Finite Limit at a Finite Point

Notes of Chapter 01
Calculus with Analytic Geometry
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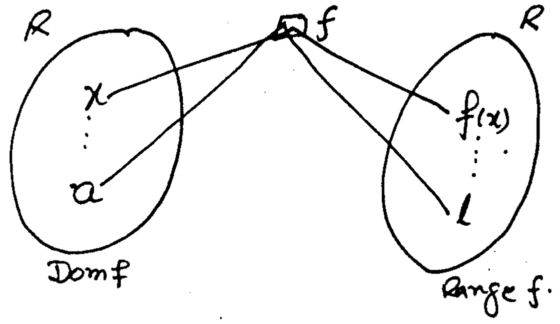

Def: Let 'a' be any real number and let f be a function from R to R. Which is defined for all values of x near a with possible exception of the point $x=a$. The function f is said to have limit l as x tends to 'a', if for every $\epsilon>0$, there exists a positive real number δ (which usually depends on ϵ) s.t $|f(x)-l|<\epsilon$ whenever $0<|x-a|<\delta$

In this case we write
$\lim_{x\to a} f(x) = l$ and say that the function f has limit l (or $f(x)$ converges to l) as x approaches a.

Exp: Limit (tends to, approaches)

$x=0$
$x = 0.1, 0.01, 0.001, 0.0001, \ldots$
x tends to 0; x approaches to 0 $(x\to 0)$
$f(x)=2+x$
When $x = 0.1, 0.01, 0.001, 0.0001, \ldots$ $x\to 0$
$f(x) = 2+.1, 2+.01, 2+.001, 2+.0001 \ldots$
$= 2.1 + 2.01 + 2.001 + 2.0001$
$f(x)\to 2$
Then we write $\lim_{x\to 0} f(x) = 2$.

Left hand Limit (L.H.L)

If we approaching to 'a' from left (from values less than a), then the limit is called left hand limit and is denoted by
$\lim_{x\to a-0}$ or $\lim_{x\to a^-}$ or $f(a-0)$

for this Limit we put $x = a - h$
and Then the new Limit will
be $h \to 0$

Right hand Limit (R.H.L)

If we are approaching to 'a' from the right side (from values greater than a) Then the limit is called the R.H.L, and is denoted by
$\underset{x \to a+0}{Lt}$ or $\underset{x \to a^+}{Lt}$ or $f(a+0)$

for this Limit put $x = a + h$
Then the new Limit will be $h \to 0$.

Example $\underset{x \to 3}{Lt} \dfrac{x^3 - 27}{x^2 - 9}$

Sol: L.H.L

$\underset{x \to 3-0}{Lt} \dfrac{x^3-27}{x^2-9} = \underset{h \to 0}{Lt} \dfrac{(3-h)^3 - 27}{(3-h)^2 - 9}$ Put $x = 3-h$

$= \underset{h \to 0}{Lt} \dfrac{\cancel{(3-h-3)}((3-h)^2 + 3(3-h) + 9)}{\cancel{(3-h-3)}(3-h+3)}$

$= \underset{h \to 0}{Lt} \dfrac{9 + h^2 - 6h + 9 - 3h + 9}{6-h}$

$= \underset{h \to 0}{Lt} \dfrac{27 + h^2 - 9h}{6-h} = \dfrac{27}{6} = \dfrac{9}{2} \to$ i.

R.H.L

$\underset{x \to 3+0}{Lt} \dfrac{x^3-27}{x^2-9} = \underset{h \to 0}{Lt} \dfrac{(3+h)^3 - 27}{(3+h)^2 - 9}$ Put $x = 3+h$

$= \underset{h \to 0}{Lt} \dfrac{\cancel{27} + 27h + 9h^2 + h^3 - \cancel{27}}{\cancel{9} + h^2 + 6h - \cancel{9}}$

$= \underset{h \to 0}{Lt} \dfrac{\cancel{h}[27 + 9h^2 + h^2]}{\cancel{h}[6+h]}$

$= \dfrac{27}{6} = \dfrac{9}{2} \to$ ii.

from i. & ii.

LH.L = R.H.L

$\underset{x \to 3}{Lt} \dfrac{x^3-27}{x^2-9} = \dfrac{9}{2}$ Ans

Some Imp. Formulae

(1) $\underset{x \to 0}{Lt} \dfrac{\sin x}{x} = 1$

(2) $\underset{x \to 0}{Lt} \dfrac{\cos x - 1}{x} = 0$

(3) $\underset{x \to 0}{Lt} \dfrac{a^x - 1}{x} = \ln a$

4) $\underset{x \to a}{Lt} \dfrac{x^n - a^n}{x - a} = n a^{n-1}$

5) $\underset{x \to \infty}{Lt} \left(1 + \dfrac{1}{x}\right)^x = e$

6) $\underset{x \to 0}{Lt} (1+x)^{\frac{1}{x}} = e$.

Note: L'Hospitals Rule (Ch # 3)

Let $f(x)$ and $g(x)$ be continuous and differentiable in the neighbourhood of $x = a$ and $f(a) = 0 = g(a)$

Then $\underset{x \to a}{Lt} \dfrac{f(x)}{g(x)} = \underset{x \to a}{Lt} \dfrac{f'(x)}{g'(x)}$

when $f'(a) = g'(a) = 0$

Repeating the arguments

$= \underset{x \to a}{Lt} \dfrac{f''(x)}{g''(x)} = \cdots\cdots$

CAUTION $\dfrac{f(x)}{g(x)}$ is not to differentiated by quotient rule.
But $f(x)$ and $g(x)$ are to be differentiated separated.