EXERCISE 1.2

Evaluate the indicated limits (Problems 1 – 30):

1. $\lim\limits_{x \to 2} \dfrac{x-2}{\sqrt{2+x}}$

2. $\lim\limits_{x \to 1} \dfrac{x^3-1}{x-1}$

3. $\lim\limits_{x \to 1} \left(\dfrac{1}{1-x} - \dfrac{3}{1-x^3} \right)$

4. If $P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, prove that

$$\lim_{x \to a} P_n(x) = P_n(a)$$

5. $\lim\limits_{x \to 0} \dfrac{\csc x - \cot x}{x}$

6. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$

7. $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$

8. $\lim\limits_{y \to x} \dfrac{y^{2/3} - x^{2/3}}{y-x}$

9. $\lim\limits_{x \to \pi} \dfrac{\tan(\sin x)}{\sin x}$

10. $\lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x}\right)$

11. $\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

12. $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+1}}{x+1}$

13. $\lim\limits_{x \to \infty} \dfrac{4x^3 - 2x^2 + 1}{3x^3 - 5}$

14. $\lim\limits_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$

15. $\lim\limits_{x \to \infty} \left(1 - \dfrac{1}{x}\right)^x$

16. $\lim\limits_{x \to \infty} \left(\dfrac{x}{1+x}\right)^x$

17. $\lim\limits_{x \to \infty} \dfrac{a^x - 1}{x}$, $(a > 1)$

18. $\lim\limits_{x \to \infty} \dfrac{x^4 - 2x^2 + 6}{x^2 + 7}$

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19. $\lim\limits_{x \to \pm\infty} \left[\dfrac{x^2}{x+1} - \dfrac{x^2}{x+3} \right]$

20. $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2 - a^2} \right)$

21. $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{x^{3/2}}$

22. $\lim\limits_{x \to \pm\infty} \dfrac{5x^3 + 3x^2 - 1}{x - 4x^4}$

23. $\lim\limits_{x \to \infty} \dfrac{3 - 2x^4}{1 + x}$

24. $\lim\limits_{x \to \pm\infty} (x^4 - x^3 + x)$

25. $\lim\limits_{x \to \pm\infty} \left[\dfrac{x^2}{x+3} - \dfrac{x^2}{x+5} \right]$

26. $\lim\limits_{x \to \pm\infty} \left[4x^3 - 3x^2 + x - 1 \right]$

27. $\lim\limits_{x \to 1-0} \dfrac{\sqrt{1 - x^2}}{1 - x}$

28. $\lim\limits_{x \to 1+0} \dfrac{x - 1}{\sqrt{x^2 - 1}}$

29. $\lim\limits_{x \to 2-0} \dfrac{\sqrt{4 - x^2}}{\sqrt{6 - 5x + x^2}}$

30. $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x}$

(24) $\lim\limits_{x \to -1} \dfrac{x^{1/3} + 1}{x + 1}$

(25) $\lim\limits_{x \to 3^-} \left(\dfrac{1}{x-3} - \dfrac{1}{|x-} \right.$

(26) $\lim\limits_{x \to 2} \dfrac{x^2 + 2x - 8}{x^2 - 4}$

31. Let $f(x) = \begin{cases} x^2 + 3 & \text{if } x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

Find $f(1+0)$ and $f(1-0)$.

32. $f(x) = \begin{cases} 3 & \text{if } x \leq -2 \\ -\dfrac{1}{2} x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

Find $f(-2-0)$, $f(-2+0)$, $f(2-0)$ and $f(2+0)$

33. Let $f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 2 \\ \sqrt{x + 7} & \text{if } x > 2 \end{cases}$

Find $\lim f(x)$ as $x \to 2$.

34. Let $f(x) = \begin{cases} \cos x & \text{if } x \leq 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

Find $\lim\limits_{x \to 0} f(x)$.

35. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ x^3 & \text{if } x > 1 \end{cases}$

Show that $\lim\limits_{x \to 1} f(x) = 1$

36. Let $f(x) = \begin{cases} x + 2 & \text{if } x \leq -1 \\ ax^2 & \text{if } x > -1 \end{cases}$

Find a so that $\lim\limits_{x \to -1} f(x)$ exists.

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37. Evaluate $\lim\limits_{x \longrightarrow 3+0} \dfrac{3-x}{|x-3|}$

38. Evaluate $\lim\limits_{x \longrightarrow 0-0} \dfrac{x}{x-|x|}$.

39. Find $\lim\limits_{h \longrightarrow 0-0} \dfrac{|-1+h|-1}{h}$,

40. Evaluate, $[\cdots]$ being the bracket function:

(i) $\lim\limits_{x \longrightarrow 1} [2x](x-1)$ (ii) $\lim\limits_{x \longrightarrow 0} [x][x+1]$.

(iii) $\lim\limits_{x \longrightarrow 0} x\left[\dfrac{1}{x}\right]$ (iv) $\lim\limits_{x \longrightarrow 0} x^3\left[\dfrac{1}{x}\right]$.

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EXERCISE No 1.2

Q.No (1)

$$\lim_{x \to 2} \frac{x-2}{\sqrt{2+x}} = \frac{2-2}{\sqrt{2+2}} = \frac{0}{2} = 0$$

Q.No (2)

$$\lim_{x \to 1} \frac{x^3-1}{x-1} \cdots (0/0) \text{ form}$$

$$\lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$$

$$= \lim_{x \to 1} (x^2+x+1)$$

$$= 1^2+1+1$$

$$= 3$$

(3) $\lim_{x \to 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right)$

So taking LCM, we get

$$\lim_{x \to 1} \left(\frac{x^2+x+1-3}{(1-x)(x^2+x+1)} \right)$$

$$\lim_{x \to 1} \left(\frac{x^2+x-2}{(1-x)(x^2+x+1)} \right)$$

$$= \lim_{x \to 1} \frac{x^2+2x-x-2}{(1-x)(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{(x+2)\cancel{(x-1)}}{-\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{x+2}{-(x^2+x+1)}$$

$$= \frac{1+2}{-(1+1+1)} = \frac{3}{-3} = -1 \text{ Ans}$$

(4) $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \to (i)$

taking the limit of both sides

$$\lim_{x \to a} P_n(x) = \lim_{x \to a} (a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n)$$

$$= a_0a^n + a_1a^{n-1} + \cdots + a_{n-1}a + a_n$$

$$= P_n(a) \quad \text{by (i)}$$

(5) $\lim_{x \to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x}$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{1-\cos x}{\sin x} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{1-\cos x}{\sin x} \times \frac{1+\cos x}{1+\cos x} \right]$$

$$= \lim_{x \to 0} \frac{1}{x} \left[\frac{1-\cos^2 x}{\sin x \,(1+\cos x)} \right]$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x \cancel{\sin x} (1+\cos x)}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x} \right) \times \frac{1}{\lim_{x \to 0} 1+\cos x}$$

$$= (1) \times \frac{1}{1+1} = \frac{1}{2} \text{ Ans}$$

(6) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

$$= \lim_{x \to 0} \frac{\sin ax}{\sin bx} \times \frac{ax}{ax} \times \frac{bx}{bx}$$

$$= \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{bx}{\sin bx} \times \frac{a\cancel{x}}{b\cancel{x}}$$

$$= \left(\lim_{x \to 0} \frac{\sin ax}{ax} \right) \left(\lim_{x \to 0} \frac{\sin bx}{bx} \right)^{-1} \times \frac{a}{b}$$

$$= (1)\,(1)^{-1} \left(\frac{a}{b} \right) = \frac{a}{b}$$

(7) $\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{1-\cos x}{x^2} \times \frac{1+\cos x}{1+\cos x}$ ——— A.

$$= \lim_{x \to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1+\cos x}$$

$$= 1 \times \frac{1}{1+1} = \frac{1}{2} \text{ Ans}$$

(8) $\lim_{y \to x} \dfrac{y^{2/3} - x^{2/3}}{y - x}$

$= \lim_{y \to x} \dfrac{(y^{1/3})^2 - (x^{1/3})^2}{(y^{1/3})^3 - (x^{1/3})^3}$

$= \lim_{y \to x} \dfrac{\cancel{(y^{1/3} - x^{1/3})}(y^{1/3} + x^{1/3})}{\cancel{(y^{1/3} - x^{1/3})}((y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2)}$

$= \lim_{y \to x} \dfrac{y^{1/3} + x^{1/3}}{y^{2/3} + y^{1/3}x^{1/3} + x^{2/3}}$

$= \dfrac{x^{1/3} + x^{1/3}}{x^{2/3} + x^{1/3} \cdot x^{1/3} + x^{2/3}}$

$= \dfrac{2x^{1/3}}{3x^{2/3}}$

$= \dfrac{2}{3(x)^{2/3 - 1/3}} = \dfrac{2}{3(x)^{1/3}}$ Ans

2nd Method Put $y = x + h$ using B. Th

(9) $\lim_{x \to \pi} \dfrac{\tan(\sin x)}{\sin x}$

Let $\sin x = \theta$

when $x \to \pi$, $\theta \to 0$

$= \lim_{\theta \to 0} \dfrac{\tan\theta}{\theta}$

$= \lim_{\theta \to 0} \dfrac{\sin\theta}{\cos\theta} \times \dfrac{1}{\theta}$

$= \lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} \cdot \dfrac{1}{\cos\theta}$

$= (1) \times \dfrac{1}{\cos 0} = 1 \times \dfrac{1}{1}$

$= 1$ Ans

(10) $\lim_{x \to 0} x \sin\dfrac{1}{x}$

$= \left(\lim_{x \to 0} x\right) \lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

$= 0 \times$ Something

$= 0$

$\because$ Sin is a bounded function

value of $\sin x$ lies b/w 1 & -1

(11) $\lim_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

Now $\lim_{x \to 0+0} \sin\dfrac{1}{x}$ Put $x = 0 + h$

$= \lim_{h \to 0} \sin\left(\dfrac{1}{0+h}\right)$

$= \lim_{h \to 0} \sin\left(\dfrac{1}{h}\right) = l$ (Say) $\to$ i

$\because l \in [-1, 1]$

Also

$\lim_{x \to 0-0} \sin\left(\dfrac{1}{x}\right)$ Put $x = 0 - h$

$= \lim_{h \to 0} \sin\left(\dfrac{1}{0-h}\right)$

$= \lim_{h \to 0} \sin\left(-\dfrac{1}{h}\right)$

$= -\lim_{h \to 0} \sin\left(\dfrac{1}{h}\right) = -l$ (Say) $\to$ ii

$\because l \in [-1, 1]$

From i, & ii

$\lim_{x \to 0+0} \sin\dfrac{1}{x} \neq \lim_{x \to 0-0} \sin\dfrac{1}{x}$

Hence $\lim_{x \to 0} \sin\dfrac{1}{x}$ does not Exist

(12) $\lim_{x \to \infty} \dfrac{\sqrt{x^2+1}}{x+1} = \lim_{x \to \infty} \dfrac{x\sqrt{1 + \frac{1}{x^2}}}{x\left(1 + \frac{1}{x}\right)}$

$= \dfrac{\sqrt{1+0}}{(1+0)} = \dfrac{1}{1} = 1$

Ans

(13) $\lim_{x\to\infty} \dfrac{4x^3-2x^2+1}{3x^3-5}$

$= \lim_{x\to\infty} \dfrac{x^3\left(4-\frac{2}{x}+\frac{1}{x^3}\right)}{x^3\left(3-\frac{5}{x^3}\right)}$

$= \lim_{x\to\infty} \dfrac{4-\frac{2}{x}+\frac{1}{x^3}}{3-\frac{5}{x^3}}$

$= \dfrac{4-0+0}{3-0} = \dfrac{4}{3}$ Ans

Alt: Put $x = \frac{1}{y}$ $\lim_{x\to\infty}, y\to 0$

(14) $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

$\lim_{x\to\infty} \left[\left(1+\frac{2}{x}\right)^{\frac{x}{2}}\right]^2$

$= e^2$ $\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

(15) $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x \to (1)$

(1) Can be written as

$\lim_{x\to\infty} \left(1+\left(-\frac{1}{x}\right)\right)^x$

$= \left[\lim_{x\to\infty}\left(1+\left(-\frac{1}{x}\right)\right)^{-x}\right]^{-1}$

$= e^{-1} = 1/e$

(16) $\lim_{x\to\infty} \left[\dfrac{x}{1+x}\right]^x$

$\lim_{x\to\infty} \left(\dfrac{1+x}{x}\right)^{-x}$

$\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^{-1}$

$= e^{-1} = 1/e$

(17) $\lim_{x\to\infty} \dfrac{a^x-1}{x}$ $\because a>1$

Let $y = a^x - 1$

When $\lim x\to\infty$, $y\to\infty$

$a^x = y+1$

Taking ln of both sides, we get

$x \ln a = \ln(y+1)$

$x = \dfrac{\ln(1+y)}{\ln a}$

$\lim_{x\to\infty} \dfrac{a^x-1}{x} = \lim_{y\to\infty} \dfrac{y}{\frac{\ln(1+y)}{\ln a}}$

(*) $= \lim_{y\to\infty} \dfrac{\ln a}{\frac{1}{y}\ln(1+y)} = \dfrac{\ln a}{\ln \lim_{y\to\infty}(1+y)^{1/y}}$

$\because \lim_{y\to\infty}(1+y)^{\frac{1}{y}} = (\infty)^0 = 1$ (assume)

$\Rightarrow 5^0 = 1$

$= \dfrac{\ln a}{\ln 1} = \dfrac{\ln a}{0} = \infty$

2nd Solution (*)

$= \ln a \lim_{y\to\infty} \dfrac{y}{\ln(1+y)}$

$= \ln a \cdot \dfrac{\infty}{\infty}$ Apply L. Hospital Rule

$= \ln a \cdot \lim_{y\to\infty} \dfrac{1}{\frac{1}{1+y}}$

$= \ln a \lim_{y\to\infty}(1+y) = \ln a \times \infty = \infty$

OR 3rd way $\lim_{x\to\infty} \dfrac{a^x-1}{x} = \dfrac{a^x \ln a - 0}{1}$ L H Rule.

$= \lim_{x\to\infty}(a^x \ln a) = \infty \times \ln a = \infty$

4th way $\lim_{x\to\infty} \dfrac{a^x-1}{x}$ Put $a = 1+h$

$\lim_{x\to\infty} \dfrac{(1+h)^x-1}{x} = \lim_{x\to\infty} \dfrac{1+xh+\frac{x(x-1)}{2!}h^2+\cdots-1}{x} = \lim_{x\to\infty} \dfrac{x\left[h+\frac{x-1}{2!}h^2+\cdots\right]}{x}$

$= h + \infty = \infty$

(18) $\underset{x\to\infty}{Lt} \dfrac{x^4-2x^2+6}{x^2+7}$

Dividing N & D by x^4

$$\underset{x\to\infty}{Lt} \dfrac{1-\frac{2}{x^2}+\frac{6}{x^4}}{\frac{1}{x^2}+\frac{7}{x^4}}$$

$$= \frac{1-0+0}{0+0} = \frac{1}{0} = \infty$$

(19) $\underset{x\to\pm\infty}{Lt} \left[\dfrac{x^2}{x+1} - \dfrac{x^2}{x+3}\right]$

$$= \underset{x\to\pm\infty}{Lt} \left[\frac{x^2(x+3)-x^2(x+1)}{(x+1)(x+3)}\right]$$

$$= \underset{x\to\pm\infty}{Lt} \frac{\cancel{x^3}+3x^2-\cancel{x^3}-x^2}{x^2+4x+3}$$

$$= \underset{x\to\pm\infty}{Lt} \frac{2x^2}{x^2+4x+3}$$

$$= \underset{x\to\pm\infty}{Lt} \frac{2\cancel{x^2}}{\cancel{x^2}(1+\frac{4}{x}+\frac{3}{x^2})}$$

$$= \frac{2}{1+0+0} = 2 \text{ Ans}$$

(20) M-1

$$\underset{x\to\infty}{Lt} \left[x - \sqrt{x^2-a^2}\right]$$

$$= \underset{x\to\infty}{Lt} \left[x - x\left(1-\frac{a^2}{x^2}\right)^{\frac{1}{2}}\right]$$

$$= \underset{x\to\infty}{Lt} \left[x - x\left\{1+\frac{1}{2}\left(-\frac{a^2}{x^2}\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(-\frac{a^2}{x^2}\right)^2+\dots\right\}\right]$$

$$= \underset{x\to\infty}{Lt}\left[\cancel{x}-\cancel{x}-\frac{a^2}{2x}-\frac{1}{8}\cdot\frac{a^4}{x^3}+\dots\right]$$

$$= 0 - 0 + 0 \dots = 0$$

M-2

$$\underset{x\to\infty}{Lt}\left[x-\sqrt{x^2-a^2}\right]\times\frac{\left[x+\sqrt{x^2-a^2}\right]}{\left[x+\sqrt{x^2-a^2}\right]}$$

$$= \underset{x\to\infty}{Lt}\frac{\cancel{x^2}-\cancel{x^2}+a^2}{x+\sqrt{x^2-a^2}}$$

$$= \frac{a^2}{\infty} = 0 \text{ Ans}$$

(21) $\underset{x\to\infty}{Lt} \dfrac{x^2+1}{x^{3/2}}$

$\div$ by x^2 N & D

$$\underset{x\to\infty}{Lt} \frac{\left[1+\frac{1}{x^2}\right]}{x^{-1/2}}$$

$$\underset{x\to\infty}{Lt} x^{\frac{1}{2}}\left[1+\frac{1}{x^2}\right]$$

$$= \infty(1+0) = \infty \text{ Ans.}$$

$\frac{3}{2}-2$

$\frac{3-4}{2}=-\frac{1}{2}$

OR Put $x=\frac{1}{y}$ $\underset{x\to\infty}{Lt}$, $y\to 0$

(22) $\underset{x\to\pm\infty}{Lt} \dfrac{5x^3+2x^2-1}{x-4x^4}$

$\div$ by x^4 N & D

$$\frac{\underset{x\to\pm\infty}{Lt}\left[\frac{5}{x}+\frac{2}{x^2}-\frac{1}{x^4}\right]}{\underset{x\to\pm\infty}{Lt}\left[\frac{1}{x^3}-4\right]}$$

$$= \frac{0+0-0}{0-4} = \frac{0}{-4} = 0 \text{ Ans}$$

2nd way Note Since the limit of quotient of Polynomial as $x\to\pm\infty$ is the same as the limit of the quotient of the highest power terms.

$$\underset{x\to\pm\infty}{Lt} \frac{5x^3}{-4x^4}$$

$$= \underset{x\to\pm\infty}{Lt} \frac{5}{-4x} = \frac{5}{\infty} = 0$$

$\boxed{\text{Note}}$

$$\underset{x\to\pm\infty}{Lt}(x^4-x^3-x)$$

$$= (\pm\infty)^4-(\pm\infty)^3-(\pm\infty)$$

$$= \infty$$

OR $\underset{x\to\pm\infty}{Lt} x^4 = \infty$

Best Take higher Power Term

Note if

$$\underset{x\to\pm\infty}{Lt}(4x^3-3x^2-1)$$
$$=\underset{x\to\pm\infty}{Lt}4x^3$$
$$=\pm\infty$$

(23) $$\underset{x\to\infty}{Lt}\frac{3-2x^4}{1+x}$$
$$=\lim_{x\to\infty}\frac{-2x^4}{x}$$
$$=\lim_{x\to\infty}-2x^3$$
$$=-\infty$$

(24) $$\underset{x\to-1}{Lt}\frac{x^{1/3}+1}{x+1}\cdots(0/0)\text{ form}$$

Put $x=-1+h$

$x\to-1,\ h\to0$

$$\underset{h\to0}{Lt}\frac{(-1+h)^{1/3}+1}{\cancel{-1}+h+\cancel{1}}$$
$$\underset{h\to0}{Lt}\frac{\cancel{-1}+\frac{1}{3}h-\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}h^2+\cdots\cancel{+1}}{h}$$
$$\underset{h\to0}{Lt}\frac{\cancel{h}\left[\frac{1}{3}-\frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}h+\cdots\right.}{\cancel{h}}$$
$$=\frac{1}{3}-0+0\cdots=\frac{1}{3}\text{ Ans}$$

(25) $$\underset{x\to3^-}{Lt}\left(\frac{1}{x-3}-\frac{1}{|x-3|}\right)$$

Put $x=3-h$

L.H.L. $|x-3|$

$=-(x-3)\qquad x-3<0$

$$\underset{x\to3^-}{Lt}\left(\frac{1}{x-3}-\frac{1}{-(x-3)}\right)$$
$$\underset{x\to3^-}{Lt}\left(\frac{2}{x-3}\right)$$
$$\underset{h\to0}{Lt}\left(\frac{2}{\cancel{3}-h-\cancel{3}}\right)=\underset{h\to0}{Lt}\left(\frac{2}{-h}\right)$$
$$=-\infty\text{ Ans}$$

(26) $$\underset{x\to-2^-}{Lt}\frac{x^2+2x-8}{x^2-4}$$
$$=\underset{x\to-2^-}{Lt}\frac{x^2+4x-2x-8}{(x-2)(x+2)}$$
$$=\underset{x\to-2^-}{Lt}\frac{(x+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}$$
$$=\underset{x\to-2^-}{Lt}\frac{x+4}{x+2}$$

Put $x=-2-h$

$h\to0,\ x\to-2$

$$=\underset{h\to0}{Lt}\frac{-2-h+4}{\cancel{-2}-h+\cancel{2}}$$
$$=\lim_{h\to0}\frac{2-h}{-h}=\underset{h\to0}{Lt}\frac{2}{-h}+1=-\infty$$

if $$=\underset{h\to0}{Lt}\frac{h-2}{h}=\infty\text{ Ans (Correct)}$$

(27) $$\underset{x\to1^-}{Lt}\frac{\sqrt{1-x^2}}{1-x}$$
$$\underset{x\to1^-}{Lt}\frac{\sqrt{1-x}\sqrt{1+x}}{1-x}$$
$$\underset{x\to1^-}{Lt}\frac{\sqrt{1+x}}{\sqrt{1-x}}$$

Put $x=1-h$ L.H. Limit

$$\underset{h\to0}{Lt}\frac{\sqrt{1+1-h}}{\sqrt{\cancel{1}-\cancel{1}+h}}=\underset{h\to0}{Lt}\frac{\sqrt{2-h}}{\sqrt{h}}$$
$$=\frac{\sqrt{2-0}}{\sqrt{0}}=\infty\text{ Ans}$$

2nd M Alt Put $x=1-h$

$$\underset{h\to0}{Lt}\frac{\sqrt{1-(1-h)^2}}{1-(1-h)}$$
$$=\infty\text{ Ans}$$

(28) $\lim_{x \to 1^+} \frac{x-1}{\sqrt{x^2-1}}$

$= \lim_{x \to 1^+} \frac{x-1}{\sqrt{(x-1)(x+1)}}$

$= \lim_{x \to 1^+} \frac{\sqrt{x-1}}{\sqrt{x+1}}$

Put $x = 1+h$ R.H.L

$= \lim_{h \to 0} \frac{\sqrt{\cancel{1}+h-\cancel{1}}}{\sqrt{1+h+1}}$

$= \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{2+h}} = \frac{0}{\sqrt{2+0}} = 0$ Ans

(29) $\lim_{x \to 2^-} \frac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$

$\lim_{x \to 2^-} \frac{\sqrt{(2-x)(2+x)}}{\sqrt{(2-x)(3-x)}}$

$\because 6-5x+x^2 = x^2-5x+6$
$= x^2-3x-2x+6$
$= x(x-3)-2(x-3)$
$= (x-3)(x-2)$
$= (2-x)(3-x)$

$= \lim_{x \to 2^-} \frac{\sqrt{2+x}}{\sqrt{3-x}}$

Put $x = 2-h$

$= \lim_{h \to 0} \frac{\sqrt{2+(2-h)}}{\sqrt{3-2+h}}$

$= \lim_{h \to 0} \frac{\sqrt{4-h}}{\sqrt{1+h}}$

$= \frac{\sqrt{4}}{\sqrt{1}} = 2$ Ans

OR Put $x = 2-h$

Hint $= \lim_{h \to 0} \frac{\sqrt{4-(2-h)^2}}{\sqrt{6-5(2-h)+(2-h)^2}} = 2$

(30) $\lim_{x \to \infty} \frac{x+\sin x}{x}$

$= \lim_{x \to \infty} \left[\frac{x}{x} + \frac{\sin x}{x} \right]$

$= 1 + \lim_{x \to \infty} \frac{\sin x}{x}$

$= 1 + \frac{l}{\infty} \quad l \in [-1, 1]$

$= 1 + 0 = 1$ Ans

(31) $f(x) = \begin{cases} x^2+3 & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$

find $\lim_{x \to 1^+} f(x)$ & $\lim_{x \to 1^-} f(x)$

$f(1+0) = \lim_{x \to 1+0} f(x) = \lim_{x \to 1} (x+1)$

$= 1+1 = 2$

$f(1-0) = \lim_{x \to 1-0} f(x) = \lim_{x \to 1} (x^2+3)$

$= 1^2+3$

$= 4$

(32) $f(x) = \begin{cases} 3 & \text{if } x \le -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$

find $\lim_{x \to \pm 2^+} f(x)$ and $\lim_{x \to \pm 2^-} f(x)$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (3) = 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (-\frac{1}{2}x^2) = -\frac{1}{2}(2)^2 = -2$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2} 3 = 3$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2} (-\frac{1}{2}x^2)$

$= -\frac{1}{2}(-2)^2 = -2$

33 $f(x) = \begin{cases} x^2-1 & \text{if } x \le 2 \\ \sqrt{x+7} & \text{if } x > 2 \end{cases}$

find $f(x)$ as $x \to 2$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \sqrt{x+7}$

$= \sqrt{2+7}$

$= \sqrt{9} = 3$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2} (x^2-1)$

$= 4-1$

$= 3$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 3$

Then $\lim_{x \to 2} f(x) = 3$ Limit Exist

(34) $f(x) = \begin{cases} \cos x & \text{if } x \le 0 \\ 1-x & \text{if } x > 0 \end{cases}$

find $\lim_{x \to 0} f(x)$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (1-x)$

$= 1-0$

$= 1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \cos x$

$= \cos(0)$

$= 1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = 1$

$\lim_{x \to 0} f(x) = 1$

Limit of fn. Exists.

(35)

$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x^3 & \text{if } x > 1 \end{cases}$

Show that

$\lim_{x \to 1} f(x) = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} x^3 = (1)^3 = 1$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} x^2 = (1)^2 = 1$

$LHL = R.H.L = 1$

$\lim_{x \to 1} f(x) = 1$

(36) v.az

$f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ ax^2 & \text{if } x > -1 \end{cases}$

find 'a', So that

$\lim_{x \to -1} f(x)$ Exists,

Because given that

$\lim_{x \to -1} f(x)$ exist

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^-} f(x)$

$\lim_{x \to -1} (ax^2) = \lim_{x \to -1} (x+2)$

$a(-1)^2 = (-1+2)$

$a = 1$

(37) $\lim_{x \to 3^-} \dfrac{3-x}{|x-3|}$

LHL. $x \to 3^-$, $|x-3| = -(x-3)$

$\lim_{x \to 3^-} \dfrac{3-x}{-(x-3)}$

$\lim_{x \to 3^-} \dfrac{-(x-3)}{-(x-3)} = \lim_{x \to 3^-} (1)$

$= 1$ Ans

(38) Evaluate

$$\lim_{x \to 0^-} \frac{x}{x - |x|}$$

for L.H. Limit

$x < 0 \quad |x| = -x$

$$\lim_{x \to 0^-} \frac{x}{x - (-x)}$$

$$\lim_{x \to 0^-} \frac{x}{2x}$$

$$\lim_{x \to 0^-} \left(\frac{1}{2}\right) = \frac{1}{2} \text{ Ans}$$

(39)

$$\lim_{h \to 0} \frac{|-1+h| - 1}{h}$$

$\because \left|-(1-h)\right| \xrightarrow{\text{Mod.}} = (1-h)$

$$\lim_{h \to 0} \frac{(1-h) - 1}{h}$$

$= -1$ Ans

Note

$$\lim_{h \to 0^-} \frac{|-1+h| - 1}{h}$$

$|-1+h| = -(-1+h)$
$= 1 - h$

$$= \lim_{h \to 0^-} \frac{1-h-1}{h}$$

$= -1$

if $\lim_{h \to 0^+} \frac{|-1+h|-1}{h}$

$|-1+h| = (-1+h)$

$$\lim_{h \to 0^+} \frac{-1+h-1}{h}$$

$$\lim_{h \to 0^+} \frac{-2+h}{h}$$

$$= \frac{-2+0}{0} = -\infty$$

(40) Evaluate [....] being the bracket fn.

i) $\lim_{x \to 1} [2x](x-1)$

$\because x \to 1-0 \; [2x] = 1$

as value of x is slightly less than 1 is and $[2x]$ is Greatest integer less than 1 is 1

$y=0 = 0 \le x < 1$

$\because [2(+0.9)]$
$= [+1.8] = 1$

$$= \lim_{x \to 1-0} [2x](x-1)$$

$= 1 (1-1) = 0$

&

$$\lim_{x \to 1+0} [2x](x-1)$$

$\because \lim_{x \to 1+0} [2x]$

x is slightly greater than 1

$x \to 1+0 \; [2x] = 2$

$\because [2(1.1)]$
$[2.2] = 2$

$= (2)(1-1)$
$= 0$

$LHL = RHL$

So $\lim_{x \to 1} f(x) = 0$

ii) $\lim_{x \to 1} [x][x+1]$

$$\lim_{x \to 1^-} [x][x+1]$$

$= (0)(1) = 0$

$\lim_{x \to 1^-} [x] = 0$

$\lim_{x \to 1^-} [x+1] = 1$

$$\lim_{x \to 1^+} [x][x+1]$$

$(1)(2) = 2$

$\lim_{x \to 1} f(x)$ does not exist

Correction

$x \to 0 \; [x][x+1]$

$$\lim_{x \to 0^-} [x][x+1]$$

$(-1)(0) = 0$

$$\lim_{x \to 0^+} [x][x+1]$$

$(0)(1) = 0$

$L.H.L = R.H.L$

$$\lim_{x \to 0} [x][x+1] = 0 \text{ Ans}$$

$\because \lim_{x \to 0^-} [x] = [-.5]$
$= -1$

$\lim_{x \to 0^-} [x+1] = [-.5+1]$
$= [.5]$
$= 0$

$\lim_{x \to 0^+} [x] = [.5] = 0$

$\lim_{x \to 0^+} [x+1] = [.5+1]$
$= [1.5]$
$= 1$

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(iii) $\underset{x\to 0}{Lt}\, x\left[\frac{1}{x}\right]$

$\Rightarrow \underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0-0}{Lt} x\left[\frac{1}{x}\right]$

$= 0\left[\frac{1}{-1}\right] = 0$

$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0+0}{Lt} x\left[\frac{1}{x}\right] = 0\left[\frac{1}{+}\right] = 0$

$\underset{x\to 0}{Lt}\, x\left[\frac{1}{x}\right] = 0$ Ans

$f(x) = [x]$

$= 0$ when $0 \le x < 1$

$= 1$ $\quad 1 \le x < 2$

$-1 \le x < 0$

$f(x) = (-1)$

(iv) $\underset{x\to 0}{Lt}\, x^3\left[\frac{1}{x}\right]$

L.H.L $f(0-0) = \underset{x\to 0-0}{Lt} x^3\left[\frac{1}{x}\right] = 0\left[\frac{1}{-1}\right] = 0$

R.H.L $f(0+0) = \underset{x\to 0+0}{Lt} x^3\left[\frac{1}{x}\right] = 0\left[+\right] = 0$

$f(0-0) = f(0+0)$,

L.H. Limit = R.H. Limit $\quad \underset{x\to 0}{Lt}\, x^3\left[\frac{1}{x}\right] = 0$ Ans

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