Exercise # 1.3

Discuss the continuity of the following functions at the indicated points/set.

1. $f(x) = |x-3|$ at $x=3$

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} |x-3|$

$= \lim\limits_{\substack{x=3-h \\ h\to 0}} |\cancel{3}-h-\cancel{3}| = \lim\limits_{h\to 0} |-h| = \lim\limits_{h\to 0} h$

$\lim\limits_{h\to 3^-} f(x) = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} |x-3| = \lim\limits_{\substack{x=3+h \\ h\to 0}} |\cancel{3}+h-\cancel{3}|$

$= \lim\limits_{h\to 0} |h| = \lim\limits_{h\to 0} h = 0$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3} f(x)$

$f(3) = |3-3| = 0$

$\lim\limits_{x\to 3} f(x) = f(3)$

hence function is continuous at $x=3$.

2. $$f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases}$$

Sol:

$f(3) = 0$

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3} \dfrac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}}$

$= \lim\limits_{x\to 3} (x+3) = 3+3$

$\lim\limits_{x\to 3} f(x) = 6$

So $\lim\limits_{x\to 3} f(x) \neq f(3)$

function is discontinuous.

3. $f(x) = \begin{cases} x-4 & \text{if } -1 \le x \le 2 \\ x^2-6 & \text{if } 2 < x < 5 \end{cases}$ at $x=2$

Sol.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x-4) = 2-4$ $\qquad$ $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-6) = 4-6$

$\lim_{x\to 2^-} f(x) = -2$ $\qquad$ $\lim_{x\to 2^+} f(x) = -2$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} f(x) = -2$$

$$f(2) = x-4 = 2-4 = -2$$

$$\lim_{x\to 2} f(x) = f(2)$$

function is continuous.

4. $f(x) = \begin{cases} \dfrac{x^3-27}{x^2-9} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x=3$

Sol. $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^3-27}{x^2-9}$

$$= \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}$$

$$= \lim_{x\to 3} \frac{(x^2+3x+9)}{x+3}$$ $\qquad f(3)=6$

$$= \frac{9+9+9}{3+3}$$

$$= \frac{27}{6}$$

$$\lim_{x\to 3} f(x) = \frac{9}{2}$$

So here $\lim_{x\to 3} f(x) \ne f(3)$ function is discontinuous.

5. $f(x) = \begin{cases} \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x=0$.

Sol.

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin\frac{1}{x} = \sin(-\infty)$ = any value b/w $[-1, 1]$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin(\frac{1}{x}) = \sin(\infty)$ = any value b/w $[-1, 1]$

Limit does not exist.

function is discontinuous.

6. $f(x) = \sin x \quad \forall\ x \in \mathbb{R}$.

Let θ be an arbitrary real no.

$\lim_{x\to\theta^-} f(x) = \lim_{x\to\theta^-} \sin x = \sin\theta.$

$\lim_{x\to\theta^+} f(x) = \lim_{x\to\theta^+} \sin x = \sin\theta$

$f(\theta) = \sin\theta$

$\lim_{x\to\theta^-} f(x) = \lim_{x\to\theta^+} f(x) = \lim_{x\to\theta} f(x) = f(\theta)$

$\Rightarrow$ $f(x)$ is continuous at $x=\theta$

$\because \theta \in \mathbb{R}$, $\Rightarrow$ $f(x)$ is continuous $\forall$ $\mathbb{R}$.

7. $f(x) = \begin{cases} \frac{x^2}{a} - a & \text{if } 0<x<a \\ 0 & \text{if } x=a \\ a-\frac{a^2}{x} & \text{if } x>a. \end{cases}$ **at $x=a$**

Sol:

$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} \left(\frac{x^2}{a} - a\right)$

$= \frac{a^2}{a} - a$

$= a-a$

$\lim_{x\to a^-} f(x) = 0$

$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} a - \frac{a^2}{x}$

$= a - \frac{a^2}{a}$

$= a-a$

$\lim_{x\to a^+} f(x) = 0$

$\Rightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x) = 0$

$f(a) = 0$

$\lim_{x\to a} f(x) = f(a) = 0$

function is continuous at $x=a$

8. Determine all the points of continuity of the function $f(x) = x - [x]$ $\forall$ $x \in \mathbb{R}$.

Sol. $[\cdots]$ being Bracket function.

when $0\le x<1$ $[x]=0$
when $1\le x<2$ $[x]=1$
when $2\le x<3$ $[x]=3$
$\vdots$

when $-1\le x<0$ $[x]=-1$
when $-2\le x<-1$ $[x]=-2$
when $-3\le x<-2$ $[x]=-3$
$\vdots$

$\Rightarrow$ $f(x) = x$ $\quad 0<x<1$
$f(x) = x-1$ $\quad 1<x<2$
$f(x) = x-2$ $\quad 2<x<3$
$\vdots$

$f(x) = x+1$ $\quad -1<x<0$
$f(x) = x+2$ $\quad -2<x<-1$
$f(x) = x+3$ $\quad -3<x<-2$
$\vdots$

At $x=1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1) = 0$

Therefore, limit does not exist. Function is not continuous at $x=1$.

At $x=0$,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0$

Limit does not exist, function is not continuous at $x=0$

Similarly, it is continuous for integral values of x, both positive and negative, but it is continuous at every other real value of x.

9. Discuss continuity at $x=1$, $f(x) = x-|x|$.

Sol. $f(x) = x - |x|$

$\lim_{x\to 1^-} x-|x| = 1-1=0$

$\lim_{x\to 1^+} x-|x| = 1-1=0$

$f(1) = 1-1 = 0$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$$

$\Rightarrow$ function is continuous at $x=1$.

10. Show that function $f: R\to R$. defined by $f(x) = \begin{cases} x & \text{if } x \text{ is irrational.} \\ 1-x & \text{if } x \text{ is rational.} \end{cases}$ is cont. at $x=\frac{1}{2}$.

Sol. $f(\frac{1}{2}) = 1-\frac{1}{2} = \frac{1}{2}$

$\lim_{x\to 1/2} f(x) = \lim_{x\to 1/2} (1-x) = 1-\frac{1}{2} = \frac{1}{2}$

$\Rightarrow \lim_{x\to 1/2} f(x) = f(1/2)$

function is continuous.

11. Show that function $f:]0,1] \to R$. defined by $f(x) = \frac{1}{x}$ is continuous on $]0,1]$. Is $f(x)$ is bounded on this interval? Explain.

Sol:

$f(x)$ is defined for all real values of x. Such that $0 < x \le 1$ and its limit exists at each such x and equals to its value there, so it is

continuous on $]0,1]$.
When $x = 1$, $f(x) = 1$ which is its lower bound. So it is bounded below. $f(x)$ increases indefinitely as x becomes small. Thus $f(x)$ is not bounded above. hence $f(x)$ is not bounded on $]0,1]$

12. Let $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, **Is f continuous at $x=0$.**

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos\left(\frac{1}{x}\right) = \text{any value} \in [-1,1]$
Thus Limit does not exist. Similarly,
$\lim_{x\to 0^+} f(x)$ does not exist.
Hence $\lim_{x\to 0} f(x)$ does not exist
So given function is not continuous. at $x=0$.

13. Let $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$

Discus continuity of 'f' at $x=a$.

Sol. $\lim_{x\to a} f(x) = \lim_{x\to a} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= \lim_{x\to a}(x-a) \lim_{x\to a} \sin\left(\frac{1}{x-a}\right)$

$= (a-a).\sin\left(\frac{1}{a-a}\right)$

$= 0.\sin\left(\frac{1}{0}\right)$

$= 0.(\text{any value from } [-1,1]$

$\lim_{x\to a} f(x) = 0$

$f(a) = 0$ By defined function

$\lim_{x\to a} f(x) = f(a) = 0$
So given f is continuous at $x=a$.

14. Let $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show f is continuous at $x=0$

Sol. $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$ | $f(0) = 0$

$= \lim_{x\to 0} x \ \lim_{x\to 0} \cos\left(\frac{1}{x}\right)$

$= 0.\ \cos\left(\frac{1}{0}\right)$

$= 0.\ $[any value b/w $[-1,1]$)

$\lim_{x\to 0} f(x) = 0$

So $\lim_{x\to 0} f(x) = f(0)$

So given function is continuous at $x=0$.

15. Let $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$ **Discus cont. of f at $x=0$.**

Sol.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\sin(1/x)$ $\boxed{f(0) = 0}$

$= \lim_{x\to 0} x^2 .\ \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

$= 0.\ \sin\left(\frac{1}{0}\right)$

$= 0$ [any value b/w $[-1,1]$]

$\lim_{x\to 0} f(x) = 0$

So $\lim_{x\to 0} f(x) = f(0) = 0$

So $f(x)$ is continuous at $x=0$

16. Let $f(x) = \begin{cases} x\sin\frac{|x|}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$. Discus continuoty.

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\sin\frac{|x|}{x} = \lim_{x\to 0^-} x \cdot \lim_{x\to 0^-} \sin\frac{|x|}{x}$

$= 0 \lim_{x\to 0} \sin\frac{(-\cancel{x})}{\cancel{x}} = 0\sin(-1) = 0$

$\lim_{x\to 0^-} f(x) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\sin\frac{|x|}{x} = \lim_{x\to 0^+} x \cdot \lim_{x\to 0^+} \sin\frac{|x|}{x}$

$= 0 \cdot \sin\frac{\cancel{x}}{\cancel{x}} = 0\sin(1) = 0$

$\lim_{x\to 0^+} f(x) = 0 = \lim_{x\to 0^-} f(x)$

$\lim_{x\to 0} f(x) = 0$

$f(0) = 0$

So $\lim_{x\to 0} f(x) = f(0)$ function is continuous at $x=0$

17. Find c, such that the function $f(x)=$

$$f(x)=\begin{cases}\dfrac{1-\sqrt{x}}{x-1} & \text{if } 0\le x<1\\ c & \text{if } x=1\end{cases}$$, is continuous at ~~x=1~~ $\forall\ x\in[0,1]$

Sol.

Let $a\in[0,1]$ $\because$ arbitrary 'a'

$$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{1-\sqrt{x}}{x-1}=\frac{1-\sqrt{a}}{a-1}=f(a)$$

Thus f is continuous at a and $\because a$ is arbitrary point of $[0,1]$. Since f is continuous on $[0,1]$. So f is continuous on $x=1$

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{(1-\sqrt{x})}{x-1}=\lim_{x\to 1}\frac{-(\sqrt{x}-1)}{(\sqrt{x})^2-(1)^2}$$

$$=\lim_{x\to 1}\frac{-\cancel{(\sqrt{x}-1)}}{(\sqrt{x}+1)\cancel{(\sqrt{x}-1)}}=\lim_{x\to 1}\frac{-1}{\sqrt{x}+1}=\frac{-1}{1+1}$$

$$\lim_{x\to 1} f(x)=-\frac{1}{2}$$

$$f(1)=c$$

Since function is continuous So

$$\lim_{x\to 1} f(x)=f(1)\Rightarrow c=-1/2$$

In Problem 18-20. find the points of discontinuoty of the given functions.

18. $$f(x)=\begin{cases}x+4 & \text{if } -6\le x<2\\ x & \text{if } -2\le x<2\\ x-4 & \text{if } 2<x<6\end{cases}$$

$$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^-}(x+4)=-2+4$$

$$\lim_{x\to -2^-} f(x)=2$$

$$\lim_{x\to -2^+} f(x)=\lim_{x\to -2^+} x=-2$$

$$\lim_{x\to -2^-} f(x)\ne\lim_{x\to -2^+} f(x)$$

Limit does not exists

So function is discontinuous.

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(x)=2$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(x-4)$$

$$=2-4$$

$$\lim_{x\to 2^+} f(x)=2$$

So $$\lim_{x\to 2^-} f(x)\ne\lim_{x\to 2^+} f(x)$$

function is discontinuous.

19. $g(x) = \begin{cases} x^3 & \text{if } x<1 \\ -4-x^2 & \text{if } 1\le x\le 10 \\ 6x^2+46 & \text{if } x>10 \end{cases}$

$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-} x^3 = 1$

$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (-4-x^2) = -4-1$

$\lim_{x\to 1^+} g(x) = -5$

$\lim_{x\to 1^-} g(x) \neq \lim_{x\to 1^+} g(x)$

function is discontinuous at $x=1$

$\lim_{x\to 10^-} g(x) = \lim_{x\to 10^-} (-4-x^2)$
$= -4-100 = -104$

$\lim_{x\to 10^+} g(x) = \lim_{x\to 10^+} (6x^2+46)$
$= 600+46$
$= 646$

$\lim_{x\to 10^-} g(x) \neq \lim_{x\to 10^+} g(x)$

function is discontinuous.

20. $f(x) = \begin{cases} x+2 & \text{if } 0\le x<1 \\ x & \text{if } 1\le x<2 \\ x+5 & \text{if } 2\le x<3 \end{cases}$

Sol. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+2) = 1+2$
$= 3$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x) = 1$

$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

function is not continuous.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x) = 2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+5) = 2+5$
$= 7$

$\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$

Function is not continuous.

21. Find constants 'a' and 'b' $\ni$ function f defined by $f(x) = \begin{cases} x^3 & \text{if } 0\le -1 \\ ax+b & \text{if } -1\le x<1 \\ x^2+2 & \text{if } x\ge 1 \end{cases}$ **is continuous on all x.**

Sol. function is continuous on $x=1$

So Limit exists.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax+b)$
$= a+b$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+2)$
$= 1+2$
$= 3$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$a+b = 3 \rightarrow (1)$

function is continuous on $x=-1$

So Limit exists.

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} x^3$
$= -1$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (ax+b)$
$= -a+b$

$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$

$-a+b = -1 \rightarrow (2)$

adding ① & ②

$$\begin{array}{r} \cancel{a} + b = 3 \\ -\cancel{a} + b = -1 \\ \hline 2b = 2 \end{array}$$

$\Rightarrow \boxed{b = 1}$

Put value of 'b' in ①

$a + b = 3$

$a + 1 = 3$

$a = 3 - 1$

$\Rightarrow \boxed{a = 2}$

Find the interval on which the given fnctn is continuous. Also find points where it is discontinuos. (Problem 22-26)

22. $f(x) = \dfrac{x^2 - 5}{x - 1}$

Sol. function $f(x) = \dfrac{x^2-5}{x-1}$ is not defined at $x = 1$. Thus function is discontinuous at $x = 1$.

Numerator is continuous at every point of $\mathbb{R}$ and so is the denumerator is $x - 1$.

Hence $f(x)$ is continuous at every point of $\mathbb{R} - \{1\}$.

23. $f(x) = \dfrac{x}{|x|}$

Sol. $f(x)$ is not defined at $x = 0$ so $f(x)$ is discontinuous at $x = 0$ and continuous at $\mathbb{R} - \{0\}$.

24. $f(x) = \dfrac{\sin x}{x}$

Sol. $f(x)$ is not defined at $x = 0$. Hence it is discontinuous at $x = 0$.

$f(x)$ is continuous on $\mathbb{R} - \{0\}$.

25. $f(x) = \dfrac{\sin x}{\cos x}$

function is not defined on $x = (2n+1)\pi/2$ where 'n' is any integer.

Thus f is discontinuous at these points and continuous on all other points of $\mathbb{R}$.

26. $f(x) = \begin{cases} \sin x & \text{if } x \le \pi/4 \\ \cos x & \text{if } x \geq \pi/4. \end{cases}$

Sol. $\lim_{x \to \pi/4^-} f(x) = \lim_{x \to \pi/4^-} \sin x = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

$\lim_{x \to \pi/4^+} f(x) = \lim_{x \to \pi/4^+} \cos x = \cos(\pi/4) = \frac{1}{\sqrt{2}}$

$f(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

Function is continuous at $x = \pi/4$. We also know that $\sin x$ and $\cos x$ are continuous at every point of $\mathbb{R}$. Hence $f(x)$ is continuous at every point of $\mathbb{R}$.

In Problem 27-34, examine whether the given function is continuous at $x=0$.

27. $f(x) = \begin{cases} (1+3x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0. \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+3x)^{1/x} = \lim_{x \to 0} \left[(1+3x)^{1/3x}\right]^3$

$= \left[\lim_{x \to 0} (1+3x)^{1/3x}\right]^3$

$\lim_{x \to 0} f(x) = e^3$

$f(0) = e^2$

$\lim_{x \to 0} f(x) \neq f(0)$ function is not continuous.

28. $f(x) = \begin{cases} (1+x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+x)^{1/x} = e$

$f(0) = 1$

$\lim_{x \to 0} f(x) \neq f(0)$

function is not continuous.

29. $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0. \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+2x)^{1/x} = \lim_{x \to 0} \left[(1+2x)^{1/2x}\right]^2$

$= \left[\lim_{x \to 0} (1+2x)^{1/2x}\right]^2$

$= e^2$

$f(0) = e^{2}$

So $\lim_{x\to 0} f(x) = f(0)$

So given $f(x)$ is continuous at $x=0$.

30 $f(x) = \begin{cases} e^{-1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-1/x} = e^{-1/0} = e^{-\infty} = 0$

$f(0) = 1$

$\lim_{x\to 0} f(x) \neq f(0)$, So given function is not continuous.

31. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

Sol. when $x \to 0^-$, $\frac{1}{x} \to -\infty$ and so $e^{1/x} \to e^{-\infty} = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{e^{1/x}}{1+e^{1/x}} = \dfrac{0}{1+0} = 0$

when $x \to 0^+$, $\frac{1}{x} \to \infty$ and $e^{1/x} \to e^{\infty} = \infty$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \dfrac{e^{1/x}}{1+e^{1/x}} = \lim_{x\to 0^+} \dfrac{\cancel{e^{1/x}}}{\cancel{e^{1/x}}\left(1+\frac{1}{e^{1/x}}\right)}$

$= \lim_{x\to 0^+} \dfrac{1}{1+\frac{1}{e^{1/x}}} = \dfrac{1}{1+\frac{1}{\infty}}$

$= \dfrac{1}{1+0}$

$\lim_{x\to 0^+} f(x) = 1$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ function is not continuous at $x=0$.

32. $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

Sol. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} = \lim_{x\to 0} \dfrac{\cancel{e^{1/x^2}}}{\cancel{e^{1/x^2}}\left(1-\frac{1}{e^{1/x^2}}\right)}$

$= \lim_{x\to 0} \dfrac{1}{1-1/e^{1/x^2}}$

$= \dfrac{1}{1-1/e^{\infty}} = \dfrac{1}{1-0} = 1$

$\lim_{x\to 0} f(x) = 1$ and $f(0) = 1$

$\lim_{x\to 0} f(x) = f(0)$

function is continuous at $x=0$.

33. $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ $\quad \because \lim_{x \to 0} \dfrac{\sin x}{x} = 1$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin 2x}{x} = 2 \lim_{x \to 0} \dfrac{\sin 2x}{2x} = 2.1 = 2$

$f(0) = 1$

$\lim_{x \to 0} f(x) \neq f(1)$

function is discontinuous.

34. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ 2/3 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin 3x}{\sin 2x} = \dfrac{3x}{2x} \lim_{x \to 0} \dfrac{\dfrac{\sin 3x}{3x}}{\dfrac{\sin 2x}{2x}}$

$= \dfrac{3}{2} \dfrac{\lim_{x \to 0} \dfrac{\sin 3x}{x}}{\lim_{x \to 0} \dfrac{\sin 2x}{2}} = \dfrac{3.1}{2.1}$

$\lim_{x \to 0} f(x) = \dfrac{3}{2}$

$f(0) = 2/3$

$\lim_{x \to 0} f(x) \neq f(0)$ Function is not continuous at $x = 0$

35. **Let $f(x) = x^2$ and $g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ |x-4| & \text{if } x > 0 \end{cases}$**

Determine whether fog and gof are continuous at $x = 0$.

Sol $fog(x) = f\{g(x)\} = f(-4) \quad \text{if } x \leq 0$

$f\{g(x)\} = f(|x-4|) \quad \text{if } x > 0$

Thus $(fog)(x) = 16 \quad \text{if } x \leq 0$

$(fog)(x) = (x-4)^2 \quad \text{if } x > 0$

Now $\lim_{x \to 0^-} (fog)(x) = 16$

$\lim_{x \to 0^+} (fog)(x) = \lim_{x \to 0^+} (x-4)^2 = (0-4)^2 = 16$

$(fog)(0) = 16$

Thus fog is continuous at $x = 0$.

Again, $(gof)(x) = g\{f(x)\} = g(x^2)$

$$(gof)(x) = -4 \quad \text{if } x^2 \leq 0$$
$$= |x^2 - 4| \quad \text{if } x^2 > 0$$

$$\lim_{x \to 0^-} (gof)(x) = -4$$

$$\lim_{x \to 0^+} (gof)(x) = \lim_{x \to 0^+} |x^2 - 4|$$
$$= |0 - 4|$$

$$\lim_{x \to 0^+} (gof)(x) = 4$$

$$\lim_{x \to 0^+} (gof)(x) \neq \lim_{x \to 0^-} (gof)(x)$$

Limit does not exist.

So $(gof)(x)$ is not continuous at $x = 0$.