EXERCISE 1.3

Discuss the continuity of the following functions at the indicated points sets **(Problems 1 – 7):**

1. $f(x) = |x-3|$ at $x = 3$

2. $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$

   at $x = 3$

3. $f(x) = \begin{cases} x-4 & \text{if } -1 < x \le 2 \\ x^2-6 & \text{if } 2 < x < 5 \end{cases}$

   at $x = 2$

4. $f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

   at $x = 3$

5. $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   at $x = 0$

6. $f(x) = \sin x$ for all $x \in \mathbf{R}$. (to do)

7. $f(x) = \begin{cases} \dfrac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - \dfrac{a^2}{x} & \text{if } x > a \end{cases}$

   at $x = a$

8. Determine the points of continuity of the function $f(x) = x - [x]$ for all $x \in \mathbf{R}$.

9. Discuss the continuity of $x - |x|$ at $x = 1$. to ask.

10. Show that the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 1-x & \text{if } x \text{ is rational} \end{cases}$$

is continuous at $x = \dfrac{1}{2}$.

11. Show that the function $f : \,]0, 1] \longrightarrow \mathbf{R}$ defined by

$$f(x) = \frac{1}{x}$$

is continuous on $]0, 1]$. Is $f(x)$ bounded on this interval? Explain.

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12. Let $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$

Is f continuous at $x = 0$?

13. Let $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq 0 \quad x \neq a \\ 0 & \text{if } x = 0 \quad x = a \end{cases}$

Discuss the continuity of f at $x = a$

14. Let $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that f is continuous at $x = 0$

15. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$.

16. Let $f(x) = \begin{cases} x\sin\left(\frac{|x|}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

17. Find c such that the function

$$f(x) = \begin{cases} \dfrac{1-\sqrt{x}}{x-1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases}$$

is continuous for all $x \in [0, 1]$

In Problems 18 – 20, find the points of discontinuity of the given function

18. $f(x) = \begin{cases} x + 4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x - 4 & \text{if } 2 \leq x \leq 6 \end{cases}$

19. $g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4 + x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2 + 46 & \text{if } x > 10 \end{cases}$

20. $f(x) = \begin{cases} x + 2 & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x < 2 \\ x + 5 & \text{if } 2 \leq x < 3 \end{cases}$

21. Find constants a and b such that the function f defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ ax + b & \text{if } -1 \leq x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}$$

is continuous for all x.

Find the interval on which the given function is continuous. Also find points where it is discontinuous. **(Problems 22–26) :**

22. $f(x) = \dfrac{x^2 - 5}{x - 1}$ **23.** $f(x) = \dfrac{x}{|x|}$

24. $f(x) = \dfrac{\sin x}{x}$ **25.** $f(x) = \tan x$

26. $f(x) = \begin{cases} \sin x & \text{if } x \le \pi/4 \\ \cos x & \text{if } x > \pi/4 \end{cases}$

In Problems 27 – 34, examine whether the given function is continuous at $x = 0$

27. $f(x) = \begin{cases} (1 + 3x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

28. $f(x) = \begin{cases} (1 + x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

29. $f(x) = \begin{cases} (1 + 2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

30. $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

31. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1 + e^{1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

32. $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2} - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

33. $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

34. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ \dfrac{2}{3} & \text{if } x = 0 \end{cases}$

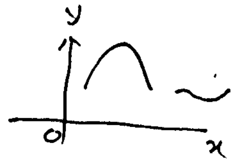
35. Let $f(x) = x^2$ and

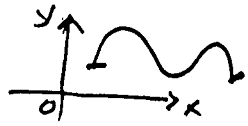
$$g(x) = \begin{cases} -4 & \text{if } x \le 0 \\ |x - 4| & \text{if } x > 0. \end{cases}$$

Determine whether fog and gof are continuous at $x = 0$.

Continuity

A function $y = f(x)$ is said to be continuous at a point $x = a \in D_f$





if (i) $f(x)$ is defined at $x = a$

(ii) $L.H.\lim_{x \to a} f(x) = R.H.\lim_{x \to a} f(x)$

i.e. limit of $f(x)$ when $x \to a$ is exist

(iii) $\lim_{x \to a} f(x) = f(a)$

EXERCISE No 1.3

Q1

$f(x) = |x-3| \rightarrow (i)$

at $x = 3$

Value

$f(x) = |x-3|$

Put $x = 3$

$f(3) = |3-3|$

$= |0|$

$= 0 \rightarrow (ii)$

R.H.L

$\lim_{x \to 3+0} f(x) = \lim_{x \to 3+0} |x-3|$

Put $x = 3+h$

$= \lim_{h \to 0} |3+h-3|$

$= 0 \rightarrow (iii)$

$\lim_{x \to 3-0} f(x) = \lim_{x \to 3-0} |x-3|$

Put $x = 3-h$

$= \lim_{h \to 0} |3-h-3|$

$= 0 \rightarrow (iv)$

(ii), (iii) & (iv)

Value = R.H.L = L.H.L

$f(x)$ is cont: at $x = 3$

Q2

$f(x) = \frac{x^2-9}{x-3}$ if $x \neq 3$

$= 0$ if $x = 3$

Value at $x = 3$ is given

$f(3) = 0 \rightarrow (i)$

$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)}$

$= \lim_{x \to 3} (x+3)$

$= 6$

(i) (ii)

$f(3) \neq \lim_{x \to 3} f(x)$

$f(x)$ is Dis-cont: at $x = 3$

(3) $f(x) = \begin{cases} x-4 & \text{if } -1 < x \leq 2 \\ x^2-6 & \text{if } 2 < x < 5 \end{cases}$ at $x = 2$

Value

$f(x) = x-4$ at $x = 2$

$f(2) = 2-4 = -2 \rightarrow (i)$

R.H.L $\lim_{x \to 2+0} f(x) = \lim_{x \to 2+0} (x^2-6)$ Put $x = 2+h$

$= \lim_{h \to 0} [(2+h)^2-6] = (2+0)^2-6 = 4-6$

LHL $\lim_{x\to 2-0} f(x) = \lim_{x\to 2-0} (x-4)$ put $x=2-h$

$= \lim_{h\to 0} (2-h-4)$

$= (2-0-4) = -2 \rightarrow iii.$

i, ii. and iii,

$\Rightarrow f(x)$ is Continuous at $x=2$

(4)

$$f(x) = \begin{cases} \frac{x^3-27}{x^2-9} & \text{if } x \neq 3 \\ 6 & \text{if } x=3 \end{cases}$$

at $x=3$

Value at $x=3$ is given and is 6

$f(3) = 6 \longrightarrow i.$

Limit

$\lim_{x\to 3} f(x) = \lim_{x\to 3} \left(\frac{x^3-27}{x^2-9}\right)$

$= \lim_{x\to 3} \frac{x^3-3^3}{x^2-3^2}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x^2+9+3x)}{\cancel{(x-3)}(x+3)}$

$= \frac{3^2+9+3(3)}{3+3} = \frac{27}{6} = \frac{9}{2} \rightarrow ii.$

i, and ii.

$f(3) \neq \lim_{x\to 3} f(x)$

$\Rightarrow f(x)$ is discontinuous at $x=3$

Q5

Hint $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 2 & \text{if } x=0 \end{cases}$ at $x=0$

Ex.1.2 Q 11 Page (4) Limit

$\lim_{x\to 0} f(x)$ does not exist.

Hence The given function is not Continuous at $x=0$

Q6 $f(x) = \sin x \quad \forall x \in R$

Sol

Let $a \in R$ we discuss the Continuity at $x=a$

$\because$ given that $x \in R$

Value $f(x) = \sin x$ Put $x=a$

$f(a) = \sin a \rightarrow i.$

RHL

$\lim_{x\to a+0} f(x) = \lim_{x\to a+0} \sin x$ Put $x=a+h$

$= \lim_{h\to 0} \sin(a+h)$

$= \sin(a+0) = \sin a \rightarrow ii.$

LHL

$\lim_{x\to a-0} f(x) = \lim_{x\to a-0} \sin x$ Put $x=a-h$

$= \lim_{h\to 0} \sin(a-h)$

$= \sin a \rightarrow iii$

i, ii, and iii,

$f(x)$ is Continuous at $x=a \in R$

But: as 'a' is arbitrary real number. So f is Continuous at all $x \in R$.

(7)

$$f(x) = \begin{cases} \frac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x=a \\ a - \frac{a^2}{x} & \text{if } x > a \end{cases}$$

at $x=a$

Sol:

i. $f(a) = 0$ (given)

$f(x)$ is defined at $x=a$

ii. LH Limit $\lim_{x\to a} f(x) = \lim_{\substack{x\to a-h \\ h\to 0}} \left(\frac{x^2}{a} - a\right)$

$$\underset{h\to 0}{Lt}\left[\frac{(a-h)^2}{a}-a\right]$$

$$=\frac{a^2}{a}-a=0$$

R.H. Limit $f(x)$ $\underset{x\to a}{Lt} = \underset{\substack{x\to a+h\\h\to 0}}{Lt}\left(a-\frac{a^2}{x}\right)$

$$=\underset{h\to 0}{Lt}\left(a-\frac{a^2}{a+h}\right)$$

$$=a-\frac{a^2}{a}=0$$

$$\Rightarrow \underset{x\to a}{LH\,Lim}\,f(x)=\underset{x\to a}{RH.Lim}\,f(x)=0$$

$$\Rightarrow \underset{x\to a}{Lt}\,f(x)=0$$

Limit of $f(x)$ exists at $x=0$

iii. $\underset{x\to c}{Lt}\,f(x)=f(a)=0$

All Three Cond., are Satisfied

$\because$ $f(x)$ is Cont: at $x=0$

Q (8)

Determine the points of Continuity of the function $f(x)=x-[x]$ for all $x\in R$.

Note $y=[x]$ is called Bracket fn (Greatest integral value of x But not greater than x (If x is Decimal)

Case-I Let $x=2.5$ (Take Fractional Value $\in R$)

Then $f(2.5)=2.5-[2.5]$

$$=2.5-2=.5\rightarrow ①$$

and $\underset{x\to 2.5}{Lt}\,f(x)=\underset{x\to 2.5}{Lt}\,x-[x]$

$$=2.5-[2.5]$$

$$=2.5-2=.5\rightarrow ②$$

From ① and ②, we get

$$f(2.5)=\underset{x\to 2.5}{Lt}\,f(x)=.5$$

fn: is Cont: at any Fractional value of $x\in R$

Case-II

When 'C' is Integer either +ve or -ve

Suppose that $x=c=5$

Then $f(x)=x-[x]$ will

Then $f(5)=5-[5]=5-5=0$

and $\underset{x\to 5-0}{Lt}\,f(x)=\underset{x\to 5-0}{Lt}\,(x-[x])$

$$=5-[5-0]$$

$$=5-4=1$$

$$\underset{x\to 5+0}{Lt}\,f(x)=\underset{x\to 5+0}{Lt}\,(x-[x])$$

$$=5-[5+0]$$

$$=5-5=0$$

$$\underset{x\to 5-0}{Lt}\,f(x)\neq\underset{x\to 5+0}{Lt}\,f(x)$$

Limit does not Exist at $x=c=5\in R$.

i.e for any +ve or −ve Integral Value of $x \in R$ fn: does not exist. Implies that fn: is dis Continous for all Integral values of x

However it is Continous at any Other Real Value of x

∵ fn: is Cont: for all decimal values

(9) $f(x) = x - |x|$ at $x=1$

$f(1) = 1 - |1| = 1-1 = 0$

$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} x - |x|$

$= 1-1 = 0$

$\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0} x - |x|$

$= 1-1$

$= 0$

∵ $\lim_{x\to 1-0} f(x) = \lim_{x\to 1+0} f(x) = f(1)$

fn: is Cont: at $x=1$

(10) Show that the fn. $f: R \to R$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 1-x & \text{if } x \text{ is Rational} \end{cases}$$

is Cont: at $x = \frac{1}{2}$

∵ Note The numbers these Can be written to the form of p/q, p and q are integer when $q \neq 0$ is Called Rational no:

i. $f(\frac{1}{2}) = 1 - x$

$= 1 - \frac{1}{2} = \frac{1}{2}$

$f(x)$ is defined at $x = \frac{1}{2}$

ii, L.H.Lt $\lim_{x\to \frac{1}{2}} f(x) = \lim_{\substack{x\to \frac{1}{2}-h \\ h\to 0}} f(x)$

$= \lim_{\substack{x\to \frac{1}{2}-h \\ h\to 0}} (x)$

$= \lim_{h\to 0} (\frac{1}{2} - h) = \frac{1}{2}$

(if x is irrational)

& $\lim_{\substack{x\to \frac{1}{2}-h \\ h\to 0}} (1-x)$ if x is rational

$= 1 - (\frac{1}{2} - h)$

$= \frac{1}{2} + h = \frac{1}{2}$

R.H.L $\lim_{x\to \frac{1}{2}} f(x) = \lim_{\substack{x\to \frac{1}{2}+h \\ h\to 0}} f(x)$

$= \lim_{\substack{x\to \frac{1}{2}+h \\ h\to 0}} (x)$ x is irrational

$= \lim_{h\to 0} (\frac{1}{2} + h) = \frac{1}{2}$

& $\lim_{\substack{x\to \frac{1}{2}-h \\ h\to 0}} (1-x)$ x is Rational

$= \lim_{h\to 0} (1 - (\frac{1}{2} - h))$

$= \lim_{h\to 0} (\frac{1}{2} - h) = \frac{1}{2}$

∵ Value = Limit. (Obviously)

$\lim_{x\to \frac{1}{2}} f(x) = f(\frac{1}{2}) = \frac{1}{2}$

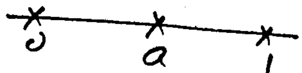
All the Three Condns are Satisfied

∴ $f(x)$ is Cont: at $x = \frac{1}{2}$

1) Show that the fn: $f :]0,1] \rightarrow R$ defined by $f(x) = \frac{1}{x}$ is Cont: on $]0,1]$. Is $f(x)$ bounded on this interval? Explain

Sol: Let a is an arbitrary real no belonging to $]0,1]$

i. Value



$f(a) = \frac{1}{a} \in R$

Limit

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} \left(\frac{1}{x}\right) \quad \text{Put } x = a-h$$

$$= \lim_{h \to 0} \left(\frac{1}{a-h}\right)$$

$$= \frac{1}{a}$$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} \left(\frac{1}{x}\right) \quad \text{Put } x = a+h$$

$$= \lim_{h \to 0} \frac{1}{a+h} = \frac{1}{a}$$

Limit exist

iii. $f(a) = \lim_{x \to a} f(x) = \frac{1}{a}$

a is arbitrary real no $\in (0,1]$

$f(x)$ is Cont: on $(0,1]$

Explain $x \to$ any value $\in]0,1]$

we see that $f(x) = 1$ when $x = 1$

$x = 1$ is its Lower bound.

But value of x become decreases from 1. The value of $f(x) \longrightarrow \infty$ (as $x \to 0$)

So fn: has not upper bound.

Thus fn: $f(x)$ is not bounded above.

Hence $f(x)$ is unbounded.

(12)

$$\text{Let } f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

f is Cont: at $x=0$

Value $f(0) = 0$ (given)

Limit

$$\lim_{x \to 0+0} f(x) \neq \lim_{x \to 0-0} f(x)$$

$\because \cos\frac{1}{x}$ (may be any value b/w $[-1,1]$)

Limit does not unique.

$\Rightarrow$ Limit does not Exist

$f(x)$ is dis Cont: at $x=0$

(13)

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & x \neq a \\ 0 & x = a \end{cases}$$

Cont: at $x=0$

Sol:

$f(a) = 0$ (given)

LH. Limit

$$\lim_{x \to a} f(x) = \lim_{\substack{x \to a-h \\ h \to 0}} (x-a)\sin\left(\frac{1}{x-a}\right)$$

$$= \lim_{h \to 0} (\cancel{a}-h-\cancel{a})\sin\left(\frac{1}{\cancel{a}-h-\cancel{a}}\right)$$

$$= \lim_{h \to 0} -h\sin\left(-\frac{1}{h}\right) = \lim_{h \to 0} h\sin\frac{1}{h}$$

$= 0 \times$ any values $[-1,1] = 0$

RH Limit

$$\lim_{x \to a} f(x) = \lim_{\substack{x \to a+h \\ h \to 0}} (x-a)\sin\left(\frac{1}{x-a}\right)$$

$$= (a+h-a)\sin\left(\frac{1}{a+h-a}\right)$$

limit exist

iii $\lim_{x \to a} f(x) = f(a) = 0$

$f(x)$ is Cont: at $x = a$

Same Q 14, 15 as 13

Q (16) $f(x) = \begin{cases} x \sin \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$

$\because f(x) = x \sin \frac{x}{x} = x \sin 1$ if $x \geq 0$

$= x \sin \frac{-x}{x} = x \sin(-1)$ $\quad x < 0$

(1) Value $f(0) = 0$ given

(2) Limit $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} x \sin 1$

$= 0 \times \sin 1 = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} x \sin(-1)$

$= 0 \times \sin(-1) = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 0$

$\lim_{x \to 0} f(x) = 0$

(3) $\lim_{x \to 0} f(x) = f(0)$

fn: is Cont: at $x = 0$

(17) Find C s.t the fn

$f(x) = \begin{cases} \frac{1-\sqrt{x}}{x-1} & \text{if } 0 \leq x < 1 \\ C & \text{if } x = 1 \end{cases}$

$f(x)$ is cont: $x \in [0,1]$

$f(1) = C \longrightarrow (i)$

$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{1-\sqrt{x}}{x-1}$

$= \lim_{x \to 1} \frac{-\cancel{(\sqrt{x}-1)}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}$

$= \lim_{x \to 1} \frac{-1}{\sqrt{x}+1} = -\frac{1}{2} \rightarrow ii$

Given that $f(x)$ is Continous at $x = 1$

Therefore $f(1) = \lim_{x \to 1} f(x)$

using i. & ii.

$C = -\frac{1}{2}$ Ans

(18) Find the points of discontinuity of the given fn.

$f(x) = \begin{cases} x+4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x-4 & \text{if } 2 \leq x \leq 6 \end{cases}$

$\because$ Continuity of the fn: at the Changing pt $x = -2, 2$.

At $\Rightarrow x = -2$

i. $f(-2) = (-2) = -2$

ii. $\lim_{x \to -2^+} f(x) = \lim_{x \to -2} (x)$

$= (-2) = -2$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} (x+4)$

$= -2+4 = 2$

$\lim_{x \to -2^+} f(x) \neq \lim_{x \to -2^-} f(x)$

$f(x)$ is discont: at $x = -2$

At $\Rightarrow x = 2$

$f(2) = 2-4 = -2$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x-4) = 2-4 = -2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (x) = 2$

$\lim_{x \to 2^+} f(x) \neq \lim_{x \to 2^-} f(x)$

$f(x)$ is discont: at $x = 2$

Thus the pt: of discont: at $x = 2, -2$

(19)

$$g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \le x \le 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$$

At $x=1$: $g(1) = -4-(1)^2 = -4-1 = -5$

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1} (x^3) = (1)^3 = 1$

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1} -4-x^3 = -4-(1)^3 = -5$

$\lim_{x\to 1^-} g(x) \neq \lim_{x\to 1^+} g(x)$

$g(x)$ is dis Cont: at $x=1$

At $x=10$: $g(10) = -4-(10)^2 = -4-100 = -104$

$\lim_{x\to 10^-} g(x) = \lim_{x\to 10} -4-x^2 = -4-(10)^2 = -104$

$\lim_{x\to 10^+} g(x) = \lim_{x\to 10^+} 6(10)^2+46 = 6(100)+46 = 600+46 = 646$

$\lim_{x\to 10^+} g(x) \neq \lim_{x\to 10^-} g(x)$

$g(x)$ is also disCont: at $x=10$

Same Q: 20
Check dis: at $x = 1, 2$.

(21)

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ ax+b & \text{if } -1 \le x < 1 \\ x^2+2 & \text{if } x \ge 1 \end{cases}$$

is Cont: for all x.

at $x=1$

$f(1) = x^2+2 = (1)^2+2 = 3 \rightarrow$ (i)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \rightarrow$ ii

$\lim_{x\to 1} x^2+2 = \lim_{x\to 1} (ax+b)$

$(1)^2+2 = a(1)+b$

$3 = a+b \rightarrow$ (iii)

Since fn: is Cont: (given)
from i, ii & iii

$a+b = 3 \rightarrow (1)$

At $x=-1$

$f(-1) = a(-1)+b = -a+b \rightarrow$ i

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1} (x)^3 = (-1)^3 = -1 \rightarrow$ ii

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1} ax+b = (a)(-1)+b = -a+b \rightarrow$ iii

from i, ii & iii

$-a+b = -1 \rightarrow (2)$

Adding (1) & (2)

$-a+b=-1$
$a+b=3$

$2b=2 \Rightarrow \boxed{b=1}$

using (1) $a+1=3$

$\boxed{a=2}$

Find the interval on which the given function is Continous. Also find points where it is Discontinous (22–26)

(22) $f(x) = \frac{x^2-5}{x-1}$

Clearly at $x=1$, The Value of $f(x)$ doesnot Exist

$f(x)$ is not Continous at $x=1$

and Continous for all $x \in \mathbb{R} - \{1\}$

(23) $f(x) = \frac{x}{|x|}$

As function is not defined at $x=0$

So is discontinuous at $x=0$

The function is Cont: at Every Value of x when $x \in \mathbb{R} - \{0\}$

(24) $f(x) = \frac{\sin x}{x}$

$f(x)$ is not defined at $x=0$ (DisCont: Pt)

Every value of $\sin x$ and x is Continuous when $x \neq 0$

i.e $x \in R - \{0\}$

(25) $f(x) = \tan x$

Since $\tan (2n+1)\frac{\pi}{2} = \infty$

Value of $\tan x$ at $x = (2n+1)\frac{\pi}{2}$ does not Exist, Therefore $f(x)$ is discontinous at $x = (2n+1)\frac{\pi}{2}$

$f(x)$ is Cont: for all $x \in R - \{(2n+1)\frac{\pi}{2}\}$

(26) $f(x) = \sin x \quad x \le \frac{\pi}{4}$

$= \cos x \quad x > \frac{\pi}{4}$

(i) $f(\frac{\pi}{4}) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

(ii) $\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}} \cos x = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}} \sin x = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^-} f(x)$

(iii) $f(\frac{\pi}{4}) = \lim_{x \to \frac{\pi}{4}} f(x)$

$f(x)$ is Cont: at $x = \frac{\pi}{4}$

both $\sin x$ & $\cos x$ are continous at Every Value of $x \in R$. Hence $f(x)$ is Continuous at every value of $x \in R$.

Examine whether the given Function is Continuous at $x=0$

(27) $f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & \text{if } x \neq 0 \\ e^2 & \text{if } x=0 \end{cases}$

Value $f(0) = e^2 \rightarrow i$

Limit

$\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+3x)^{\frac{1}{x}}$

$= \left[\lim_{x\to 0} (1+3x)^{\frac{1}{3x}}\right]^3$

$= e^3 \rightarrow ii$

i, and ii,

f(x) is not Continous at $x=0$

(28) $f(x) = \begin{cases} (1+x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x=0 \end{cases}$

Value $f(0) = 1 \rightarrow i$,

Limit $\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+x)^{1/x}$

$= e \rightarrow ii$

from i, & ii,

Function f(x) is discontinuous at $x=0$.

(29) $f(x) = (1+2x)^{1/x} \quad x \neq 0$

$= e^2 \quad \text{if } x=0$

Value $f(0) = e^2 \rightarrow i$

Limit $\lim_{x\to 0} f(x) = \lim_{x\to 0} (1+2x)^{\frac{1}{x}}$

$= \left[\lim_{x\to 0} (1+2x)^{\frac{1}{2x}}\right]^2 = e^2 \rightarrow ii$

from i, & ii

f(x) is Continuous at $x=0$

(30) $f(x) = e^{-1/x^2} \quad x \neq 0$

$= 1 \quad \text{if } x=0$

Value

at $x=0$ is given and is 1

$f(0) = 1 \rightarrow i$

Limit

$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x^2}$

$= \lim_{x\to 0} \frac{1}{e^{1/x^2}} = \frac{1}{e^{\infty}}$

$= \frac{1}{\infty} = 0 \rightarrow ii$

OR

L.H.L

$\lim_{x\to 0-0} = \lim_{x\to 0-0} e^{-1/x^2}$ Put $x = 0-h$

$= \lim_{h\to 0} e^{-\frac{1}{(0-h)^2}} = \lim_{h\to 0} \frac{1}{e^{\frac{1}{(0-h)^2}}}$

$= \lim_{h\to 0} \frac{1}{e^{(\frac{1}{0-0})^2}} = \frac{1}{e^{\infty}} = \frac{1}{\infty}$

$= 0$

f(x) is discontinuous at $x=0$.

(31) $f(x) = \frac{e^{-1/x}}{1+e^{1/x}} \quad \text{if } x \neq 0$

$= 1 \quad \text{if } x=0$

Since $f(0) = 1$ (given)

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{-1/x}}{1+e^{1/x}}$

$$= \lim_{x\to 0} \frac{1/e^{1/x}}{1+e^{1/x}}$$

$$= \frac{1}{\lim_{x\to 0} e^{1/x}(1+e^{1/x})}$$

$$= \frac{1}{e^{1/0}(1+e^{1/0})} = \frac{1}{\infty} = 0$$

Since

$f(0) \neq \lim_{x\to 0} f(x)$

function is not Continuous at $x=0$

Note

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} \frac{e^{\frac{1}{h}}}{1+e^{-1/h}}$$

Put $x = 0-h$

$$= \frac{e^{\frac{1}{h}}}{\frac{e^{1/h}+1}{1/e^{\frac{1}{h}}}} = \frac{e^{\frac{2}{h}}}{1+e^{1/h}}$$

$\lim = \infty$ does not Exist

$$\lim_{x\to 0+0} = \lim_{h\to 0} \frac{e^{-1/h}}{1+e^{1/h}} = \frac{1}{e^{2/h}(1+e^{-1/h})}$$

$x = 0+h$

$= 0$

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(32)

$$f(x) = \frac{e^{1/x^2}}{e^{\frac{1}{x^2}}-1} \quad \text{if } x \neq 0$$

$$= 1 \quad \text{if } x = 0$$

Value $f(0) = 1$

Limit

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{\frac{1}{x^2}}}{e^{1/x^2}-1}$$

$$= \lim_{x\to 0} \frac{\cancel{e^{\frac{1}{x^2}}}}{\cancel{e^{1/x^2}}\left[1 - \frac{1}{e^{1/x^2}}\right]}$$

$$= \lim_{x\to 0} \frac{1}{1 - \frac{1}{e^{1/x^2}}} = \frac{1}{1-0} = 1$$

$f(0) = \lim_{x\to 0} f(x)$

fn: is Continuous at $x=0$

(33)

$$f(x) = \frac{\sin 2x}{x} \quad \text{if } x \neq 0$$

$$= 1 \quad \text{if } x = 0$$

Value $f(0) = 1$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{x}$$

$$= 2\left[\lim_{x\to 0} \frac{\sin 2x}{2x}\right]$$

$$= 2(1) = 2$$

$\lim_{x\to 0} f(x) \neq f(0)$

$f(x)$ is not Cont: at $x=0$.

(34)

$$f(x) = \begin{cases} \frac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ \frac{2}{3} & \text{if } x = 0 \end{cases}$$

Value $f(0) = 2/3$

Limit $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$

$$= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{3x}{1} \cdot \frac{1}{\frac{\sin 2x}{2x}} \cdot \frac{1}{2x}$$

$$= \lim_{x\to 0} \frac{3}{2}(1) \cdot \frac{1}{(1)}$$

$= 3/2$

$f(0) \neq \lim_{x\to 0} f(x)$

fn. is discont. at $x=0$

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(38) Let $f(x) = x^2$

and

$$g(x) = \begin{cases} -4 & \text{if } x \le 0 \\ |x-4| & \text{if } x > 0 \end{cases}$$

Determine whether $f\circ g$ and $g\circ f$ are Continuous at $x=0$

$$f\circ g(x) = f(g(x))$$

$$= \begin{cases} f(-4) = (-4)^2 = 16 & \text{if } x \le 0 \\ f|x-4| = (\pm(x-4))^2 = (x-4)^2 & \text{if } x > 0 \end{cases}$$

Value $f\circ g(0) = f(g(0))$

$= f(-4)$

$= (-4)^2 = 16$

OR $f\circ g(0) = 16$ given.

$\lim_{x\to 0^+} f\circ g(x) = \lim_{x\to 0} (x-4)^2$

$= (0-4)^2 = 16$

$\lim_{x\to 0^-} f\circ g(x) = \lim_{x\to 0} 16 = 16$

$\lim_{x\to 0} f\circ g(x) = f\circ g(0)$

$f\circ g(x)$ is Continuous at $x=0$

$$g\circ f(x) = g(f(x)) = \begin{cases} g(f(x)) = g(x^2) = -4 & \text{if } x^2 \le 0 \\ g[f(x)] = g(x^2) = |x^2-4| & \text{if } x^2 > 0 \end{cases}$$

Value

$g\circ f(0) = -4$

Limit

$\lim_{x\to 0^+} g\circ f(x) = \lim_{x\to 0} |x^2-4|$

$= |0-4| = 4$

$\lim_{x\to 0^-} g\circ f(x) = \lim_{x\to 0} (-4)$

$= -4$

$\lim_{x\to 0^+} g\circ f(x) \neq \lim_{x\to 0^-} g\circ f$

$\lim_{x\to 0} g\circ f(x) \neq g\circ f(0)$

$g\circ f(x)$ is discont. at $x=0$

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