Exercise #8.5

Q#1: Show that the straight line $\frac{x+3}{2}=\frac{y-4}{-7}=\frac{z}{3}$ is parallel to the plane $4x+2y+2z=9$.

Solution:

Given equation of line and plane

$$\frac{x+3}{2}=\frac{y-4}{-7}=\frac{z}{3} \quad ----(L)$$

$$\& \quad 4x+2y+2z=9 \quad ----(P)$$

Direction ratios of line L are $a_1=2, \quad b_1=-7, \quad c_1=3$

Now direction ratios of normal vector of plane P are $a_2=4, \quad b_2=2, \quad c_2=2$

We have to prove P ∥ L

If P ∥ L then normal vector of plane P is perpendicular to the line L. we have

$$a_1a_2+b_1b_2+c_1c_2=0$$

$$(2)(4)+(-7)(2)+(3)(2)=0$$

$$8-14+6=0$$

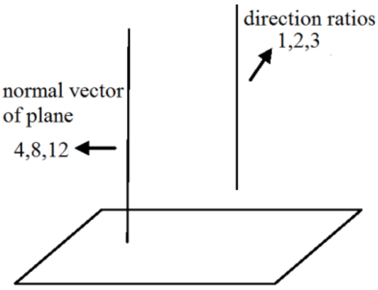
$$0=0$$

Hence proved that the line L and plane P are parallel.

Q#2: Show that the straight line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is perpendicular to the plane $4x+8y+12z+19=0$.

Solution:

Given equation of line and plane are



$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \quad ----(L)$$

$$4x+8y+12z+19=0 ----(P)$$

Direction ratios of line L are $a_1=1, b_1=2, c_1=3$

and direction ratios of normal vector of plane P are $a_2=4, b_2=8, c_2=12$

We have to prove P ⊥ L

If P ⊥ L then normal vector of plane P is parallel to the line L. we have

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

$$\frac{1}{4}=\frac{2}{8}=\frac{3}{12}$$

$$\Rightarrow \frac{1}{4}=\frac{1}{4}=\frac{1}{4}$$

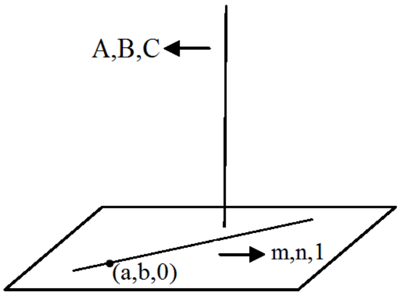
Hence proved that the line L and plane P are perpendicular.

Q#3:Find the condition that the straight line $x = mz + a, y = nz + b$ may lie the plane $Ax + By + Cz + D = 0$.

Solution: Given equations of line and plane are

$$x = mz + a, \qquad y = nz + b$$

$$\& \quad Ax + By + Cz + D = 0 \quad ----(P)$$



From given equation of line is

$$x = mz + a \quad \Rightarrow z = \frac{x-a}{m} \quad \dots\dots(1)$$

$$y = nz + b \quad \Rightarrow z = \frac{y-b}{n} \quad \dots\dots(2)$$

$$\Rightarrow \frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1}$$

$$Let \quad \frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = t \quad (say)$$

Then
$$\begin{cases} x = a + mt \\ y = b + nt \\ z = t \end{cases}$$

Given that line L lies on the plane P then each point of line lies on plane. Hence point $(a + mt, b + nt, t)$ lies on plane.So equation (P) becomes

$$\Rightarrow A(a + mt) + B(b + nt) + Ct + D = 0$$

$$\Rightarrow Aa + Amt + Bb + Bnt + Ct + D = 0$$

$$\Rightarrow [Aa + Bb + D] + t[Am + Bn + C] = 0$$

This eq. must be satisfied for every value of

$$\Rightarrow Aa + Bb + D = 0 \qquad \& \qquad Am + Bn + C = 0$$

This is the required condition for which the given line lies on the given plane.

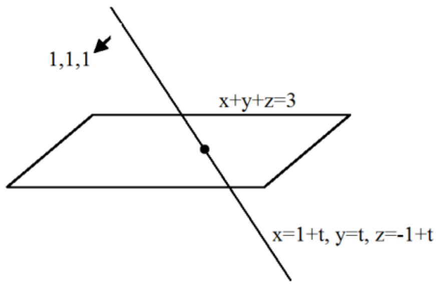
Q#4: Determine the point , if any, common to the straight line $x = 1 + t, y = t, z = -1 + t$ and the Plane $x + y + z = 3$.

Solution:

Given equation of straight line and plane are

$$\left.\begin{array}{l} x = 1 + t \\ y = t \\ z = -1 + t \end{array}\right\} \quad ----(L)$$

$$\& \quad x + y + z = 3 \quad ----(P)$$



Let a point $(1 + t, t, -1 + t)$ is common point of line L and plane P.

This point satisfies the equation of the plane then

$$\Rightarrow 1 + t + t + (-1 + t) = 3$$

$$\Rightarrow 1 + t + t - 1 + t = 3 \qquad \Rightarrow 3t = 3 \qquad \Rightarrow \boldsymbol{t = 1}$$

Hence the common point of the line L and plane P is $(1 + t, t, -1 + t) = (1 + 1, 1, -1 + 1,) = (2,1,0)$.

Q#5: Find an equation of the plane through the point (x_1,y_1,z_1) and through the straight line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

Solution: Given equation of straight line

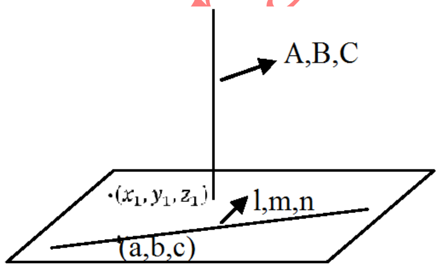
$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

Let $A,B\,\&C$ are the direction ratios of normal vector of the plane Then equation of the plane through given line will be written as

$$A(x-a)+B(y-b)+C(z-c)=0\quad ----(1)$$

$$Al+Bm+Cn=0\quad ----(2)$$

Required equation of the plane passes through the point (x_1,y_1,z_1). then eq. (1) will become



$$A(x_1-a)+B(y_1-b)+C(z_1-c)=0\quad ----(3)$$

Eliminating A,B &C from eq. (1),(2) &(3) we have

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ x_1-a & y_1-b & z_1-c \end{vmatrix}=0$$

or $$\sum(x-a)\begin{vmatrix} m & n \\ y_1-b & z_1-c \end{vmatrix}=0$$

$$\sum(x-a)\,[m(z_1-c)-n(y_1-b)]=0$$

This is the required equation of plane.

Q#6: Find an equation of the plane passing through straight line $x+2z=4, y-z=8$ and parallel to the straight line

$$\frac{x-3}{2}=\frac{y+4}{3}=\frac{z-7}{4}$$

Solution: Given equation of straight line

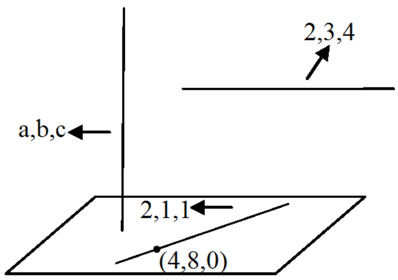
$$x+2z=4, y-z=8\quad ----L_1$$

$$\frac{x-3}{2}=\frac{y+4}{3}=\frac{z-7}{4}\quad ----L_2$$

Consider line L_1 $\quad x+2z=4\ ,\quad y-z=8$

$$\Rightarrow 2z=4-x\ ,\qquad z=y-8$$

$$\Rightarrow z=\frac{x-4}{-2}\ ---(1)\ ,\qquad z=\frac{y-8}{1}\ ---(2)$$



Equating (1) & (2)

$$\frac{x-4}{-2}=\frac{y-8}{1}=\frac{z}{1}\qquad (symmetric\ form)$$

Let A,B &C are direction ratios of normal vector of the plane

The equation will be written as

$$A(x-4)+B(y-8)+C(z-0)=0\quad ----(1)$$

Where $$A(-2)+B(1)+C(1)=0\quad ----(2)$$

As required equation of the plane is parallel to the line L_2

$$A(2)+B(3)+C(4)=0\quad ----(3)$$

Eliminating A,B &C from eq.(1),(2) &(3) we have

$$\begin{vmatrix} x-4 & y-8 & z \\ -2 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$(x-4)\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - (y-8)\begin{vmatrix} -2 & 1 \\ 2 & 4 \end{vmatrix} + z\begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} = 0$$

$$\Rightarrow (x-4)(1) - (y-8)(-10) + z(-8) = 0$$

$$\Rightarrow x - 4 - (y-8)(-10) - 8z = 0$$

$$\Rightarrow x - 4 + 10y - 80 - 8z = 0$$

$$\Rightarrow x + 10y - 8z - 84 = 0$$ is required equation of the plane.

Q#7: Find an equation of the plane passing through the point (α, β, γ) and parallel to each of the straight lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{and} \quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$$

Solution:

Let equation of the plane passes through the point (α, β, γ) with direction ratios A,B &C

$$(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \quad ----(1)$$

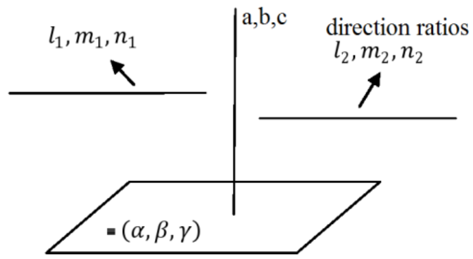
Equation of the plane is parallel to the given lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad ----L_1$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad ----L_2$$

We know that normal vectors of

the plane is perpendicular to lines L_1 & L_2



Therefore,

$$Al_1 + Bm_1 + Cn_1 = 0 \quad ----(2)$$

$$Al_2 + Bm_2 + Cn_2 = 0 \quad ----(3)$$

Now eliminating A,B &C from eq.(1),(2) &(3) we have

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$\sum(x-\alpha)\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \sum(x-\alpha)(m_1 n_2 - m_2 n_1) = 0$$

This is the required equation of plane.

Q#8: Find an equation of the plane through the straight line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ and parallel to the straight line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

Solution: Given equations of the straight lines are

$$ax+by+cz+d=0=a'x+b'y+c'z+d' \quad ----L_1$$

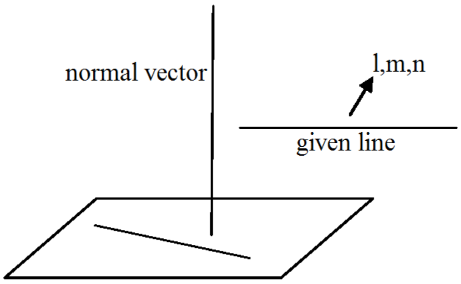
& $$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ------------ \quad L_2$$

Let equation of the plane through the straight line $ax+by+cz+d=0=a'x+b'y+c'z+d'$ can be written as

$$(ax+by+cz+d)+k(a'x+b'y+c'z+d')=0 \quad ----(P)$$

$$ax+by+cz+d+a'kx+b'ky+c'kz+d'k=0$$

$$(a+a'k)x+(b+b'k)y+(c+c'k)z+(d+d'k)=0$$



Here $A=(a+a'k)$, $B=(b+b'k)$, $C=(c+c'k)$

are the direction ratios of the normal vector of the plane.

This plane is parallel to the line L_2

Then normal vector of the plane is perpendicular to the line L_2

$$\Rightarrow (a+a'k)l+(b+b'k)m+(c+c'k)n=0$$

$$\Rightarrow al+a'kl+bm+b'km+cn+c'kn=0$$

$$\Rightarrow a'kl+b'km+c'kn+al+bm+cn=0$$

$$\Rightarrow k(a'l+b'm+c'n)+al+bm+cn=0$$

$$\Rightarrow k(a'l+b'm+c'n)=-(al+bm+cn)$$

$$\Rightarrow k=\frac{-(al+bm+cn)}{(a'l+b'm+c'n)}$$

Using value of k in equation P

$$(ax+by+cz+d)-\frac{(al+bm+cn)}{(a'l+b'm+c'n)}(a'x+b'y+c'z+d')=0$$

$$(a'l+b'm+c'n)(ax+by+cz+d)-(al+bm+cn)(a'x+b'y+c'z+d')=0$$

$$(a'l+b'm+c'n)(ax+by+cz+d)=(al+bm+cn)(a'x+b'y+c'z+d')$$

This is the required equation of the plane.

Q#9: Prove that the straight lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplanar.

Solution:

Given equations of straight lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad ----L_1$$

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad ----L_2$$

We have to prove that these straight lines are coplanar.

As we know that the straight lines $\frac{x-a_1}{l_1}=\frac{y-b_1}{m_1}=\frac{z-c_1}{n_1}$ & $\frac{x-a_2}{l_2}=\frac{y-b_2}{m_2}=\frac{z-c_2}{n_2}$ are coplanar if

$$\begin{vmatrix} a_2-a_1 & b_2-b_1 & c_2-c_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

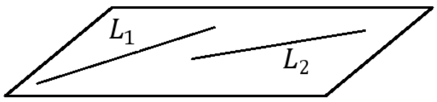
From L_1 & L_2 $\quad we\ have \quad (a_1,b_1,c_1)=(1,2,3)$ & $(a_2,b_2,c_2)=(2,3,4)$

$l_1=2,\quad m_1=3\quad \&\quad n_1=4\qquad for\ line\ L_1$

$l_2=3,\quad m_2=4\quad \&\quad n_2=5\qquad for\ line\ L_2$

Then

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$$



$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0$$

Expanding by R_1

$$\Rightarrow 1(15-16)-1(10-12)+1(8-9)=0$$

$$\Rightarrow \quad 1(-1)-1(-2)+1(-1)=0$$

$$\Rightarrow \quad -1+2-1=0$$

$$\Rightarrow \quad 0=0$$

Hence given straight lines are coplanar.

Q#10: prove that the straight lines $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ *and* $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ intersect.Also find the point of intersection and the plane through them.

Solution: Given equations of straight lines

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}----L_1$$

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}\quad ----L_2$$

Let $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}=t\ (say)$ & $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}=s\ (say)$

Now parametric equations of given straight lines are

$$\left.\begin{aligned} x&=1+2t \\ y&=-1-3t \\ z&=-10+8t \end{aligned}\right\} ------(L_1) \qquad \& \qquad \left.\begin{aligned} x&=4+s \\ y&=-3-4s \\ z&=-1+7s \end{aligned}\right\} ------(L_2)$$

Any point on line L_1 is $(1+2t,-1-3t,-10+8t)$

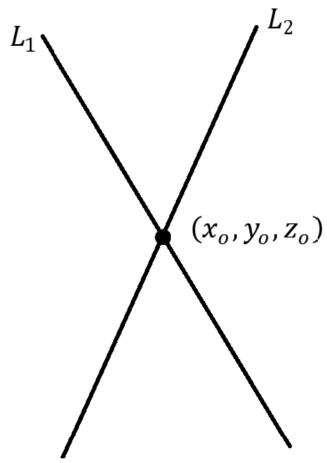
Any point on line L_2 is $(4+s,-3-4s,-1+7s)$

Let a point $P(x_o,y_o,z_o)$ is a point of intersection of lines L_1 & L_2.So this point will satisfy equations (1) &(2).

$x_o=1+2t \qquad\qquad x_o=4+s$

$y_o=-1-3t \qquad\qquad y_o=-3-4s$

$z_o=-10+8t \qquad\qquad z_o=-1+7s$



Equating above

$$1 + 2t = 4 + s \quad \Rightarrow \quad 2t - s = 3 \quad ---(3)$$

$$-1 - 3t = -3 - 4s \quad \Rightarrow \quad 3t - 4s = 2 \quad ---(4)$$

$$-10 + 8t = -1 + 7s \quad \Rightarrow \quad 8t - 7s = 9 \quad ---(5)$$

Now solving eq.(3) &(4)

Multiplying eq.(3) by 4 and subtracting eq.(3) &(4)

$$8t - 4s - 3t + 4s = 12 - 2 \quad \Rightarrow 5t = 10 \quad \Rightarrow \boldsymbol{t = 2}$$

Putting value of $\boldsymbol{t}$ in eq.(3)

$$2(2) - s = 3 \quad \Rightarrow 4 - s = 3 \quad \Rightarrow \boldsymbol{s = 1}$$

we see that these values of **t** & **s** satisfies eq.(5)

hence given lines intersect each other and their point of intersection is $(x_o, y_o, z_o) = (5, -7, 6)$

Now we have to find the equation of plane through these lines.

A plane through these lines must be contain the point of intersection of these lines.

If A,B & C are direction ratios of normal vector of the plane then equation of plane through $(5, -7, 6)$ is

$$A(x-5) + B(y+7) + C(z-6) = 0 \quad ----(I)$$

As this plane contain both lines , so

$$2A - 3B + 8C = 0 \quad ----(II)$$

$$A - 4B + 7C = 0 \quad ----(III)$$

Eliminating A,B &C from eq.(I) (II)& (III) we have

$$\begin{vmatrix} x-5 & y+7 & z-6 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-5)(-21+32) - (y+7)(14-8) + (z-6)(-8+3) = 0$$

$$\Rightarrow \quad (x-5)(11) - (y+7)(6) + (z-6)(-5) = 0$$

$$\Rightarrow \quad 11x - 55 - 6y - 42 - 5z + 30 = 0$$

$$\Rightarrow \quad 11x - 6y - 5z - 67 = 0 \quad \textit{is required equation of plane.}$$

Checked by: Sir Hameed ullah ***(hameedmath2017 @ gmail.com)***

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