Exercise #8.6

Q#1: Show that the shortest distance between the lines $x+a=2y=-12z$ and $x=y+2a=6(z-a)$ is $2a$.

Solution: Given lines are

$$\left.\begin{array}{r} x+a=2y=-12z \\ x=y+2a=6(z-a) \end{array}\right]$$

$$\left.\frac{x+a}{1}=\frac{y}{\frac{1}{2}}=\frac{z}{-\frac{1}{12}}\right]$$

$$\frac{x+a}{12}=\frac{y}{6}=\frac{z}{-1} \quad ----\ ---(1)$$

$$x-y-2a=0=x-6z+6a \quad ------(2)$$

Now equation of a plane through line (2) is

$$(x-y-2a)+k(x-6z+6a\,)=0 \quad \Rightarrow (1+k)x-y-6kz-2a+6ka=0$$

Now direction ratios of normal vector of this plane are $1+k,-1,-6k$

Then $12(1+k)+6(-1)-1(-6k)=0 \;\Rightarrow 12+12k-6+6k=0$

$$18k+6=0 \;\Rightarrow 3k+1=0 \;\Rightarrow\; 3k=-1 \;\Rightarrow\; k=-\frac{1}{3}$$

Putting in equation of plane

$$(x-y-2a)-\tfrac{1}{3}(x-6z+6a\,)=0$$

$$3x-3y-6a-x+6z-6a=0$$

$2x-3y+6z-12a=0$ is required plane through line (2) and parallel to line (1)

Let d be the required shortest distance then

$$d=Distance\ of\ point\ (-a,0,0)\ from\ plane\ 2x-3y+6z-12a=0$$

$$d=\frac{|2(-a)-0+0-12a|}{\sqrt{(2)^2+(-3)^2+(6)^2}} \;\Rightarrow d=\frac{|-2a-12a|}{\sqrt{4+9+36}} \;\Rightarrow d=\frac{|-14a|}{\sqrt{49}} \;\Rightarrow d=\frac{14a}{7} \;\Rightarrow d=2a$$

Q#2: Find the shortest distance between the x-axis and the straight line $\mathbf{ax+by+cz+d=0=a'x+b'y+c'z+d'}$.

Solution: We know that equation of x-axis in symmetric form is

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{0} \quad ----(1)$$

Now given line is

$$ax+by+cz+d=0=a'x+b'y+c'z+d' \quad ----------(2)$$

Then equation of a plane containing this line is

$$(ax+by+cz+d)+k(a'x+b'y+c'z+d')=0$$

$$\Rightarrow (a+ka')x+(b+kb')y+(c+kc')z+(d+kd')=0$$

As this plane is parallel to x-axis

So $1(a+ka')=0 \Rightarrow a+ka'=0\ \ ka'=-a\ \ k=-\frac{a}{a'}$

Putting in equation of plane

$$(ax+by+cz+d)-\frac{a}{a'}(a'x+b'y+c'z+d')=0$$

$$aa'x+a'by+a'cz+a'd-aa'x-ab'y-ac'z-ad'=0$$

$$(a'b-ab')y+(a'c-ac')z+(a'd-ad')=0$$ is equation of plane containing line (2)

Let d' be the required shortset distance then

$$d'=\textit{Distance of point } (0,0,0) \textit{ from plane}$$

$$d'=\frac{|(a'b-ab')(0)+(a'c-ac')(0)+(a'd-ad')|}{\sqrt{(a'b-ab')^2+(a'c-ac')^2}}$$

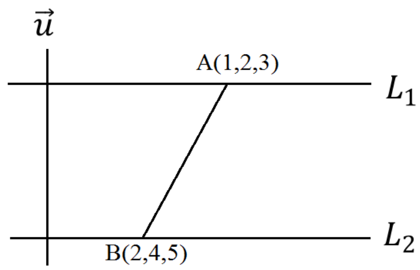
$$\Rightarrow d'=\frac{a'd-ad'}{\sqrt{(a'b-ab')^2+(a'c-ac')^2}} \textit{required distance}$$

Q#3: Show that the shortest distance between the straight lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ **and** $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ **is** $\frac{1}{\sqrt{6}}$ **and equations of the straight line perpendicular to both are** $11x+2y-7z+6=0=7x+y-5z+7$.

Solution: Given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}\ \ ------(1)$$

$$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}\ \ ------(2)$$



A point on line (1) is $A(1,2,3)$

A point on line (2) is $B(2,4,5)$

$\overrightarrow{AB}=(2-1)\hat{\imath}+(4-2)\hat{\jmath}+(5-3)\hat{k} \Rightarrow \overrightarrow{AB}=\hat{\imath}+2\hat{\jmath}+2\hat{k}$

Here direction ratios of line (1) are 2,3,4 & direction ratios of line (2) are 3,4,5

Let $\vec{u}$ be a vector perpendicular to both given lines then

$$\vec{u}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}$$

Expanding from R_1

$$\vec{u}=(15-16)\hat{\imath}-(10-12)\hat{\jmath}+(8-9)\hat{k}$$

$$\Rightarrow \vec{u}=-\hat{\imath}+2\hat{\jmath}-\hat{k}$$

Let d be the required shortest distance between lines then

$$d=\frac{\overrightarrow{AB}.\vec{u}}{|\vec{u}|} \Rightarrow d=\frac{(\hat{\imath}+2\hat{\jmath}+2\hat{k}).(-\hat{\imath}+2\hat{\jmath}-\hat{k})}{\sqrt{1+4+1}}=\frac{1(-1)+2(2)+2(-1)}{\sqrt{6}} \Rightarrow d=\frac{-1+4-2}{\sqrt{6}} \Rightarrow d=\frac{1}{\sqrt{6}}$$

$\textit{is required distance}$

Now equations of line perpendicular to both given lines is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}$$

$(x-1)(-3-8)-(y-2)(-2+4)+(z-3)(7)=0=(x-2)(-4-10)-(y-4)(-3+5)+(z-5)(6+4)$

$(x-1)(-11)-(y-2)(2)+(z-3)(7)=0=(x-2)(-14)-(y-4)(2)+(z-5)(10)$

$-11x-2y+7z+11+4-21=0=-14x-2y+10z+28+8-50$

$-11x-2y+7z-6=0=-14x-2y+10z-14$

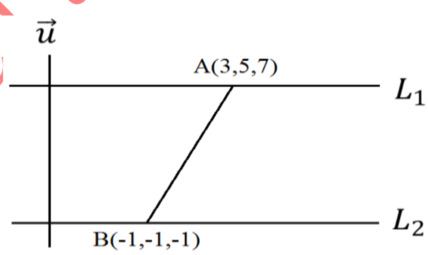
$11x+2y-7z+6=0=7x+y-5z+7$ is required equation.

Q#4: Find the shortest distance between the lines
$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ **and** $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$
Find equations of the straight line perpendicular to both the given straight lines and also its points of intersection with the given straight lines.

Solution: given lines are

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1} \quad ------(1)$$

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1} \quad ------(2)$$



A point on line (1) is $A(3,5,7)$

A point on line (2) is $B(-1,-1,-1)$

$\overrightarrow{AB}=(-1-3)\hat{\imath}+(-1-5)\hat{\jmath}+(-1-7)\hat{k} \implies \overrightarrow{AB}=-4\hat{\imath}-6\hat{\jmath}-8\hat{k}$

Here direction ratios of line (1) are $1,-2,1$ & direction ratios of line (2) are $7,-6,1$

Let $\vec{u}$ be a vector perpendicular to both lines then

$$\vec{u}=\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}$$

Expanding from R_1

$$\vec{u}=(-2+6)\hat{\imath}-(1-7)\hat{\jmath}+(-6+14)\hat{k}$$

$$\implies \vec{u}=4\hat{\imath}+6\hat{\jmath}+8\hat{k}$$

Let d be the required distance between lines then

$$d=\frac{\overrightarrow{AB}.\vec{u}}{|\vec{u}|} \implies d=\frac{(-4\hat{\imath}-6\hat{\jmath}-8\hat{k}).(4\hat{\imath}+6\hat{\jmath}+8\hat{k})}{\sqrt{(4)^2+(6)^2+(8)^2}} \implies d=\frac{-16-36-64}{\sqrt{16+36+64}} \implies d=\frac{-116}{\sqrt{116}}$$

$d=-\sqrt{116} \implies d=-\sqrt{4\times 29} \implies d=-2\sqrt{29} \implies d=2\sqrt{29}$ (In magnitude)

Now given lines are

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}=t \quad ----(1)$$

$$\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}=s \quad ----(2)$$

Parametric equations of given lines are

$$\left.\begin{aligned} x &= 3+t \\ y &= 5-2t \\ z &= 7+t \end{aligned}\right] \quad ----(1) \qquad \& \qquad \left.\begin{aligned} x &= -1+7s \\ y &= -1-6s \\ z &= -1+s \end{aligned}\right] \quad ----(2)$$

Any point on line (1) is $P(3+t, 5-2t, +t)$

Any point on line (2) is $Q(-1+7s, -1-6s, -1+s)$

Direction ratios of line PQ are $\quad 3+t+1-7s, 5-2t+1+6s, 7+t+1-s$

$$= t-7s+4, -2t+6s+6, t-s+8$$

If PQ is the line of shortest then PQ is perpendicular to both lines

So
$$\left.\begin{aligned} 1(t-7s+4)-2(-2t+6s+6)+1(t-s+8) &= 0 \\ 7(t-7s+4)-6(-2t+6s+6)+1(t-s+8) &= 0 \end{aligned}\right]$$

$$\left.\begin{aligned} t-7s+4+4t-12s-12+t-s+8 &= 0 \\ 7t-49s+28+12t-36s-36+t-s+8 &= 0 \end{aligned}\right] \Rightarrow \left.\begin{aligned} 6t-20s &= 0 \\ 20t-86s &= 0 \end{aligned}\right]$$

$$\Rightarrow t = 0 \quad \& \quad s = 0$$

Hence coordinates of points P & Q are $\quad P(3,5,7)$ & $Q(-1,-1,-1)$

Now equation of line of shortest distance is

$$\frac{x-3}{3+1} = \frac{y-5}{5+1} = \frac{z-7}{7+1} \Rightarrow \frac{x-3}{4} = \frac{y-5}{6} = \frac{z-7}{8} \Rightarrow \frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4} \quad \textit{is required line.}$$

Q#5: Find the coordinates of the point on the join of $(-3,7,-13)$& $(-6,1,-10)$ which is nearest to the intersection of the planes $\quad 2x-y-3z+32=0$ and $3x+2y-15z-8=0$.

Solution: Equation of the line through $(-3,7,-13)$& $(-6,1,-10)$ is

$$\frac{x+3}{-3+6} = \frac{y-7}{7-1} = \frac{z+13}{-13+10} \qquad \Rightarrow \frac{x+3}{3} = \frac{y-7}{6} = \frac{z+13}{-3}$$

$$Let \quad \frac{x+3}{1} = \frac{y-7}{2} = \frac{z+13}{-1} = t$$

$$\left.\begin{aligned} x &= -3+t \\ y &= 7+2t \\ z &= -13-t \end{aligned}\right] \quad ----(1)$$

Any point on line (1) is $P(-3+t, 7+2t, -13-t)$

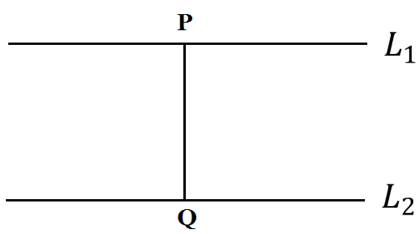
Also given equation of line is $\quad 2x-y-3z+32 = 0 = 3x+2y-15z-8$

Let l, m, n be the direction cosines of this line

Since it lies on both planes ,so by condition of perpendicularity

$$\left.\begin{aligned} 2l-m-3n &= 0 \\ 3l+2m-15n &= 0 \end{aligned}\right]$$

$$\frac{l}{15+6} = \frac{-m}{-30+9} = \frac{n}{4+3} \qquad \Rightarrow \frac{l}{3} = \frac{m}{3} = \frac{n}{1}$$



So direcion ratios of given line are 3,3,1

To find a point on line , put $z = 0$ in above equations

$$\left.\begin{aligned}2x - y + 32 = 0\\ 3x + 2y - 8 = 0\end{aligned}\right]$$

$$\frac{x}{8-64} = \frac{-y}{-16-96} = \frac{1}{4+3} \Rightarrow \frac{x}{-56} = \frac{-y}{-112} = \frac{1}{7} \quad \Rightarrow x = -\frac{56}{7} = -8 \quad \Rightarrow y = \frac{112}{7} = 16$$

So $(-8,16,0)$ is a point on given line

Now equation of given line through $(-8,16,0)$ & with direction ratios 3,3,1 is

$$\frac{x+8}{3} = \frac{y-16}{3} = \frac{z}{1} = s$$

$$\left.\begin{aligned}x = -8 + 3s\\ y = 16 + 3s\\ z = s\end{aligned}\right] \quad ----(2)$$

Any poit on line (2) is $Q(-8+3s, 16+3s, s)$

Now direction ratios of line PQ are $-3 + t + 8 - 3s, 7 + 2t - 16 - 3s, -13 - t - s$

$$= t - 3s + 5, 2t - 3s - 9, -t - s - 13$$

If PQ is perpendicular to both lines (1) & (2)

Then by condition of perpendicularity

$$\left.\begin{aligned}1(t-3s+5) + 2(2t-3s-9) - 1(-t-s-13) = 0\\ 3(t-3s+5) + 3(2t-3s-9) + 1(-t-s-13) = 0\end{aligned}\right] \Rightarrow \left.\begin{aligned}t - 3s + 5 + 4t - 6s - 18 + t + s + 13 = 0\\ 3t - 9s + 15 + 6t - 9s - 27 - t - s - 13\end{aligned}\right]$$

$6t - 8s = 0$ ---------- (I)

$8t - 19s - 25 = 0$ ---------- (II)

Multiplying (I) by 4 & (II) by 3 and subtracting (II) from (I)

$25s = -75 \quad \Rightarrow \boldsymbol{s = -3}$

Put in (I) $\quad 6t - 8(-3) = 0 \Rightarrow 6t + 24 = 0 \quad \Rightarrow t + 4 = 0 \quad \Rightarrow t = -4$

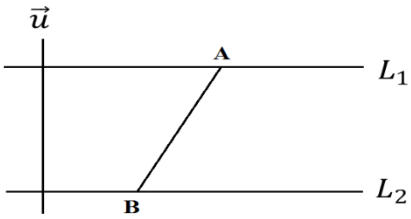
Put $\boldsymbol{t = -4}$ in coordinates of $\quad P(-3-4, 7-8, -13+4) \quad \Rightarrow P(-7,-1,-9)$

Q#6: Find the length and equations of the common perpendicular of the lines
L : $6x + 8y + 3z - 13 = 0$, $x + 2y + z - 3 = 0$
M : $3x - 9y + 5z = 0$, $x + y - z = 0$

Solution: given lines are

$L \;:\; 6x + 8y + 3z - 13 = 0 \;, \quad x + 2y + z - 3 = 0$

$M \;:\; 3x - 9y + 5z = 0 \;, \; x + y - z = 0$



We will write both equations in symmetric form.

Let l_1, m_1, n_1 be direction cosines of line L.Since it lies on both planes.Hence by condition of perpendicularity

$$\left.\begin{aligned}6l_1 + 8m_1 + 3n_1 = 0\\ l_1 + 2m_1 + n_1 = 0\end{aligned}\right]$$

$$\frac{l_1}{8-6} = \frac{-m_1}{6-3} = \frac{n_1}{12-8} \quad \Rightarrow \frac{l_1}{2} = \frac{m_1}{-3} = \frac{n_1}{4}$$

So direction ratios of line L are 2,-3,4

To find a point on line L , put $z = 0$

$$\begin{bmatrix} 6x + 8y - 13 = 0 \\ x + 2y - 3 = 0 \end{bmatrix}$$

$$\frac{x}{-24+26} = \frac{-y}{-18+13} = \frac{1}{12-8} \quad \Rightarrow \frac{x}{2} = \frac{y}{5} = \frac{1}{4} \quad \Rightarrow x = \frac{1}{2}, \qquad y = \frac{5}{4}$$

So a point on line L is $\left(\frac{1}{2}, \frac{5}{4}, 0\right)$

Now eqaution of line L through $\left(\frac{1}{2}, \frac{5}{4}, 0\right)$ & having direction ratios 2,-3,4 is

$$\frac{x - 1/2}{2} = \frac{y - 5/4}{-3} = \frac{z}{4} \quad ----(1)$$

Next suppose l_2, m_2, n_2 be the direction cosines of line M. Since it lies on both planes , so by condition of perpendicularity

$$\begin{bmatrix} 3l_2 - 9m_2 + 5n_2 = 0 \\ l_2 + m_2 - n_2 = 0 \end{bmatrix}$$

$$\frac{l_2}{9-5} = \frac{-m_2}{-3-5} = \frac{n_2}{3+9} \quad \Rightarrow \frac{l_2}{4} = \frac{m_2}{8} = \frac{n_2}{12} \quad \Rightarrow \frac{l_2}{1} = \frac{m_2}{2} = \frac{n_2}{3}$$

So direction ratios of line M are 1,2,3

To find a point on line M , put $z = 0$

$$\begin{bmatrix} 3x - 9y = 0 \\ x + y = 0 \end{bmatrix} \Rightarrow \begin{matrix} x - 3y = 0 & ---(I) \\ -x + y = 0 & ---(II) \end{matrix}$$

Subtracting we have $\quad -4y = 0 \ \Rightarrow \boldsymbol{y = 0}$

Put in (I) $\quad x - 0 = 0 \ \Rightarrow \boldsymbol{x = 0}$

Now a point on line M is (0,0,0)

Hence equation of line M through (0,0,0) having direction ratios 1,2,3

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad ----(2)$$

Now we want to find shortest distance between lines (1) & (2)

A point on line (1) is $A\left(\frac{1}{2}, \frac{5}{4}, 0\right)$

A point on line (2) is $B(0,0,0)$

$$\overrightarrow{AB} = -\frac{1}{2}\hat{\imath} - \frac{5}{4}\hat{\jmath} + 0\hat{k}$$

Let $\vec{u}$ be a vector perpendicular to both lines then $\quad \vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

Expanding from R_1

$$\vec{u} = (-9-8)\hat{\imath} - (6-4)\hat{\jmath} + (4+3)\hat{k} \quad \Rightarrow \vec{u} = -17\hat{\imath} - 2\hat{\jmath} + 7\hat{k}$$

Suppose d be the required shortset distance between lines then

$$d=\frac{\overrightarrow{AB}.\vec{u}}{|\vec{u}|}=\frac{\left(-\frac{1}{2}\hat{\imath}-\frac{5}{4}\hat{\jmath}+0\hat{k}\right).\left(-17\hat{\imath}-2\hat{\jmath}+7\hat{k}\right)}{\sqrt{(-17)^2+(-2)^2+(7)^2}} \Rightarrow d=\frac{\frac{17}{2}+\frac{5}{2}+0}{\sqrt{289+4+49}} \Rightarrow d=\frac{22/2}{\sqrt{342}} \Rightarrow d=\frac{11}{\sqrt{342}}\text{m}$$

Now equation of common perpendicular is

$$\begin{vmatrix} x-\frac{1}{2} & y-\frac{5}{4} & z \\ 2 & -3 & 4 \\ -17 & -2 & 7 \end{vmatrix}=0=\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ -17 & -2 & 7 \end{vmatrix}$$

$$\left(x-\frac{1}{2}\right)(-21+8)-\left(y-\frac{5}{4}\right)(14+68)+z(-4-51)=0=x(14+6)-y(7+51)+z(-2+34)$$

$$\left(x-\frac{1}{2}\right)(-13)-\left(y-\frac{5}{4}\right)(82)+z(-55)=0=20x-58y+32z$$

$$-13x-82y-55z+\frac{13}{2}+\frac{205}{2}=0=20x-58y+32z$$

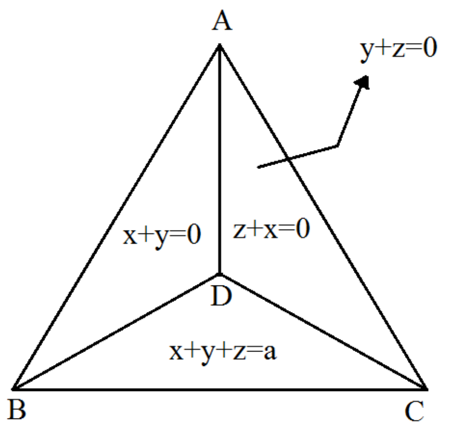
$$-13x-82y-55z+\frac{218}{2}=0=10x-29y+16z$$

$$-13x-82y-55z+109=0=10x-29y+16z$$

$$13x+82y+55z-109=0=10x-29y+16z \quad \text{is required line}$$

Q#7: Show that the shortest distance between ant two opposite edges of the tetrahedron formed by the planes $\mathbf{y+z=0}$, $\mathbf{z+x=0}$, $\mathbf{x+y=0}$, $\mathbf{x+y+z=a}$ is $\frac{2a}{\sqrt{6}}$ and that the three straight lines of the shortest distances intersect at the point $(-a,-a,-a)$.

Solution: Suppose the planes $y+z=0$, $z+x=0$, $x+y=0$ & $x+y+z=a$ be represented by ABC, ACD, ABD & BCD respectively.



The equation of line AC is $\left.\begin{matrix} y+z=0 \\ z+x=0 \end{matrix}\right]$ $or\ y=-z$ & $x=-z$

$$or\ \frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \quad -----(1)\ \text{is symmetric form of } AC.$$

Now the equation of opposite edges BD is

$$\left.\begin{matrix} x+y=0 \\ x+y+z=a \end{matrix}\right]$$

Let l,m,n be the direction cosines of this line.

Since it llies on both planes.So by condition of perpendicularity

$$\left.\begin{matrix} l+m+0n=0 \\ l+m+n=0 \end{matrix}\right]$$

$$\frac{l}{1-0}=\frac{-m}{1-0}=\frac{n}{1-1} \Rightarrow \frac{l}{1}=\frac{m}{-1}=\frac{n}{0}$$

So direction ratios of line is $1,-1,0$

To find a point on this line , put $x=0$ in above equations $\left.\begin{matrix} 0+y=0 \\ 0+y+z=a \end{matrix}\right]$ $or\ y=0$ & $y+z=a \Rightarrow z=a$

So a point on this line BD is $(0,0,a)$, Hence equation of this line is

$$\frac{x}{1}=\frac{y}{-1}=\frac{z-a}{0} \quad -----(2)$$

Now we will find shortest distance between line (1) & (2)

A point on line (1) is $A(0,0,0)$

A point on line (2) is $B(0,0,a)$

Now $\overrightarrow{AB} = 0\hat{\imath} + 0\hat{\jmath} + a\hat{k}$

Let $\vec{u}$ be a vector perpendicular to both lines then $\vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$

Expanding from R_1 $\vec{u} = (0-1)\hat{\imath} - (0+1)\hat{\jmath} + (-1-1)\hat{k} \Rightarrow \vec{u} = -\hat{\imath} - \hat{\jmath} - 2\hat{k}$

Let d be the shortest distance between lines then

$$d = \frac{\overrightarrow{AB}.\vec{u}}{|\vec{u}|} = \frac{(0\hat{\imath} + 0\hat{\jmath} + a\hat{k}).(-\hat{\imath} - \hat{\jmath} - 2\hat{k})}{\sqrt{1+1+4}} \Rightarrow d = \frac{0+0-2a}{\sqrt{6}} \Rightarrow d = \frac{2a}{\sqrt{6}} \quad \textit{is required distance.}$$

Similarly we can show that the shortest distance between opposite edges AB,CD & BC,AD is also $\frac{2a}{\sqrt{6}}$

Now equation of line of shortest distance between opposite edges AC & BD is

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & -1 & -2 \end{vmatrix} = 0 = \begin{vmatrix} x & y & z-a \\ 0 & -1 & 0 \\ -1 & -1 & -2 \end{vmatrix}$$

$$x(-2-1) - y(-2-1) + z(-1+1) = 0 = x(2-0) - y(-2+0) + (z-a)(-1-1)$$

$$x(-3) - y(-3) + z(0) = 0 = x(2) - y(-2) + (z-a)(-2)$$

$$-3x + 3y + 0z = 0 = 2x + 2y - 2z + 2a$$

$$x - y = 0 = x + y - z + a$$

We see that the point $(-a,-a,-a)$ satisfies this equation , So this point lies on the line of shortest distance between AC & BD. Similarly $(-a,-a,-a)$ also lies on the other two lines of shortest distance.

Hence it lies on the intersection of all three lines of shortest distance.

Q#8: Find the shortest distance between the straight line joining the points A$(3,2,-4)$& B$(1,6,-6)$ and the straight line joining the points C$(-1,1,-2)$& D$(-3,1,-6)$.Also find equation of the line of shortest distance and coordinates of the feet of the common perpendicular.

Solution: Equation of line passing through $A(3,2,-4)$& $B(1,6,-6)$ is

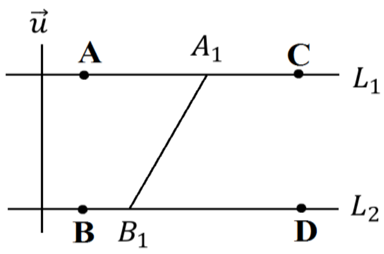
$$\frac{x-3}{1-3} = \frac{y-2}{6-2} = \frac{z+4}{-6+4} \Rightarrow \frac{x-3}{-2} = \frac{y-2}{4} = \frac{z+4}{-2} \Rightarrow \frac{x-3}{1} = \frac{y-2}{-2} = \frac{z+4}{1} \; ----(1)$$

& equation of line through $C(-1,1,-2)$& $D(-3,1,-6)$ is

$$\frac{x+1}{-3+1} = \frac{y-1}{1-1} = \frac{z+2}{-6+2} \Rightarrow \frac{x+1}{-2} = \frac{y-1}{0} = \frac{z+2}{-4} \Rightarrow \frac{x+1}{1} = \frac{y-1}{0} = \frac{z+2}{2} \; ------(2)$$

A point on line (1) is $A_1(3,2,-4)$

A point on line (2) is $B_1(-1,1,-2)$



$\overrightarrow{A_1B_1} = (-1-3)\hat{\imath} + (1-2)\hat{\jmath} + (-2+4)\hat{k} \Rightarrow \overrightarrow{A_1B_1} = -4\hat{\imath} - \hat{\jmath} + 2\hat{k}$

Let $\vec{u}$ be a vector perpendicular to both lines (1) & (2) then

$$\vec{u} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ 1 & 0 & 2 \end{vmatrix}$$

Expanding by R_1

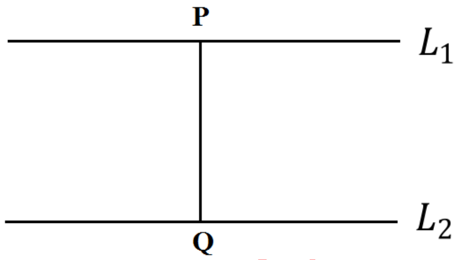
$$\vec{u} = (-4-0)\hat{\imath} - (2-1)\hat{\jmath} + (0+2)\hat{k} \implies \vec{u} = -4\hat{\imath} - \hat{\jmath} + 2\hat{k}$$

Let d be the required shortest distance between lines (1) & (2) then

$$d = \frac{\overrightarrow{A_1B_1}.\vec{u}}{|\vec{u}|} = \frac{(-4\hat{\imath}-\hat{\jmath}+2\hat{k}).(-4\hat{\imath}-\hat{\jmath}+2\hat{k})}{\sqrt{16+1+4}} \implies d = \frac{16+1+4}{\sqrt{21}} \implies d = \frac{21}{\sqrt{21}} \implies \boldsymbol{d = \sqrt{21}}$$

As lines are

$$\frac{x-3}{1} = \frac{y-2}{-2} = \frac{z+4}{1} = t \quad ------(1)$$



$$\& \quad \frac{x+1}{1} = \frac{y-1}{0} = \frac{z+2}{2} = s \quad --------(2)$$

Any point on line (1) is $P(3+t, 2-2t, -4+t)$

Any point on line (2) is $Q(-1+s, 1, -2+2s)$

Direction ratios of PQ are $3+t+1-s, 2-2t-1, -4+t+2-2s$

Direction ratios of line PQ are $t-s+4, -2t+1, t-2s-2$

Suppose PQ is line of shortest distance then PQ is perpendicular to both lines (1) & (2)

So by condition of perpendicularity

$$\left.\begin{matrix} 1(t-s+4) - 2(-2t+1) + 1(t-2s-2) = 0 \\ 1(t-s+4) + 0(-2t+1) + 2(t-2s-2) = 0 \end{matrix}\right]$$

$$\left.\begin{matrix} t-s+4+4t-2+t-2s-2 = 0 \\ t-s+4+2t-4s-4 = 0 \end{matrix}\right] \implies \left.\begin{matrix} 6t-5s = 0 \\ 3t-5s = 0 \end{matrix}\right] \implies \boldsymbol{t = 0} \quad \& \quad s = 0$$

So coordinates of feet of perpendicular P & Q are $P(3,2,-4)$ & $Q(-1,1,-2)$.

Now equation of common perpendicular PQ is

$$\frac{x-3}{-1-3} = \frac{y-2}{1-2} = \frac{z+4}{-2+4} \implies \frac{x-3}{-4} = \frac{y-2}{-1} = \frac{z+4}{2} \quad or \quad \frac{x-3}{4} = \frac{y-2}{1} = \frac{z+4}{-2}$$

Checked by: Sir Hameed ullah (***hameedmath2017 @ gmail.com)***

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