Exercise 1.2

Evaluate the indicated Limits. (1-30)

1. $\lim_{x\to 2} \dfrac{x-2}{\sqrt{2+x}}$

$= \dfrac{\lim_{x\to 2} x-2}{\lim_{x\to 2}\sqrt{2+x}}$

$= \dfrac{2-2}{\sqrt{2+2}} = \dfrac{0}{\sqrt{4}}$

$= 0$

2. $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$

$= \lim_{x\to 1} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$

$= \lim_{x\to 1} (x^2+x+1)$

$= 1+1+1$

$= 3$

3. $\lim_{x\to 1}\left(\dfrac{1}{1-x} - \dfrac{3}{1-x^3}\right)$

$= \lim_{x\to 1}\left(\dfrac{1}{1-x} - \dfrac{3}{(1-x)(x^2+x+1)}\right)$

$= \lim_{x\to 1}\left(\dfrac{x^2+x+1-3}{(1-x)(x^2+x+1)}\right)$

$= \lim_{x\to 1}\left(\dfrac{x^2+x-2}{(1-x)(x^2+x+1)}\right)$

$= \lim_{x\to 1}\left(\dfrac{x^2+2x-x-2}{(1-x)(x^2+x+1)}\right)$

$= \lim_{x\to 1}\left(\dfrac{x(x+2)-1(x+2)}{(1-x)(x^2+x+1)}\right)$

$= \lim_{x\to 1} \dfrac{(x-1)(x+2)}{(1-x)(x^2+x+1)}$

$= \lim_{x\to 2} \dfrac{-\cancel{(1-x)}(x+2)}{\cancel{(1-x)}(x^2+x+1)}$

$= -\dfrac{(1+2)}{1+1+1}$

$= -3/3$

$= -1$

4. if $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$

prove that $\lim_{x\to a} P_n(x) = P_n(a)$

Sol.

$\lim_{x\to an} P(x) = \lim_{x\to a}\left[a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n\right]$

$= a_0a^n + a_1a^{n-1} + \cdots + a_{n-1}a + a_n$

$\lim_{n\to a} P_n(x) = P_n(a)$

5. $\lim_{x\to 0} \dfrac{\csc - \cot x}{x}$

$= \lim_{x\to 0} \dfrac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x\to 0} \dfrac{1-\cos x}{x\sin x}$

$= \lim_{x\to 0} \dfrac{1-\cos x}{x\sin x} \times \dfrac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0} \dfrac{1-\cos^2 x}{x\sin x(1+\cos x)}$

$= \lim_{x\to 0} \dfrac{\sin^{\cancel{2}} x}{x\cancel{\sin x}(1+\cos x)}$

$= \lim_{x\to 0} \dfrac{\sin x}{x(1+\cos x)}$

$= \lim_{x\to 0}\left(\dfrac{\sin x}{x}\right) \cdot \lim_{x\to 0} \dfrac{1}{1+\cos x}$

$= 1 \cdot \dfrac{1}{1+1}$

$= \dfrac{1}{2}$

6. $\lim_{x\to 0} \dfrac{\sin ax}{\sin bx}$.

$= \lim_{x\to 0} \dfrac{a\cancel{x} \cdot \frac{\sin ax}{ax}}{b\cancel{x} \cdot \frac{\sin bx}{bx}}$

$= \dfrac{a}{b} \dfrac{\lim_{x\to 0} \frac{\sin ax}{ax}}{\lim_{x\to 0} \frac{\sin bx}{bx}} = \dfrac{a}{b}\,\dfrac{1}{1}$

$= a/b$

$\lim_{x\to 0}\frac{\sin x}{x}=1$

7. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$

$=\lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$

$=\lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$

$=\lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$

$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\lim_{x\to 0}\frac{1}{1+\cos x}$

$=1\cdot\frac{1}{1+1}$

$=1/2$

9. $\lim_{x\to\pi}\frac{\tan(\sin x)}{\sin x}$

Let $\sin x=\theta$

when $x\to\pi$

$\theta\to 0$

$=\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta\cdot\theta}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{1}{\cos\theta}$

$=1\cdot\frac{1}{1}$

$=1$

12. $\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}$

$\lim_{x\to\infty}\frac{\sqrt{x^2(1+1/x^2)}}{x(1+1/x)}$

$=\lim_{x\to\infty}\frac{\cancel{x}\sqrt{1+1/x^2}}{\cancel{x}(1+1/x)}$

$=\frac{\sqrt{1+1/\infty}}{1+1/\infty}$

$=\frac{\sqrt{1+0}}{1+0}$

$=1$

8. $\lim_{y\to x}\frac{y^{2/3}-x^{2/3}}{y-x}$

$=\lim_{y\to x}\frac{(y^{1/3})^2-(x^{1/3})^2}{(y^{1/3})^3-(x^{1/3})^3}$

$=\lim_{y\to x}\frac{\cancel{(y^{1/3}-x^{1/3})}(y^{1/3}+x^{1/3})}{\cancel{(y^{1/3}-x^{1/3})}(y^{2/3}+x^{1/3}y^{1/3}+x^{2/3})}$

$=\lim_{y\to x}\frac{y^{1/3}+x^{1/3}}{y^{2/3}+x^{1/3}y^{1/3}+x^{2/3}}$

$=\frac{x^{1/3}+x^{1/3}}{x^{2/3}+x^{1/3}x^{1/3}+x^{2/3}}=\frac{2x^{1/3}}{x^{2/3}+x^{2/3}+x^{2/3}}$

$=\frac{2x^{1/3}}{3x^{2/3}}$

$=\frac{2}{3}x^{-1/3}$ or $=\frac{2}{3x^{1/3}}$

10. $\lim_{x\to 0}x\sin(1/x)$

$=\lim_{x\to 0}\frac{\sin(1/x)}{1/x}$

$=\lim_{x\to 0}x\,[-1,1]$ → any no.

$=0\,[-1,1]$

$=0$

11. $\lim_{x\to 0}\sin(1/x)$

= Some no in $[-1,1]$

Limit does not exist.

13. $\lim_{x\to\infty}\frac{4x^3-2x^2+1}{3x^3-5}$

$=\lim_{x\to\infty}\frac{\cancel{x^3}(4-\frac{2}{x}+\frac{1}{x^3})}{\cancel{x^3}(3-\frac{5}{x^3})}$

$=\frac{4-\frac{2}{\infty}+\frac{1}{\infty}}{3-\frac{5}{\infty}}$

$=\frac{4-0+0}{3-0}$

$=4/3$

$$\because \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

14. $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x/2}\right)^x\right]^{\frac{2}{2}}$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x/2}\right)^{x/2}\right]^2$$

$$=(e)^2$$

$$=e^2$$

17. $\lim_{x\to\infty}\frac{a^x-1}{x}\quad (a>1)$

Let $a^x-1=z$

$\Rightarrow a^x = z+1$

$x=\log_a(z+1)$

if $x\to\infty$, then $x\to\infty$

Therefore,

$$\lim_{x\to\infty}\frac{a^x-1}{x}=\lim_{z\to\infty}\frac{z}{\log_a(1+z)}$$

$$=\lim_{z\to\infty}\frac{1}{\frac{1}{z}\log_a(1+z)}$$

$$=\lim_{z\to\infty}\frac{1}{\log_a(1+z)^{1/z}}$$

$$=\frac{1}{(\infty)^0}\qquad \boxed{\text{assume } (\infty)^0=1}$$

$$=1$$

19. $\lim_{x\to\pm\infty}\left[\frac{x^2}{x+1}-\frac{x^2}{x+3}\right]$

$$=\lim_{x\to\pm\infty}\left[\frac{x^2(x+3)-x^2(x+1)}{(x+1)(x+3)}\right]$$

$$=\lim_{x\to\pm\infty}\left[\frac{\cancel{x^3}+3x^2-\cancel{x^3}-x^2}{x^2+3x+x+3}\right]$$

$$=\lim_{x\to\pm\infty}\left[\frac{2x^2}{x^2+4x+3}\right]$$

$$=\lim_{x\to\pm\infty}\left[\frac{2\cancel{x^2}}{\cancel{x^2}\left(1+\frac{4}{x}+\frac{3}{x^2}\right)}\right]$$

$$=\frac{2}{1+0+0}$$

$$=2$$

15. $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{(-x)}\right)^{-1x}\right]^{-1}$$

$$=e^{-1}$$

$$=\frac{1}{e}$$

16. $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$

$$=\lim_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$$

$$=\lim_{x\to\infty}\left(\frac{1}{x}+1\right)^{-x}$$

$$=\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^{-1}$$

$$=(e)^{-1}$$

$$=1/e$$

18. $\lim_{x\to\infty}\frac{x^4-2x^2+6}{x^2+7}$

$$=\lim_{x\to\infty}\frac{x^{\cancel{4}\,2}\left(1-\frac{2}{x^2}+\frac{6}{x^4}\right)}{\cancel{x^2}\left(1+\frac{7}{x^2}\right)}$$

$$=\lim_{x\to\infty}\frac{x^2\left(1-\frac{2}{x^2}+\frac{6}{x^4}\right)}{(1+7/x^2)}$$

$$=\infty$$

20. $\lim_{x\to\infty}\left(x-\sqrt{x^2-a^2}\right)$

$$=\lim_{x\to\infty}\frac{(x-\sqrt{x^2-a^2})\cdot(x+\sqrt{x^2-a^2})}{(x+\sqrt{x^2-a^2})}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}-\cancel{x^2}+a^2}{x+\sqrt{x^2-a^2}}$$

$$=\lim_{x\to\infty}\frac{a}{x+\sqrt{x^2-a^2}}$$

$$=\frac{a}{\infty}$$

$$=0$$

21. $\lim_{x\to\infty} \frac{x^2+1}{x^{3/2}}$

$= \lim_{x\to\infty} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^{3/2}}$

$= \lim_{x\to\infty} \frac{1+1/x^2}{x^{3/2-2}}$

$= \lim_{x\to\infty} \frac{1+1/x^2}{x^{-1/2}}$

$= \lim_{x\to\infty} x^{1/2}\left(1+\frac{1}{x^2}\right)$

$= \infty\left(1+\frac{1}{\infty}\right)$

$= \infty(1+0)$

$= \infty$

22. $\lim_{x\to\infty} \frac{5x^3+2x^2-1}{x-4x^4}$

$= \lim_{x\to\infty} \frac{x^3\left(5+\frac{2}{x}-\frac{1}{x^3}\right)}{x^4\left(\frac{1}{x^3}-4\right)}$

$= \lim_{x\to\infty} \frac{5+\frac{2}{x}-\frac{1}{x^3}}{x\left(\frac{1}{x^3}-4\right)}$

$= \frac{5+\frac{2}{\infty}-\frac{1}{\infty}}{\infty\left(\frac{1}{\infty}-4\right)}$

$= \frac{5+0-0}{\infty(0-4)}$

$= \frac{5}{\infty}$

$= 0$

23. $\lim_{x\to\infty} \frac{3-2x^4}{1+x}$

$= \lim_{x\to\infty} \frac{x^{\cancel{4}3}\left(\frac{3}{x^4}-2\right)}{\cancel{x}\left(1+\frac{1}{x}\right)}$

$= \lim_{x\to\infty} \frac{x^3\left(3/x^4-2\right)}{\left(1+\frac{1}{x}\right)}$

$= \frac{\infty\left(3/\infty-2\right)}{\left(1+1/\infty\right)}$

$= \frac{\infty(0-2)}{1-0}$

$= \infty$

24. $\lim_{x\to-1} \frac{x^{1/3}+1}{x+1}$

$= \lim_{x\to-1} \frac{x^{1/3}+1}{(x^{1/3})^3+(1)^{1/3^3}}$

$= \lim_{x\to-1} \frac{\cancel{x^{1/3}+1}}{\cancel{(x^{1/3}+1)}(x^{2/3}+1-x^{1/3})}$

$= \lim_{x\to-1} \frac{1}{x^{2/3}-x^{1/3}+1}$

$= \frac{1}{(-1)^{2/3}-(-1)^{1/3}+1} = \frac{1}{+1-(-1)+1}$

$= \frac{1}{+1+1+1}$

$= \frac{1}{3}$

25. $\lim_{x\to 3^-} \left(\frac{1}{x-3}-\frac{1}{|x-3|}\right)$

$= \lim_{x\to 3^-} \left(\frac{1}{x-3}-\frac{1}{-(x-3)}\right)$

$= \lim_{x\to 3^-} \left(\frac{1}{x-3}+\frac{1}{x-3}\right)$

$= \lim_{x\to 3} \left(\frac{2}{x-3}\right)$

$= \frac{2}{3-3}$

$= -\infty$

26. $\lim_{x\to -2} \frac{x^2+2x-8}{x^2-4}$

$= \frac{(-2)^2+2(-2)-8}{(-2)^2-4}$

$= \frac{\cancel{4}-\cancel{4}-8}{4-4}$

$= -\frac{8}{0}$

$= -\infty$

27. $\lim\limits_{x\to1^-} \dfrac{\sqrt{1-x^2}}{1-x}$

$= \lim\limits_{x\to1^-} \dfrac{\sqrt{(1+x)(1-x)}}{(1-x)}$

$= \lim\limits_{x\to1^-} \dfrac{\sqrt{1+x}\,\cancel{\sqrt{1-x}}}{\sqrt{1-x}\,\cancel{\sqrt{1-x}}}$

$= \lim\limits_{x\to1^-} \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$

$= \dfrac{\sqrt{1+1}}{1-1} = \dfrac{1}{0}$

$= \infty$

28. $\lim\limits_{x\to1^+} \dfrac{x-1}{\sqrt{x^2-1}}$

$= \lim\limits_{x\to1^+} \dfrac{x-1}{\sqrt{(x+1)(x-1)}}$

$= \lim\limits_{x\to1^+} \dfrac{\sqrt{x-1}\,\cancel{\sqrt{x-1}}}{\sqrt{x+1}\,\cancel{\sqrt{x-1}}}$

$= \lim\limits_{x\to1^+} \dfrac{\sqrt{x-1}}{\sqrt{x+1}}$

$= \dfrac{\sqrt{1-1}}{\sqrt{1+1}}$

$= \dfrac{0}{\sqrt{2}}$

$= 0$

29. $\lim\limits_{x\to2^-} \dfrac{\sqrt{4-x^2}}{\sqrt{6-5x+x^2}}$

$= \lim\limits_{x\to2^-} \dfrac{\sqrt{(2-x)(2+x)}}{\sqrt{6-3x-2x+x^2}}$

$= \lim\limits_{x\to2^-} \dfrac{\sqrt{2-x}\,\sqrt{2+x}}{\sqrt{3(2-x)-x(2-x)}}$

$= \lim\limits_{x\to2^-} \dfrac{\sqrt{2-x}\,\sqrt{2+x}}{\sqrt{(2-x)(3-x)}}$

$= \lim\limits_{x\to2^-} \dfrac{\cancel{\sqrt{2-x}}\,\sqrt{2+x}}{\cancel{\sqrt{2-x}}\,\sqrt{3-x}}$

$= \lim\limits_{x\to2^-} \dfrac{\sqrt{2+x}}{\sqrt{3-x}}$

$= \dfrac{\sqrt{4}}{1}$

$= 2$

10. $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x}$

$= \lim\limits_{x\to\infty} \left(\dfrac{\cancel{x}}{\cancel{x}} + \dfrac{\sin x}{x}\right)$

$= \lim\limits_{x\to\infty} \left(1 + \dfrac{\sin x}{x}\right)$

$= \lim\limits_{x\to\infty} (1) + \lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

$= 1 + 0$

$= 0$

31. Let $f(x) = \begin{cases} x^2+3 & \text{if } x \le 1 \\ x+1 & \text{if } x > 1 \end{cases}$

(i) find $\lim\limits_{x\to1^+} f(x)$ and (ii) $\lim\limits_{x\to1^-} f(x)$

(i) $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} (x+1) = 1+1 = 2$

(ii) $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} (x^2+3) = 1+3 = 4$

32. $f(x) = \begin{cases} 3 & \text{if } x \le -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \geq 2 \end{cases}$

find $\lim\limits_{x \to \pm 2^+} f(x)$ & $\lim\limits_{x \to \pm 2^-} f(x)$.

Sol. $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} (3) = 3$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} \left(-\frac{1}{2}x^2\right)$

$= -\frac{1}{2}(-2)^2$

$= -\frac{1}{2}(4)$

$= -2$

$\lim\limits_{x \to +2^-} f(x) = \lim\limits_{x \to +2^-} \left(-\frac{1}{2}x^2\right)$

$= -\frac{1}{2}(2)^2$

$= -2.$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3) = 3$

33. $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ \sqrt{x+7} & \text{if } x > 2 \end{cases}$

Sol. find $\lim\limits_{x \to 2} f(x)$.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 - 1)$

$= 4 - 1$

$= 3$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \sqrt{x+7}$

$= \sqrt{2+7} = \sqrt{9}$

$= 3$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} f(x)$

34. $f(x) = \begin{cases} \cos x & \text{if } x \le 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

find $\lim\limits_{x \to 0} f(x)$.

Sol: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x$

$= \cos 0°$

$= 1$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 - x)$

$= 1 - 0$

$= 1$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} f(x)$

$\lim\limits_{x \to 0} f(x) = 1$

35. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x^3 & \text{if } x > 1 \end{cases}$

Sol.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2$

$= 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^3)$

$= 1$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1} f(x)$

$\lim\limits_{x \to 1} f(x) = 1$

36. $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ ax^2 & \text{if } x > -1 \end{cases}$

if limit exists. Find a.

Sol. $\lim\limits_{x \to -1^-} f(x) = \lim\limits_{x \to -1^-} (x+2) = (-1+2)$

$= 1$

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^+} ax^2 = (-1)^2 a$

$= a$

Limit exist so

$\lim\limits_{x \to -1^+} f(x) = \lim\limits_{x \to -1^-} f(x)$

$\Rightarrow a = 1$

37. Evaluate.

$\lim\limits_{x \to 3^-} \frac{3-x}{|x-3|}$

$= \lim\limits_{x \to 3^-} \frac{3-x}{-(x-3)}$

$= \lim\limits_{x \to 3^-} \frac{\cancel{3-x}}{+\cancel{(3-x)}}$

$= 1$

38. Evaluate.
$$\lim_{x\to 0^-} \frac{x}{x-|x|}.$$

Sol. $= \lim_{x\to 0^-} \frac{x}{x-(-x)}$

$= \lim_{x\to 0^-} \frac{x}{x+x}$

$= \lim_{x\to 0^-} \frac{\cancel{x}}{2\cancel{x}}$

$= \lim_{x\to 0^-} \frac{1}{2}$

$= \frac{1}{2}$

39. Find. $\lim_{h\to 0} \frac{|-1+h|-1}{h}$

$-1+0=-1$ negative value.

$= \lim_{h\to 0} \frac{-(-1+h)-1}{h}$

$= \lim_{h\to 0} \frac{\cancel{1}-h-\cancel{1}}{h}$

$= \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}}$

$= \lim_{h\to 0}(-1)$

$= -1$

40. Evaluate, [....] being the bracket function.

(i) $\lim_{x\to 1} [2x](x-1)$

$\lim_{x\to 1^-} [2x](x-1)$

$= \lim_{x\to 1^-} [2x] \lim_{x\to 1^-}(x-1)$

$= (1)(0)$

$= 0$

$\lim_{x\to 1^+} [2x](x-1) = \lim_{x\to 1^+}[2x]\lim_{x\to 1^+}(x-1)$

$= 2(0)$

$= 0$

$\Rightarrow \lim_{x\to 1} [2x](x-1) = 0$

(ii) $\lim_{x\to 0} [x][x+1]$

$\lim_{x\to 0^-}[x][x+1] = \lim_{x\to 0^-}[x]\lim_{x\to 0^-}[x+1]$

$= (-1)(0)$

$= 0$

$\lim_{x\to 0^+}[x][x+1] = \lim_{x\to 0^+}[x]\lim_{x\to 0^+}[x+1]$

$= (0)(1)$

$= 0$

$\Rightarrow \lim_{x\to 0^-}[x][x+1] = \lim_{x\to 0^+}[x][x+1] = \lim_{x\to 0}[x][x+1]$

$\Rightarrow \lim_{x\to 0}[x][x+1] = 0$

(iii) $\lim_{x\to 0} x\left[\frac{1}{x}\right].$

By def. of Bracket function.

$$\frac{1}{x}-1 \le \left[\frac{1}{x}\right] \le \frac{1}{x}$$

when $x<0$

$$x\left(\frac{1}{x}-1\right) \geqslant x\left[\frac{1}{x}\right] \geqslant \frac{1}{x}x$$

$$1 \le x\left[\frac{1}{x}\right] \le 1-x$$

$$\lim_{x\to 0^-}(1) = 1$$

$$\lim_{x\to 0^-}(1-x) = 1-0 = 1$$

By Sandwich Theorem

$$\lim_{x\to 0^-} x\left[\frac{1}{x}\right] = 1$$

when $x>0$

$$x\left(\frac{1}{x}-1\right) \le x\left[\frac{1}{x}\right] \le x\cdot\frac{1}{x}$$

$$1-x \le x\left[\frac{1}{x}\right] \le 1$$

$$\lim_{x\to 0^+}(1-x) = 1-0 = 1$$

$$\lim_{x\to 0^+}(1) = 1$$

By Sandwich Theorem.

$$\lim_{x\to 0^+} x\left[\frac{1}{x}\right] = 1$$

So

$$\Rightarrow \lim_{x\to 0} x\left[\frac{1}{x}\right] = 1.$$

(iv) $\lim\limits_{x\to 0} x^3\left[\frac{1}{x}\right]$.

By definition of Bracket function.

$$\frac{1}{x}-1 \le \left[\frac{1}{x}\right] \le \frac{1}{x}$$

when $x<0$

$$x^3\left(\frac{1}{x}-1\right) \geqslant x^3\left[\frac{1}{x}\right] \geqslant x^3\cdot\frac{1}{x}$$

$$x^2-x^3 \geqslant x^3\left[\frac{1}{x}\right] \geqslant x^2$$

$$\lim_{x\to 0^-}(x^2-x^3) = 0$$

$$\lim_{x\to 0^-}(x^2) = 0$$

$\Rightarrow$ By Sandwich Theorem

$$\lim_{x\to 0^-} x^3\left[\frac{1}{x}\right] = 0$$

when $x \geqslant 0$

$$x^3\left(\frac{1}{x}-1\right) \le x^3\left[\frac{1}{x}\right] \le x^3\frac{1}{x}$$

$$x^2-x^3 \le x^3\left[\frac{1}{x}\right] \le x^2$$

$$\lim_{x\to 0^+}(x^2-x^3) = 0$$

$$\lim_{x\to 0^+}(x^2) = 0$$

By Sandwich Theorem

$$\lim_{x\to 0^+} x^3\left[\frac{1}{x}\right] = 0$$

So

$$\Rightarrow \lim_{x\to 0} x^3\left[\frac{1}{x}\right] = 0$$