Q#1: Show that the straight line \( \frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3} \) is parallel to the plane \( 4x + 2y + 2z = 9 \).

Solution:

Given equation of line and plane

\[
\frac{x + 3}{2} = \frac{y - 4}{-7} = \frac{z}{3} \quad -(L)
\]
\[
& 4x + 2y + 2z = 9 \quad -(P)
\]

Direction ratios of line L are \( a_1 = 2, \ b_1 = -7, \ c_1 = 3 \)

Now direction ratios of normal vector of plane P are \( a_2 = 4, \ b_2 = 2, \ c_2 = 2 \)

We have to prove \( P \parallel L \)

If \( P \parallel L \) then normal vector of plane P is perpendicular to the line L. we have

\[
\begin{align*}
\frac{a_1a_2}{b_1b_2} + \frac{b_1b_2}{c_1c_2} + \frac{c_1c_2}{a_1a_2} &= 0 \\
\frac{(2)(4)}{(-7)(2)} + \frac{(-7)(2)}{(3)(2)} + \frac{(3)(2)}{(2)(4)} &= 0 \\
8 - 14 + 6 &= 0 \\
0 &= 0
\end{align*}
\]

Hence proved that the line L and plane P are parallel.

Q#2: Show that the straight line \( \frac{x}{1} = \frac{y}{2} = \frac{z}{3} \) is perpendicular to the plane \( 4x + 8y + 12z + 19 = 0 \).

Solution:

Given equation of line and plane are

\[
\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \quad -(L)
\]
\[
4x + 8y + 12z + 19 = 0 \quad -(P)
\]

Direction ratios of line L are \( a_1 = 1, b_1 = 2, c_1 = 3 \)

and direction ratios of normal vector of plane P are \( a_2 = 4, b_2 = 8, c_2 = 12 \)

We have to prove \( P \perp L \)

If \( P \perp L \) then normal vector of plane P is parallel to the line L. we have

\[
\begin{align*}
\frac{a_1}{a_2} &= \frac{b_1}{b_2} = \frac{c_1}{c_2} \\
\frac{1}{4} &= \frac{2}{8} = \frac{3}{12}
\end{align*}
\]

\[\Rightarrow \frac{1}{4} = \frac{1}{4} = \frac{1}{4}\]

Hence proved that the line L and plane P are perpendicular.
Q#3: Find the condition that the straight line \( x = mz + a, y = nz + b \) may lie the plane \( Ax + By + Cz + D = 0 \).

**Solution:** Given equations of line and plane are

\[
x = mz + a, \quad y = nz + b
\]

\& \quad Ax + By + Cz + D = 0 \quad \cdots \cdots (P)

From given equation of line is

\[
x = mz + a \Rightarrow z = \frac{x-a}{m} \quad \cdots \cdots (1)
\]

\[
y = nz + b \Rightarrow z = \frac{y-b}{n} \quad \cdots \cdots (2)
\]

\[
\Rightarrow \frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1}
\]

Let \( \frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = t \) (say)

\[
\begin{cases}
\quad x = a + mt \\
\quad y = b + nt \\
\quad z = t
\end{cases}
\]

Given that line L lies on the plane P then each point of line lies on plane. Hence point \((a + mt, b + nt, t)\) lies on plane. So equation \((P)\) becomes

\[
\Rightarrow [Aa + Bb + D] + t[Am + Bn + C] = 0
\]

This eq. must be satisfied for every value of

\[
\Rightarrow Aa + Bb + D = 0 \quad \& \quad Am + Bn + C = 0
\]

This is the required condition for which the given line lies on the given plane.

Q#4: Determine the point, if any, common to the straight line \( x = 1 + t, y = t, z = -1 + t \) and the Plane \( x + y + z = 3 \).

**Solution:**

Given equation of straight line and plane are

\[
\begin{align*}
\quad x &= 1 + t \\
\quad y &= t \\
\quad z &= -1 + t
\end{align*}
\]

\& \quad x + y + z = 3 \quad \cdots \cdots (P)

Let a point \((1 + t, t, -1 + t)\) is common point of line L and plane P.

This point satisfies the equation of the plane then

\[
\Rightarrow 1 + t + t + (-1 + t) = 3
\]

\[
\Rightarrow 1 + t + t - 1 + t = 3 \quad \Rightarrow 3t = 3 \quad \Rightarrow t = 1
\]

Hence the common point of the line L and plane P is \((1 + t, t, -1 + t) = (1 + 1, 1, -1 + 1) = (2, 1, 0)\).
Q#5: Find an equation of the plane through the point \((x_1,y_1,z_1)\) and through the straight line
\[
\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}
\]

Solution: Given equation of straight line
\[
\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}
\]
Let \(A, B \& C\) are the direction ratios of normal vector of the plane Then equation of the plane through given line will be written as
\[
A(x-a) + B(y-b) + C(z-c) = 0 \quad -(1)
\]

Required equation of the plane passes through the point \((x_1,y_1,z_1)\), then eq. (1) will become
\[
A(x_1-a) + B(y_1-b) + C(z_1-c) = 0 \quad -(3)
\]
Eliminating \(A,B \& C\) from eq. (1),(2) & (3) we have
\[
\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}
\]
This is the required equation of plane.

Q#6: Find an equation of the plane passing through straight line \(x + 2z = 4, y - z = 8\) and parallel to the straight line
\[
\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}
\]

Solution: Given equation of straight line
\[
\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4} \quad L_1
\]
\[
\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4} \quad L_2
\]
Consider line \(L_1\)
\[
\frac{x+2z}{2} = \frac{y-z}{8} \quad \Rightarrow 2z = 4 - x, \quad y-z = 8
\]
\[
\Rightarrow z = \frac{x-4}{2} \quad -(1), \quad z = \frac{y-8}{1} \quad -(2)
\]
Equating (1) & (2)
\[
\frac{x-4}{2} = \frac{y-8}{1} = \frac{z}{1} \quad \text{(symmetric form)}
\]
Let \(A,B \& C\) are direction ratios of normal vector of the plane
The equation will be written as
\[
A(x-4) + B(y-8) + C(z-0) = 0 \quad -(1)
\]

Where
\[
A(-2) + B(1) + C(1) = 0 \quad -(2)
\]
As required equation of the plane is parallel to the line \(L_2\)
\[
A(2) + B(3) + C(4) = 0 \quad -(3)
\]
Eliminating A, B & C from eq. (1), (2) & (3) we have

\[
\begin{vmatrix}
  x - 4 & y - 8 & z \\
  -2 & 1 & 1 \\
  2 & 3 & 4
\end{vmatrix} = 0
\]

\[
(x - 4) \begin{vmatrix}
  1 & 1 \\
  4 & 1
\end{vmatrix} - (y - 8) \begin{vmatrix}
  -2 & 1 \\
  2 & 3
\end{vmatrix} + z \begin{vmatrix}
  -2 & 1 \\
  2 & 3
\end{vmatrix} = 0
\]

\[
\Rightarrow (x - 4)(1) - (y - 8)(-10) + z(-8) = 0
\]

\[
\Rightarrow x - 4 + 10y - 80 - 8z = 0
\]

\[
\Rightarrow x + 10y - 8z - 84 = 0
\]

is required equation of the plane.

**Q#7:** Find an equation of the plane passing through the point \((\alpha, \beta, \gamma)\) and parallel to each of the straight lines

\[
\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \text{and} \quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}
\]

**Solution:**

Let equation of the plane passes through the point \((\alpha, \beta, \gamma)\) with direction ratios A, B & C

\[
(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0
\]

Equation of the plane is parallel to the given lines

\[
\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \text{and} \quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}
\]

We know that normal vectors of

the plane is perpendicular to lines \(L_1\) & \(L_2\)

Therefore,

\[
Al_1 + Bl_2 + Cn_1 = 0
\]

\[
Al_2 + Bl_2 + Cn_2 = 0
\]

Now eliminating A, B & C from eq. (1), (2) & (3) we have

\[
\begin{vmatrix}
  x - \alpha & y - \beta & z - \gamma \\
  l_1 & m_1 & n_1 \\
  l_2 & m_2 & n_2
\end{vmatrix} = 0
\]

or

\[
\sum(x - \alpha) \begin{vmatrix}
  m_1 & n_1 \\
  m_2 & n_2
\end{vmatrix} = 0
\]

\[
\Rightarrow \sum(x - \alpha)(m_1n_2 - m_2n_1) = 0
\]

This is the required equation of plane.
Q#8: Find an equation of the plane through the straight line 
\[ ax + by + cz + d = 0 \]
and parallel to the straight line 
\[ \frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \]

**Solution:** Given equations of the straight lines are

\[ ax + by + cz + d = 0 = a'x + b'y + c'z + d' \]

&

\[ \frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \]

Let equation of the plane through the straight line 
\[ ax + by + cz + d = 0 = a'x + b'y + c'z + d' \]
can be written as

\[ (ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \]

\[ (a + a'k)x + (b + b'k)y + (c + c'k)z + (d + d'k) = 0 \]

Here \( A = (a + a'k) \), \( B = (b + b'k) \), \( C = (c + c'k) \)
are the direction ratios of the normal vector of the plane.

This plane is parallel to the line \( L_2 \)

Then normal vector of the plane is perpendicular to the line \( L_2 \)

\[ (a + a'k)l + (b + b'k)m + (c + c'k)n = 0 \]

\[ \Rightarrow k(a' l + b' m + c' n) = -(a + b m + c n) \]

Using value of \( k \) in equation \( P \)

\[ (ax + by + cz + d) - \frac{(a + b m + c n)}{(a' l + b' m + c' n)}(a'x + b'y + c'z + d') = 0 \]

\[ (a' l + b' m + c' n)(ax + by + cz + d) - (a + b m + c n)(a'x + b'y + c'z + d') = 0 \]

\[ (a' l + b' m + c' n)(ax + by + cz + d) - (a + b m + c n)(a'x + b'y + c'z + d') = 0 \]

This is the required equation of the plane.

Q#9: Prove that the straight lines 
\[ \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \] and \( \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \) are coplanar.

**Solution:**

Given equations of straight lines are

\[ \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = -L_1 \]

\[ \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} = -L_2 \]

We have to prove that these straight lines are coplanar.
As we know that the straight lines \( \frac{x-a_1}{l_1} = \frac{y-b_1}{m_1} = \frac{z-c_1}{n_1} \) \& \( \frac{x-a_2}{l_2} = \frac{y-b_2}{m_2} = \frac{z-c_2}{n_2} \) are coplanar if

\[
\begin{vmatrix}
  a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\
  l_1 & m_1 & n_1 \\
  l_2 & m_2 & n_2 
\end{vmatrix} = 0
\]

From \( L_1 \& L_2 \) we have \((a_1, b_1, c_1) = (1,2,3) \) \& \((a_2, b_2, c_2) = (2,3,4)\)

\( l_1 = 2, \  m_1 = 3 \ \& \ n_1 = 4 \) for line \( L_1 \)

\( l_2 = 3, \  m_2 = 4 \ \& \ n_2 = 5 \) for line \( L_2 \)

Then
\[
\begin{vmatrix}
  2 - 1 & 3 - 2 & 4 - 3 \\
  2 & 3 & 4 \\
  3 & 4 & 5 
\end{vmatrix} = 0
\]

Expanding by \( R_1 \)
\[
\Rightarrow 1(15 - 16) - 1(10 - 12) + 1(8 - 9) = 0
\]
\[
\Rightarrow 1(-1) - 1(-2) + 1(-1) = 0
\]
\[
\Rightarrow -1 + 2 - 1 = 0
\]
\[
\Rightarrow 0 = 0
\]

Hence given straight lines are coplanar.

Q#10: prove that the straight lines \( \frac{x-4}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \) and \( \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \) intersect. Also find the point of intersection and the plane through them.

**Solution:** Given equations of straight lines \( L_1 \) and \( L_2 \)

Let
\[
\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = t \text{ (say)}
\]
\[
\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = s \text{ (say)}
\]

Now parametric equations of given straight lines are

\[
\begin{align*}
  x & = 1 + 2t \\
  y & = -1 - 3t \\
  z & = -10 + 8t
\end{align*}
\]

\[
\begin{align*}
  x & = 4 + s \\
  y & = -3 - 4s \\
  z & = -1 + 7s
\end{align*}
\]

Any point on line \( L_1 \) is \((1 + 2t, -1 - 3t, -10 + 8t)\)

Any point on line \( L_2 \) is \((4 + s, -3 - 4s, -1 + 7s)\)

Let a point \( P(x_0, y_0, z_0) \) is a point of intersection of lines \( L_1 \& L_2 \). So this point will satisfy equations (1) & (2).

\[
\begin{align*}
  x_0 & = 1 + 2t \\
  y_0 & = -1 - 3t \\
  z_0 & = -10 + 8t
\end{align*}
\]

\[
\begin{align*}
  x_0 & = 4 + s \\
  y_0 & = -3 - 4s \\
  z_0 & = -1 + 7s
\end{align*}
\]
Equating above

\[ 1 + 2t = 4 + s \quad \Rightarrow \quad 2t - s = 3 \quad \text{--- (3)} \]
\[ -1 - 3t = -3 - 4s \quad \Rightarrow \quad 3t - 4s = 2 \quad \text{--- (4)} \]
\[ -10 + 8t = -1 + 7s \quad \Rightarrow \quad 8t - 7s = 9 \quad \text{--- (5)} \]

Now solving eq.(3) & (4)

Multiplying eq.(3) by 4 and subtracting eq.(3) & (4)

\[ 8t - 4s - 3t + 4s = 12 - 2 \quad \Rightarrow \quad 5t = 10 \quad \Rightarrow \quad t = 2 \]

Putting value of \( t \) in eq.(3)

\[ 2(2) - s = 3 \quad \Rightarrow \quad 4 - s = 3 \quad \Rightarrow \quad s = 1 \]

we see that these values of \( t \) & \( s \) satisfies eq.(5)

hence given lines intersect each other and their point of intersection is \((x_o, y_o, z_o) = (5, -7, 6)\)

Now we have to find the equation of plane through these lines.

A plane through these lines must contain the point of intersection of these lines.

If \( A, B \) & \( C \) are direction ratios of normal vector of the plane then equation of plane through \((5, -7, 6)\) is

\[ A(x - 5) + B(y + 7) + C(z - 6) = 0 \quad \text{--- (I)} \]

As this plane contain both lines, so

\[ 2A - 3B + 8C = 0 \quad \text{--- (II)} \]
\[ A - 4B + 7C = 0 \quad \text{--- (III)} \]

Eliminating \( A, B \) & \( C \) from eq.(I) (II) & (III) we have

\[ \begin{vmatrix}
  x - 5 & y + 7 & z - 6 \\
  2 & -3 & 8 \\
  1 & -4 & 7
\end{vmatrix} = 0 
\]

\[ \Rightarrow (x - 5)(-21 + 32) - (y + 7)(14 - 8) + (z - 6)(-8 + 3) = 0 
\]
\[ \Rightarrow \quad (x - 5)(11) - (y + 7)(6) + (z - 6)(-5) = 0 
\]
\[ \Rightarrow \quad 11x - 55 - 6y - 42 - 5z + 30 = 0 
\]
\[ \Rightarrow \quad 11x - 6y - 5z - 67 = 0 \quad \text{is required equation of plane.} 
\]

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