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Exercise 2.1

Show that function $f(x) = |x| + |x-1|$ is continuous for every value of x but is not differentiable at $x=0$ and $x=1$.

Sol. **Continuity:**

$f(x) = |x| + |x-1|$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} |x|+|x-1|$$

$x = a-h$, when $x \Rightarrow 0$ then $h \to 0$

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} |a-h| + |a-h-1|$$
$$= |a-0| + |a-0-1|$$
$$= |a| + |a-1|$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} |x| + |x-1|$$

$x = a+h$; when $x \Rightarrow 0$ then $h \to 0$

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} |a+h| + |a+h-1|$$
$$= |a+0| + |a+0-1|$$
$$= |a| + |a-1|$$

$f(a) = |a| + |a-1|$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a)$$

function is continuous.

Differentiability:

$f(x) = |x| + |x-1|$

$$Lf'(1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$
$$= \lim_{x\to 1^-} \frac{|x|+|x-1| - (|1|+|1-1|)}{x-1}$$
$$= \lim_{x\to 1^-} \frac{|x|+|x-1| - (|1|+0)}{x-1}$$
$$= \lim_{x\to 1^-} \frac{|x|+|x-1| - |1|}{(x-1)}$$

$x = 1-h$

when $x \to 1$ then $h \to 0$

Differentiability:

$f(x) = |x| + |x-1|$

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{x\to 0^-} \frac{|x|+|x-1| - (|0|+|0-1|)}{x-0}$$
$$= \lim_{x\to 0^-} \frac{|x|+|x-1| - |-1|}{x}$$
$$Lf'(0) = \lim_{x\to 0^-} \frac{|x|+|x-1|-1}{x}$$

$x = 0-h$

when $x\to 0 \Rightarrow h \to 0$

$$Lf'(0) = \lim_{h\to 0} \frac{|0-h| + |0-h-1| - 1}{0-h}$$
$$= \lim_{h\to 0} \frac{|-h| + |-(1+h)| - 1}{-h}$$
$$= \lim_{h\to 0} \frac{\cancel{h} + \cancel{1} + h - \cancel{1}}{-h}$$
$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}} = \lim_{h\to 0} (-2)$$

$Lf'(0) = -2$

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$
$$= \lim_{x\to 0^+} \frac{|x|+|x-1| - 1}{x}$$

$x = 0+h$, when $x\to 0 \Rightarrow h\to 0$

$$Rf'(0) = \lim_{h\to 0} \frac{|0+h| + |0+h-1| - 1}{0+h}$$
$$= \lim_{h\to 0} \frac{h + |-(1-h)| - 1}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h} + \cancel{1} - \cancel{h} - \cancel{1}}{h} = \lim_{h\to 0} (0)$$
$$= 0$$

$Rf'(0) \neq Lf'(0)$

So function is not differentiable at $x=0$

$$Lf'(0) = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$
$$= 0\cdot[-1,1]$$
$$= 0$$

$$Rf'(0) = \lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right) - 0}{x-0}$$
$$= \lim_{x\to 0} x\sin(1/x)$$
$$= 0\cdot[-1,1]$$
$$= 0$$

$Lf'(0) = Rf'(0)$ is differentiable at $x=0$

4. $f = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & x\neq a \\ 0 & x=a \end{cases}$

continuous and differentiable at $x=a$.

Sol: Continuous:

(i) $f(a) = 0$

(ii) $\lim_{x\to a} f(x) = \lim_{x\to a} (x-a)\sin\left(\frac{1}{x-a}\right)$
$$= (a-a)\sin\left(\frac{1}{a-a}\right)$$
$$= 0\sin\left(\frac{1}{0}\right)$$
$$= 0\,[-1,1]$$
$$= 0$$

(iii) $\lim_{x\to a} f(x) = f(a)$

function is continuous at $x=a$.

Differentiable:

$$Lf'(a) = \lim_{x\to a^-} \frac{(x-a)\sin\left(\frac{1}{x-a}\right) - 0}{x-a}$$
$$= \lim_{x\to a^-} \frac{\cancel{(x-a)}\sin\left(\frac{1}{x-a}\right)}{\cancel{(x-a)}}$$
$$= \lim_{x\to a^-} \sin\left(\frac{1}{x-a}\right)$$
$$= \sin\left(\frac{1}{a-a}\right) = \sin\left(\frac{1}{0}\right)$$
$$= \sin(\infty)$$
$$= [-1,1]$$

$Lf'(a)$ does not exist.

$\Rightarrow$ function is not differentiable

5. $f(x) = \begin{cases} x\tan^{-1}\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$

check continuity and differentiability: (3)

Sol. Continuity:

(i) $f(0) = 0$

(ii) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\tan^{-1}\left(\frac{1}{x}\right)$
$$= 0\cdot\tan^{-1}\left(\frac{1}{0}\right)$$
$$= 0\cdot[-1,1]$$
$$= 0$$

(iii) $\lim_{x\to 0} f(0) = f(0)$

f is continuous at $x=0$.

Differentiable:

$$Lf'(x) = \lim_{x\to 0} \frac{x\tan^{-1}(1/x) - 0}{x-0}$$
$$= \lim_{x\to 0} \frac{x\tan^{-1}(1/x)}{x}$$
$$= \lim_{x\to 0} \tan^{-1}(1/x)$$
$$= [-1,1]$$

Limit does not exist

$\Rightarrow$ function is not differentiable

6. Find $Lf'(2)$ and $Rf'(2)$

$f(x) = |x^2 - 4|$

$$Lf'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x-2}$$
$$= \lim_{x\to 2} \frac{|x^2-4| - 0}{x-2}$$

put $x = 2-h$, when $x\to 2$, $h\to 0$

$$Lf'(2) = \lim_{h\to 0} \frac{|(2-h)^2 - 4|}{2-h-2}$$
$$= \lim_{h\to 0} \frac{|4+h^2-4h-4|}{-h}$$
$$= \lim_{h\to 0} \frac{|h^2-4h|}{-h} = \lim_{h\to 0} \frac{|\ldots|}{-h}$$
$$= \lim_{h\to 0} \frac{-h^2+4h}{-h}$$

$f(2) = |2^2-4| = 0$

from continuity

put $a=1$ in ①

$$\frac{\pi}{6}+b=\frac{\sqrt{3}}{2}$$

$$b=\frac{\sqrt{3}}{2}-\frac{\pi}{6}$$

$$b=\frac{3\sqrt{3}-\pi}{6}$$

$$\boxed{a=1}$$

$$\boxed{b=\frac{3\sqrt{3}-\pi}{6}}$$

9. $$f(x)=\begin{cases} x\tanh\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$$

$$\tanh(x)=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

$$\tanh\left(\frac{1}{x}\right)=\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$$

$$\tanh\left(\frac{1}{x}\right)=\frac{e^{1/x}-\frac{1}{e^{1/x}}}{e^{1/x}+\frac{1}{e^{1/x}}}$$

$$\tanh\left(\frac{1}{x}\right)=\frac{\frac{e^{2/x}-1}{e^{1/x}}}{\frac{e^{2/x}+1}{e^{1/x}}}$$

$$\tanh\left(\frac{1}{x}\right)=\frac{e^{2/x}-1}{e^{2/x}+1}$$

$$f(x)=\begin{cases} x\cdot\frac{e^{2/x}-1}{e^{2/x}+1} & x\neq 0 \\ 0 & x=0 \end{cases}$$

Continuity:

(i) $f(0)=0$

(ii) $$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x\,\frac{e^{2/x}-1}{e^{2/x}+1}$$

put $x=0-h$, when $x\to 0$, $h\to 0$

$$\lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0-h)\,\frac{e^{\frac{2}{0-h}}-1}{e^{\frac{2}{0+h}}+1}$$

$$=\lim_{h\to 0}(-h)\,\frac{e^{-2/h}-1}{e^{2/h}+1}$$

$$=(0)\,\frac{e^{-2/0}-1}{e^{-2/0}+1}$$

$$=(0).\,\frac{e^{-\infty}-1}{e^{+\infty}+1}$$

$$=(0)\,\frac{0-1}{0+1}$$

$$=(0)(-1)$$

$$=0$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x.\,\frac{e^{2/x}-1}{e^{2/x}+1}$$

$$=\lim_{x\to 0^+} x\,\frac{e^{2/x}\left(1-e^{-2/x}\right)}{e^{2/x}\left(1+e^{-2/x}\right)}$$

$$=\lim_{x\to 0^+} x\,\frac{1-e^{-2/x}}{1+e^{-2/x}}$$

$$=(0)\,\frac{1-e^{-2/0}}{1+e^{-2/0}}$$

$$=(0)\,\frac{1-e^{-\infty}}{1+e^{-\infty}}$$

$$=(0)\,\frac{1-0}{1+0}$$

$$=(0)(1)$$

$$=0$$

$$\Rightarrow \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=0$$

$$\Rightarrow \lim_{x\to 0} f(x)=0$$

and

(iii) $\lim_{x\to 0} f(x)=f(0)$

$\Rightarrow$ function is continuous at $x=0$.

Differentiability:

$$Lf'(0)=\lim_{x\to 0^-}\frac{x.\frac{e^{2/x}-1}{e^{2/x}+1}-0}{x-0}$$

$$=\lim_{x\to 0^-}\frac{\cancel{x}.\frac{e^{2/x}-1}{e^{2/x}+1}}{\cancel{x}}$$

$$=\lim_{x\to 0^-}\frac{e^{2/x}-1}{e^{2/x}+1}$$

Same method from continuity

$$=\frac{e^{-\infty}-1}{e^{-\infty}+1}$$

$$=\frac{0-1}{0+1}$$

$$Lf'(0)=-1$$

$$Rf'(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x\frac{e^{2/x}-1}{e^{2/x}+1}-0}{x-0}$$

$$=\lim_{x\to 0^+}\frac{\cancel{x}\frac{e^{2/x}-1}{e^{2/x}+1}}{\cancel{x}}$$

$$=\lim_{x\to 0}\frac{e^{2/x}-1}{e^{2/x}+1}=\lim_{x\to 0}\frac{e^{2/x}\left(1-e^{-2/x}\right)}{e^{2/x}\left(1+e^{-2/x}\right)}$$

$$=\lim_{x\to 0}\frac{1-e^{-2/x}}{1+e^{-2/x}}=\frac{1-e^{-\infty}}{1+e^{-\infty}}=\frac{1-0}{1+0}$$

$$Rf'(0)=1$$

$\because Rf'(0)\neq Lf'(0)$

$\Rightarrow$ function is not differentiable at $x=0$