances of the two cars from O are

$$x = 5^2 + 5 = 30$$

$$y = 5^2 + 15 = 40$$

& from (1) $\quad s^2 = 30^2 + 40^2$

$$= 900 + 1600$$

$$s^2 = 2500$$

So $S = 50$

Putting values in eq. (2)

$$50\frac{ds}{dt} = 30\times 11 + 40\times 13$$

or $$50\frac{ds}{dt} = 330 + 520$$

$$\frac{ds}{dt} = \frac{850}{50}$$

or $$\frac{ds}{dt} = 17$$

Therefore, the distance between the two cars is changing at the rate of 17 ft./sec.

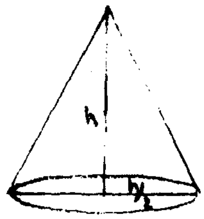
14. Sand falls from a container at the rate of 10 ft^3/min and forms a conical pile whose height is always double the radius of the base. How fast is the height increasing when the pile is 5 ft high?

Sol. Let h be the height of the pile at any instant t. Radius of the pile $= \frac{h}{2}$. Volume V of the pile is

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 . h$$

$$= \frac{1}{3}\pi . \frac{h^3}{4}$$

$$V = \frac{\pi}{12}h^3$$



Now $$\frac{dV}{dt} = \frac{\pi}{12} . 3h^2 . \frac{dh}{dt}$$

or $$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} \quad \text{——— (1)}$$

It is given that $\frac{dV}{dt} = 10$ & we want to find $\frac{dh}{dt}$ when $h = 5$

Putting values in (1)

$$10 = \frac{\pi}{4}(5)^2 . \frac{dh}{dt}$$

$$40 = 25\pi . \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{40}{25\pi}$$

$$\frac{dh}{dt} = \frac{8}{5\pi} = \frac{8}{5\times 3.14} = 0.51$$

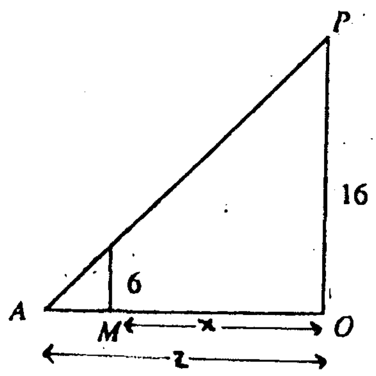
So the height of pile is changing at the rate of 0.51 ft/minute.

15. A 6 ft tall man is walking toward a lamp post 16 ft high at a speed of 5 ft/sec. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?

Sol. Let x be man's distance from the lamp post OP and z the distance of the tip of his shadow from O.

i.e. $OM = x,\quad OA = z$

From the similar triangles, we have



$$\frac{16}{z} = \frac{6}{z - x}$$

$$16z - 16x = 6z$$

$$16z - 6z = 16x$$

$$10z = 16x$$

$$5z = 8x$$

$$5\frac{dz}{dt} = 8\frac{dx}{dt}$$

It is given that $\frac{dx}{dt} = 5$

$$\text{So } 5\frac{dz}{dt} = 8 \times 5$$

$$\frac{dz}{dt} = 8$$

Therefore the tip of man's shadow is moving at the rate of 8 ft./sec.

If y is the length of the shadow then $MA = y$. From the similar triangles we have

$$\frac{16}{x + y} = \frac{6}{y}$$

i.e., $8y = 5x$

Therefore $8\dfrac{dy}{dt} = 5\dfrac{dx}{dt}$.

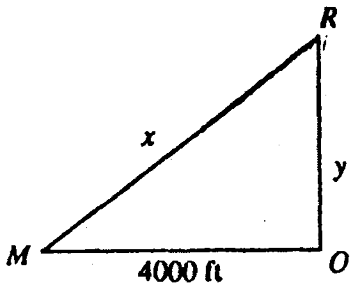
Substituting $\dfrac{dx}{dt} = 5$, we find that

$$\frac{dy}{dt} = \frac{25}{8}$$

Thus the shadow is changing at the rate of $\dfrac{25}{8}$ ft/sec.

16. At a distance of 4000 ft from a launching site, a man is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/sec. when it is at an altitude of 3000ft, how fast is the distance between the rocket and the man changing at this instant?

Sol. Let y be altitude of the rocket and x be the distance between the man and the rocket at any instant t.



We have

$$x^2 = y^2 + 4000^2 \qquad (1)$$

When $y = 3000$ ft,

we have from (1),

$$x^2 = 3000^2 + 4000^2$$
$$= 9000000 + 16000000$$
$$x^2 = 25000000$$
$$\Rightarrow x = 5000$$

Diff. (1) w.r.t. t

$$2x\frac{dx}{dt} = 2y\cdot\frac{dy}{dt}$$
$$x\frac{dx}{dt} = y\frac{dy}{dt}$$

It is given that $\frac{dy}{dt} = 600$ when $y = 3000$

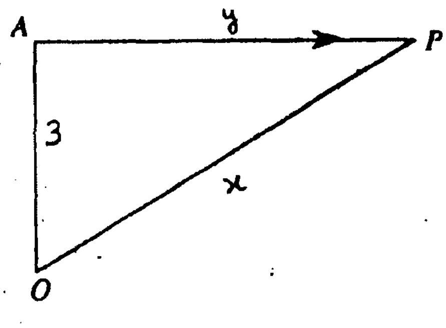
So $5000\cdot\frac{dx}{dt} = 3000\times 600$

$$\frac{dx}{dt} = \frac{3000\times 600}{5000} = \frac{3}{5}\times 600 = 3\times 120 = 360$$

Thus the distance between the rocket and the man is changing at the rate of 360 ft/sec.

17. An airplane flying horizontally at an altitude of 3 miles and a speed of 480 miles/hr. passes directly above an observer on the ground. How fast is the distance of the observer to the airplane increasing after 30 sec?

Sol. Let O be the observer on the ground and P be the plane at some instant t. Let



$$OP = x, \quad AP = y$$

It is given that $OA = 3$

From the right triangle, we have

$$3^2 + y^2 = x^2 \qquad (1)$$

The distnace travelled by the plane 30 sec. after it has passed above the observer $= \frac{480}{60\times 60}\times 30 = 4$ miles. Put in (1)

$$9 + (4)^2 = x^2$$
$$25 = x^2$$
$$x = 5$$

We have to find $\frac{dx}{dt}$ at the instant when $t = 30$ sec. and $\frac{dy}{dt} = 480$.

Now diff. (1) w.r.t. t

$$2y.\frac{dy}{dt} = 2x.\frac{dx}{dt}$$

$$y\frac{dy}{dt} = x\frac{dx}{dt}$$

$$x\frac{dx}{dt} = y.\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{y}{x}.\frac{dy}{dt}$$

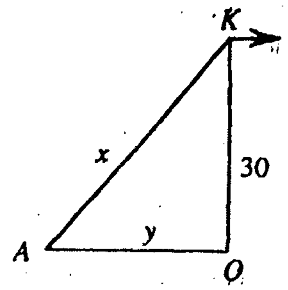
So at $t = 30$, $\frac{dx}{dt} = \frac{4}{5} \times 480$

$$= 384$$

The rate of change of distance of the plane from the observer = 384 miles/hr.

18. A boy flies a kite at an altitude of 30 m. If the kite flies horizontally away from the boy at the rate of 2 m/sec, how fast is the string being let out when the length of the string released is 70 m?

Sol. Let x be the length of the string let out at some instant t, K be the kite at an altitude of 30 m and let $AO = y$. The kite flies horizontally away from the boy at the rate of 2 m/sec.



From ΔAOK, we have

$$x^2 = 30^2 + y^2 \quad \text{------ (1)}$$

Diff. (1) w.r.t. t

$$2x.\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$x.\frac{dx}{dt} = y.\frac{dy}{dt} \quad \text{------ (2)}$$

When $x = 70$ then from (1)

$$(70)^2 = (30)^2 + y^2$$

$$4900 = 900 + y^2$$

$$4900 - 900 = y^2$$

$$4000 = y^2$$

$$y = \sqrt{400 \times 10}$$

$$y = 20\sqrt{10}$$

At this time $\frac{dy}{dt} = 2$

So from (2)

$$70.\frac{dx}{dt} = 20\sqrt{10} \times 2$$

$$\frac{dx}{dt} = \frac{40\sqrt{10}}{70} = \frac{4\sqrt{10}}{7}$$

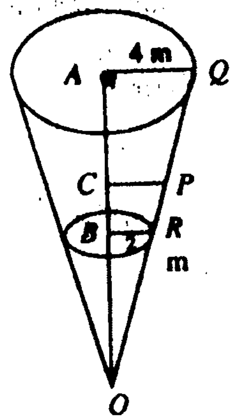
Hence the string is being let out at the rate of $\frac{4\sqrt{10}}{7}$ m/sec.

19. A water tank is in the shape of frustrum of a cone with height 6 m and upper and lower radii 4 m and 2 m, respectively. If water pours into the tank at the rate of 20 m^3/min, how fast is the water level rising when the water is half way up?

Sol. Extend the tank downward so as to form a cone. Let $BO = x$ m so that the height of the cone is $x + 6$.

Suppose that at some instant water level is at C where $BC = y$ and let $CP = r$.

From similar Δ's AOQ and BOR, we get



$$\frac{6+x}{4} = \frac{x}{2}$$

$$4x = 12+2x$$

$$4x-2x = 12$$

$$2x = 12$$

$$x = 6$$

so $\boxed{BO = 6}$

Again from similar Δs COP & BOR

$$\frac{y+6}{r} = \frac{6}{2}$$

$$6r = 2y+12$$

$$r = \frac{2(y+6)}{6}$$

$$r = \frac{y+6}{3}$$

Now the volume of frustum with upper radius r & lower radius 2 is

$$V = \frac{1}{3}\pi r^2(y+6) - \frac{1}{3}\pi(2)^2 \times 6$$

$$= \frac{\pi}{3}\cdot\frac{(y+6)^2}{9}\cdot(y+6) - \frac{\pi}{3}\times 24$$

$$V = \frac{\pi}{27}(y+6)^3 - 8\pi$$

Diff. w.r.t. t

$$\frac{dv}{dt} = \frac{\pi}{27}\cdot 3(y+6)^2\cdot\frac{dy}{dt}$$

$$\frac{dv}{dt} = \frac{\pi}{9}\times(y+6)^2\cdot\frac{dy}{dt}$$

As it is given that $\frac{dv}{dt} = 20$ & we want to find $\frac{dy}{dt}$ when water is half way up i.e., when $y = 3$

Hence from above eq.

$$20 = \frac{\pi}{9}(3+6)^2\cdot\frac{dy}{dt}$$

$$20 = \frac{\pi}{9}\times 81\cdot\frac{dy}{dt}$$

$$\frac{20\times 9}{81\pi} = \frac{dy}{dt}$$

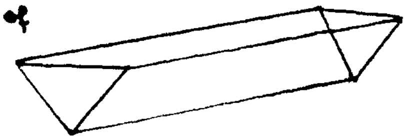
$$\frac{dy}{dt} = \frac{20}{9\pi}$$

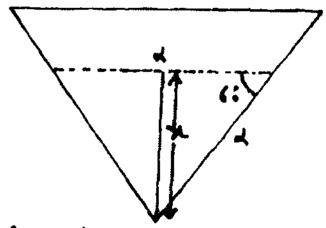
Hence water level is rising at the rate of $\frac{20}{9\pi}$ m/min.

20. A 12 m long water trough, with vertical cross-sections in the shape of equilateral triangles (one vertex down) is being filled at the rate of $4m^3/min$. How fast is the water level rising at the instant when the depth of the water is $1\frac{1}{2}$ m?

Sol. Suppose the water is x feet deep

then area of vertical cross-section of water is





Area of vertical cross-section of water is

$A = \frac{1}{2}\cdot d\cdot x$

But $\frac{x}{d} = \sin 60°$

$\Rightarrow \frac{d}{x} = \frac{1}{\sin 60}$

$d = x\,\text{Cosec}\,60°$

So $A = \frac{1}{2}\cdot x\,\text{Cosec}\,60\cdot x = \frac{x^2}{2\sin 60}$

$$A = \frac{x^2}{2\sin 60°}$$

$$= \frac{x^2}{2\times\frac{\sqrt{3}}{2}}$$

$$A = \frac{x^2}{\sqrt{3}}$$

then Volume of water at this time is $V = \frac{x^2}{\sqrt{3}}\times 12$

$$V = \frac{12x^2}{\sqrt{3}}$$

Diff. w.r.t. t

$$\frac{dV}{dt} = \frac{24x}{\sqrt{3}}\cdot\frac{dx}{dt}$$

We want to find $\frac{dx}{dt}$ at the time when $x = \frac{3}{2}$ & $\frac{dV}{dt} = 4$

So Putting values

$$4 = \frac{24}{\sqrt{3}}\times\frac{3}{2}\times\frac{dx}{dt}$$

$$4 = 12\sqrt{3}\cdot\frac{dx}{dt}$$

$$\frac{4}{12\sqrt{3}} = \frac{dx}{dt}$$

or $$\frac{dx}{dt} = \frac{1}{3\sqrt{3}}$$

So the water level is rising at the rate of

$\frac{1}{3\sqrt{3}}$ m/min.