Exercise #8.4

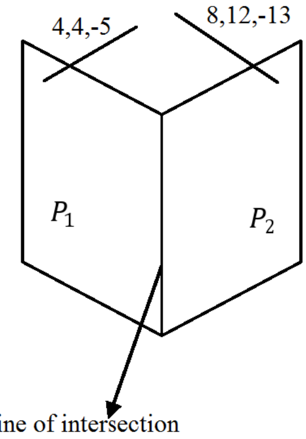
Q#1: Find that the planes $4x+4y-5z=12$, $8x+12y-13z=32$ intersect and equations of their line of intersection can be written in the form $\frac{x-1}{2}=\frac{y-2}{2}=\frac{z}{4}$.

Solution:

Given equations of planes

$$4x+4y-5z=12 \text{ --------- } (1)$$

$$8x+12y-13z=32 \text{ --------- } (2)$$



Direction ratios of normal vector of plane (1) are $a_1=4$, $b_1=4$, $c_1=-5$

Direction ratios of normal vector of plane (2) are $a_2=8$, $b_2=12$, $c_2=-13$

As direction ratios of both normal vectors are not proportional.

Hence both normal vectors are not parallel. So given planes are not parallel.

Hence given planes intersect.

Now we have to find equation of their line of intersection.

For this put $z=0$ in eq. (1) & eq. (2)

$$(1) \Rightarrow \quad 4x+4y-5(0)=12 \quad \Rightarrow \quad 4x+4y-12=0 \quad \Rightarrow \quad x+y-3=0 \text{ -------- } (3)$$

$$(2) \Rightarrow \quad 8x+12y-13(0)=32 \quad \Rightarrow 8x+12y-32=0 \quad \Rightarrow \quad 2x+3y-8=0 \text{ ------- } (4)$$

$$\frac{x}{\begin{vmatrix}1 & -3\\ 3 & -8\end{vmatrix}}=\frac{-y}{\begin{vmatrix}1 & -3\\ 2 & -8\end{vmatrix}}=\frac{1}{\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix}} \Rightarrow \frac{x}{-8+9}=\frac{-y}{-8+6}=\frac{1}{3-2} \Rightarrow \frac{x}{1}=\frac{-y}{-2}=\frac{1}{1} \quad \Rightarrow \boldsymbol{x=1, y=2}$$

Hence P(1,2,0) is a first point on line of intersection.

Again put $x=0$ in eq. (1) & eq. (2)

$$(1) \Rightarrow \quad 4(0)+4y-5z=12 \quad \Rightarrow \quad 4y-5z-12=0 \quad \Rightarrow \quad 4y-5z-12=0 \text{ -------- } (5)$$

$$(2) \Rightarrow \quad 8(0)+12y-13z=32 \quad \Rightarrow 12y-13z-32=0 \quad \Rightarrow 12y-13z-32=0 \text{ ------- } (6)$$

$$\frac{y}{\begin{vmatrix}-5 & -12\\ -13 & -32\end{vmatrix}}=\frac{-z}{\begin{vmatrix}4 & -12\\ 12 & -32\end{vmatrix}}=\frac{1}{\begin{vmatrix}4 & -5\\ 12 & -13\end{vmatrix}} \Rightarrow \frac{y}{160-156}=\frac{-z}{-128+144}=\frac{1}{-52+60} \Rightarrow \frac{y}{4}=\frac{-z}{16}=\frac{1}{8} \quad \Rightarrow \boldsymbol{y=\frac{1}{2}, z=-2}$$

Hence $Q\left(0,\frac{1}{2},-2\right)$ is second point on the line of intersection.

Hence equation of line through $P(1,2,0)$ & $Q\left(0,\frac{1}{2},-2\right)$ is

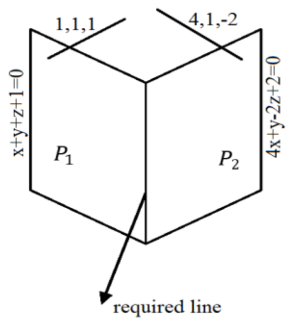
$$\frac{x-1}{0-1}=\frac{y-2}{\frac{1}{2}-2}=\frac{z-0}{-2-0} \quad \Rightarrow \quad \frac{x-1}{-1}=\frac{y-2}{\frac{-3}{2}}=\frac{z}{-2} \quad \Rightarrow \quad \frac{x-1}{1}=\frac{y-2}{\frac{3}{2}}=\frac{z}{2}$$

Dividing by 2 we have $\Rightarrow \frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}$ *is a required equation.*

Q#2: Find a symmetric form for the line $x+y+z+1=0=4x+y-2z+2$.

Solution:

Given equation of line is



$$x+y+z+1=0 \text{ ------------ } (1)$$

$$4x+y-2z+2=0 \text{ ----------- } (2)$$

For this put $z=0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} x+y+1=0 \\ 4x+y+2=0 \end{aligned}\right\}$$

$$\frac{x}{\begin{vmatrix}1&1\\1&2\end{vmatrix}}=\frac{-y}{\begin{vmatrix}1&1\\4&2\end{vmatrix}}=\frac{1}{\begin{vmatrix}1&1\\4&1\end{vmatrix}} \implies \frac{x}{2-1}=\frac{-y}{2-4}=\frac{1}{1-4} \implies \frac{x}{1}=\frac{-y}{-2}=\frac{1}{-3} \implies \boldsymbol{x=-\frac{1}{3}, y=-\frac{2}{3}}$$

Hence $P(-\frac{1}{3},-\frac{2}{3},0)$ is a first point on line of intersection.

Again put $x=0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} y+z+1=0 \\ y-2z+2=0 \end{aligned}\right\}$$

$$\frac{y}{\begin{vmatrix}1&1\\-2&2\end{vmatrix}}=\frac{-z}{\begin{vmatrix}1&1\\1&2\end{vmatrix}}=\frac{1}{\begin{vmatrix}1&1\\1&-2\end{vmatrix}} \implies \frac{y}{2+2}=\frac{-z}{2-1}=\frac{1}{-2-1} \implies \frac{y}{4}=\frac{-z}{1}=\frac{1}{-3} \implies \boldsymbol{y=\frac{-4}{3}, z=\frac{1}{3}}$$

So $Q\left(0,\frac{-4}{3},\frac{1}{3}\right)$ is second point on the line of intersection.

Then the equation of line through $P(-\frac{1}{3},-\frac{2}{3},0)$ and $Q\left(0,\frac{-4}{3},\frac{1}{3}\right)$ is

$$\frac{x-(-\frac{1}{3})}{0-(-\frac{1}{3})}=\frac{y-(-\frac{2}{3})}{-\frac{4}{3}-(-\frac{2}{3})}=\frac{z-0}{\frac{1}{3}-0} \implies \frac{x+\frac{1}{3}}{0+\frac{1}{3}}=\frac{y+\frac{2}{3}}{-\frac{4}{3}+\frac{2}{3}}=\frac{z}{\frac{1}{3}} \implies \frac{x+\frac{1}{3}}{1/3}=\frac{y+\frac{2}{3}}{-2/3}=\frac{z}{1/3}$$

Dividing by 3 we have

$$\implies \frac{x+\frac{1}{3}}{1}=\frac{y+\frac{2}{3}}{-2}=\frac{z}{1}$$ Is required equation of line in symmetric form.

Q#3: Show that the lines
L : $x + 2y - z - 7 = 0 = y + z - 2x - 6$
M : $3x + 6y - 3z - 8 = 0 = 2x - y - z$ are parallel.

Solution: Given line L is

$$x + 2y - z - 7 = 0 \text{ ------------- } (1)$$

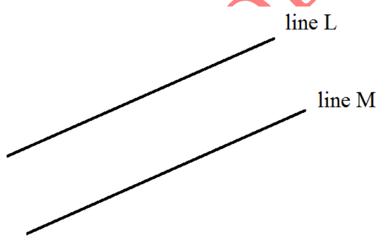
$$y + z - 2x - 6 = 0 \quad \Rightarrow -2x + y + z - 6 = 0 \quad \Rightarrow 2x - y - z + 6 = 0 \text{ --------------- } (2)$$

For this put $z = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} x + 2y - 7 = 0 \\ 2x - y + 6 = 0 \end{aligned}\right]$$

$$\frac{x}{\begin{vmatrix} 2 & -7 \\ -1 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -7 \\ 2 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} \Rightarrow \frac{x}{12-7} = \frac{-y}{6+14} = \frac{1}{-1-4} \quad \Rightarrow \frac{x}{5} = \frac{-y}{20} = \frac{1}{-5} \Rightarrow \frac{x}{1} = \frac{-y}{4} = \frac{1}{-1} \Rightarrow \boldsymbol{x = -1\,, y = 4}$$

Hence $P(-1,4,0)$ is the first point on line L.



Again put $x = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} 2y - z - 7 = 0 \\ -y - z + 6 = 0 \end{aligned}\right]$$

$$\frac{y}{\begin{vmatrix} -1 & -7 \\ -1 & 6 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} 2 & -7 \\ -1 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix}} \Rightarrow \frac{y}{-6-7} = \frac{-z}{12-7} = \frac{1}{-2-1} \quad \Rightarrow \frac{y}{-13} = \frac{-z}{5} = \frac{1}{-3} \quad \Rightarrow \boldsymbol{y = \frac{13}{3}\,, z = \frac{5}{3}}$$

So $Q\left(0,\frac{13}{3},\frac{5}{3}\right)$ is second point on the line L.

Direction ratios from P to Q are

$\Rightarrow a_1 = 0 - (-1) = 1$

$\Rightarrow b_1 = \frac{13}{3} - 4 = \frac{1}{3}$

$\Rightarrow c_1 = \frac{5}{3} - 0 = \frac{5}{3}$

Given line M is

$$3x + 6y - 3z - 8 = 0 \text{ ------------- } (1)$$

$$2x - y - z = 0 \text{ ------------ } (2)$$

For this put $z = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} 3x + 6y - 8 = 0 \\ 2x - y - 0 = 0 \end{aligned}\right]$$

$$\frac{x}{\begin{vmatrix} 6 & -8 \\ -1 & 0 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & -8 \\ 2 & 0 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 6 \\ 2 & -1 \end{vmatrix}} \quad \Rightarrow \frac{x}{-8} = \frac{-y}{16} = \frac{1}{-15} \quad \Rightarrow \boldsymbol{x = \frac{8}{15}\,, y = \frac{16}{15}}$$

Hence $P(\frac{8}{15},\frac{16}{15},0)$ is the first point on line M.

Again put $x = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} 6y - 3z - 8 = 0 \\ -y - z + 0 = 0 \end{aligned}\right]$$

$$\frac{y}{\begin{vmatrix} -3 & -8 \\ -1 & 0 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} 6 & -8 \\ -1 & 0 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 6 & -3 \\ -1 & -1 \end{vmatrix}} \Rightarrow \frac{y}{-8} = \frac{-z}{-8} = \frac{1}{-9} \quad \Rightarrow \frac{y}{8} = \frac{-z}{8} = \frac{1}{9} \quad \Rightarrow \boldsymbol{y = \frac{8}{9}\,, z = -\frac{8}{9}}$$

So $Q\left(0, \frac{8}{9}, -\frac{8}{9}\right)$ is second point on the line M.

Direction ratios from P to Q are

$\Rightarrow a_2 = 0 - \left(\frac{8}{15}\right) = -\frac{8}{15}$

$\Rightarrow b_2 = \frac{8}{9} - \frac{20}{15} = \frac{-8}{45}$

$\Rightarrow c_2 = -\frac{8}{9} - 0 = -\frac{8}{9}$

For parallel condition we have to prove

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Now putting values

$$\frac{1}{-\frac{8}{15}} = \frac{1/3}{-\frac{8}{45}} = \frac{5/3}{-\frac{8}{9}} \qquad \Rightarrow -\frac{15}{8} = -\frac{15}{8} = -\frac{15}{8}$$

Hence proved that lines L and M are parallel.

Q#4: Show that the lines
L : $x + 2y - 1 = 0 = 2y - z - 1$
M : $x - y - 1 = 0 = x - 2z - 3$ are perpendicular.

Solution:

Now do yourself as above by using this formula

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ *where* $a_1, b_1, c_1 \,\&\, a_2, b_2, c_2$ are the direction ratios of line L and M respectively.

Q#5: Find equations of the straight line through the point $(1, 2, 3)$ and parallel to the line $x - y + 2z - 5 = 0 = 3x + y + z + 6$.

Solution:

Given equation of line is

$x - y + 2z - 5 = 0$ ------------- (1)

$3x + y + z + 6 = 0$ ------------- (2)

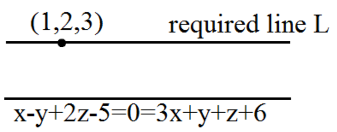
For this put $z = 0$ in eq. (1) & eq. (2)

$$\left.\begin{array}{r} x - y - 5 = 0 \\ 3x + y + 6 = 0 \end{array}\right]$$

$$\frac{x}{\begin{vmatrix} -1 & -5 \\ 1 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & -5 \\ 3 & 6 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}}$$

$$\frac{x}{-6+5} = \frac{-y}{6+15} = \frac{1}{1+3} \qquad \Rightarrow \frac{x}{-1} = \frac{-y}{21} = \frac{1}{4} \Rightarrow \boldsymbol{x = \frac{-1}{4}}, \boldsymbol{y = \frac{-21}{4}}$$

Hence $P\left(\frac{-1}{4}, \frac{-21}{4}, 0\right)$ is the first point on given line.



Again put $x = 0$ in eq. (1) & eq. (2)

$$\left.\begin{array}{r} -y + 2z - 5 = 0 \\ y + z + 6 = 0 \end{array}\right]$$

$$\frac{y}{\begin{vmatrix}2 & -5\\1 & 6\end{vmatrix}}=\frac{-z}{\begin{vmatrix}-1 & -5\\1 & 6\end{vmatrix}}=\frac{1}{\begin{vmatrix}-1 & 2\\1 & 1\end{vmatrix}}\Rightarrow\frac{y}{12+5}=\frac{-z}{-6+5}=\frac{1}{-1-2}\quad\Rightarrow\frac{x}{17}=\frac{-z}{-1}=\frac{1}{-3}\Rightarrow \boldsymbol{y}=\frac{\mathbf{-17}}{\mathbf{3}}\ ,\boldsymbol{z}=\frac{\mathbf{-1}}{\mathbf{3}}$$

Hence $Q\left(0,\frac{-17}{3},\frac{-1}{3}\right)$ is the second point on given line.

As we know that direction ratios of given line is proportional to the required line for parallel condition.

Now direction ratios from $P\left(\frac{-1}{4},\frac{-21}{4},0\right)$ to $Q\left(0,\frac{-17}{3},\frac{-1}{3}\right)$ are

$$a_1=x_2-x_1=0-\left(-\frac{1}{4}\right)=\frac{1}{4}$$

$$a_2=y_2-y_1=-\frac{17}{3}-\left(-\frac{21}{4}\right)=-\frac{17}{3}+\frac{21}{4}=-\frac{5}{12}$$

$$a_3=z_2-z_1=-\frac{1}{3}-0=-\frac{1}{3}$$

Now required line passing through the point (1,2,3) having direction ratios $\left(\frac{1}{4},-\frac{5}{12},-\frac{1}{3}\right)$ will be written as

$$\frac{x-1}{\frac{1}{4}}=\frac{y-2}{-\frac{5}{12}}=\frac{z-3}{-\frac{1}{3}}$$

Now dividing by -12 ,we get

$$\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}\qquad \textit{is required equation of line.}$$

Q#6: Find equations of the plane through the straight line $\mathbf{x+y-z=0=2x-y+3z-5}$ and perpendicular to the coordinate planes.

Solution: Given equation of line is

$$x+y-z=0\qquad\&\qquad 2x-y+3z-5=0$$

Then the equation of plane through the given line is

$$(x+y-z)+k(2x-y+3z-5)=0\quad ---------(1)$$

$$x+y-z+2kx-ky+3kz-5k=0$$

$$(1+2k)x+(1-k)y+(-1+3k)z-5k=0$$

Direction ratios of the normal vector of the plane (1) are $1+2k\,,1-k\,,-1+3k$

Case (I): For yz − plane

As plane (1) is perpendicular to $yz-plane$ whose equation is $x+0y+0z=0\quad\Rightarrow x=0$

For perpendicular condition

$$(1+2k)(1)+(1-k)(0)+(-1+3k)(0)=0$$

$$(1+2k)(1)=0\qquad\Rightarrow 1=-2k\qquad\Rightarrow \boldsymbol{k}=-\frac{\mathbf{1}}{\mathbf{2}}$$

Putting value of k in eq. (1)

$$(x+y-z)-\frac{1}{2}(2x-y+3z-5)=0$$

$$2x+2y-2z-(2x-y+3z-5)=0$$

$$2x+2y-2z-2x+y-3z+5=0$$

$3y-5z+5=0$ is required eq. of plane.

Case (II): For xz − plane

As plane (1) is perpendicular to $xz - plane$ whose equation is $0x + y + 0z = 0 \quad \Rightarrow y = 0$

For perpendicular condition

$(1 + 2k)(0) + (1 - k)(1) + (-1 + 3k)(0) = 0 \Rightarrow (1 - k)(1) = 0 \qquad \Rightarrow \boldsymbol{k = 1}$

Putting value of k in eq. (1)

$(x + y - z) + 1(2x - y + 3z - 5) = 0$

$x + y - z + 2x - y + 3z - 5 = 0$

$3x + 2z - 5 = 0$ is required eq. of plane.

Case (III): For xy − plane

As plane (1) is perpendicular to $xy - plane$ whose equation is $0x + 0y + z = 0 \quad \Rightarrow z = 0$

For perpendicular condition

$(1 + 2k)(0) + (1 - k)(0) + (-1 + 3k)(1) = 0 \Rightarrow (-1 + 3k)(1) = 0 \qquad \Rightarrow 3k = 1 \qquad \Rightarrow \boldsymbol{k = \frac{1}{3}}$

Putting value of k in eq. (1)

$(x + y - z) + \frac{1}{3}(2x - y + 3z - 5) = 0$

$3x + 3y - 3z + 2x - y + 3z - 5 = 0$

$4x + 2y - 5 = 0$ is required eq. of plane.

Q#7: Find an equation of the plane containing the line x = 2t , y = 3t , z = 4t and the intersection of the planes x + y + z = 0 and 2y − z = 0.

Solution: Given equation of plane

$x + y + z = 0 \qquad \& \qquad 2y - z = 0$

The equation of a plane through the intersection of planes

$x + y + z + k(2y - z) = 0$ --------- (A)

$x + y + z + 2ky - kz = 0$

$x + (1 + 2k)y + (1 - k)z = 0$ Here direction ratios of the normal vector of the plane (A) are $1, 1 + 2k, 1 - k$

As this plane contains the line $x = 2t , y = 3t , z = 4t$ whose direction ratios are 1, 2, 3

Now by using perpendicular condition

$(1)(2) + (1 + 2k)(3) + (1 - k)(4) = 0$

$2 + 3 + 6k + 4 - 4k = 0 \qquad \Rightarrow 2k = -9 \qquad \Rightarrow \boldsymbol{k = -\frac{9}{2}}$

Using value of k in equation (A)

$x + y + z - \frac{9}{2}(2y - z) = 0$

$2x + 2y + 2z - 9(2y - z) = 0$

$2x + 2y + 2z - 18y + 9z = 0$

$2x - 16y + 11z = 0$ is required equation of plane.

Q#8: Write an equation of the family of planes having x-intercept 5, y-intercept 2 and a nonzero z-intercept. Find the member of the family which is perpendicular to the plane $3x - 2y + z - 4 = 0$.

Solution:

Suppose c be the non-zero z-intercept, then the equation of required family of planes in intercept form is

$$\frac{x}{5}+\frac{y}{2}+\frac{z}{c}=1 \quad ----(1) \qquad where \quad c \neq 0$$

Direction ratios of normal vector of the plane (1) are $\frac{1}{5}, \frac{1}{2}, \frac{1}{c}$

If a member of family, (1) is perpendicular to the plane $3x - 2y + z - 4 = 0$ then by using perpendicular condition

$$(3)\left(\frac{1}{5}\right)-(2)\left(\frac{1}{2}\right)+(1)\left(\frac{1}{c}\right)=0$$

$$\frac{3}{5}-1+\frac{1}{c}=0 \quad \Rightarrow -\frac{2}{5}+\frac{1}{c}=0 \quad \Rightarrow \frac{1}{c}=\frac{2}{5} \quad \Rightarrow \boldsymbol{c}=\frac{\mathbf{5}}{\mathbf{2}}$$

Now using value of c in eq. (1)

$$\frac{x}{5}+\frac{y}{2}+\frac{z}{\frac{5}{2}}=1 \quad \Rightarrow \frac{x}{5}+\frac{y}{2}+\frac{2z}{5}=1 \quad required\ member\ of\ the\ family$$

Q#9: Find an equation of the plane passing through the point $(2,-3,1)$ and containing the line $x - 3 = 2y = 3z - 1$.

Solution:

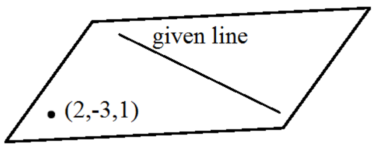
Given equation of line is

$$x-3=2y=3z-1$$

$$\Rightarrow x-3=2y \qquad \Rightarrow x-2y-3=0$$

$$\Rightarrow 2y=3z-1 \qquad \Rightarrow 2y-3z+1=0$$

Now required equation of plane containing above line is



$$x-2y-3+k(2y-3z+1)=0 \text{ ------------- } (1)$$

$$x-2y-3+2ky-3kz+k=0$$

$$x+(-2+2k)y-3kz+(k-3)=0$$

This equation of plane passes through the point (2,-3,1). So this point is satisfied the equation of plane

$$2+(-2+2k)(-3)-3k(1)+(k-3)=0$$

$$2+6-6k-3k+k-3=0 \qquad \Rightarrow -8k+5=0 \qquad \Rightarrow 8k=5 \qquad \Rightarrow \boldsymbol{k}=\frac{\mathbf{5}}{\mathbf{8}}$$

Putting value of k in equation (1)

$$x-2y-3+\frac{5}{8}(2y-3z+1)=0$$

$$8x-16y-24+10y-15z+5=0$$

$8x-6y-15z-19=0$ is required equation of plane

Q#10: Find an equation of the plane passing through the line of intersection of the planes $2x - y + 2z = 0$ and $x + 2y - 2z - 3 = 0$ and at unit distance from the origin.

Solution: Given equation of plane is

$$2x - y + 2z = 0$$

$$x + 2y - 2z - 3 = 0$$

Let required equation of plane through the line of intersection of given planes is

$$(2x - y + 2z) + k(x + 2y - 2z - 3) = 0 \quad \text{----------} \ (1)$$

$$2x - y + 2z + kx + 2ky - 2kz - 3k = 0$$

$$(2 + k)x + (-1 + 2k)y + (2 - 2k)z - 3k = 0$$

As given this plane is at unit distance from the origin $O(0,0,0)$

$$\frac{|(2 + k)(0) + (-1 + 2k)(0) + (2 - 2k)(0) - 3k|}{\sqrt{(2 + k)^2 + (2k - 1)^2 + (2 - 2k)^2}} = 1$$

$$|-3k| = \sqrt{(2 + k)^2 + (2k - 1)^2 + (2 - 2k)^2}$$

$$3k = \sqrt{4 + 4k + k^2 + 4k^2 + 1 - 4k + 4 + 4k^2 - 8k}$$

Squaring on both sides we have

$$9k^2 = 4 + k^2 + 4k^2 + 1 + 4 + 4k^2 - 8k$$

$$9k^2 = 9k^2 + 9 - 8k \qquad \Rightarrow 8k = 9 \qquad \Rightarrow \boldsymbol{k = \frac{9}{8}}$$

Using value of k in equation (1)

$$(2x - y + 2z) + \frac{9}{8}(x + 2y - 2z - 3) = 0$$

$$16x - 8y + 16z + 9x + 18y - 18z - 27 = 0$$

$25x + 10y - 2z - 27 = 0$ is required equation of plane.

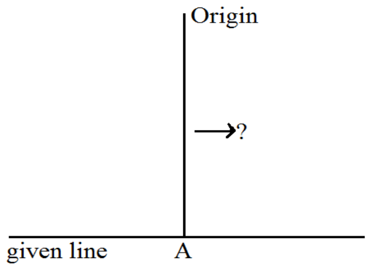
Q#11: Find equations of the perpendicular from the origin to the line $x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$. Also find the coordinates of the foot of the perpendicular.

Solution:

Given equations of line are

$$x + 2y + 3z + 4 = 0 \quad \text{-----------------} \ (1)$$

$$2x + 3y + 4z + 5 = 0 \quad \text{----------------} \ (2)$$



For this put $z = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} x + 2y + 4 = 0 \\ 2x + 3y + 5 = 0 \end{aligned}\right\}$$

$$\frac{x}{\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} \quad \Rightarrow \frac{x}{10-12} = \frac{-y}{5-8} = \frac{z}{3-4} \quad \Rightarrow \frac{x}{-2} = \frac{-y}{-3} = \frac{1}{-1} \Rightarrow \boldsymbol{x = 2, y = -3}$$

Hence P$(2, -3, 0)$ is the first point on line (1).

Again put $x = 0$ in eq. (1) & eq. (2)

$$\left.\begin{aligned} 2y + 3z + 4 = 0 \\ 3y + 4z + 5 = 0 \end{aligned}\right]$$

$$\frac{y}{\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix}} = \frac{-z}{\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}} \Rightarrow \frac{y}{15-16} = \frac{-z}{10-12} = \frac{1}{8-9} \Rightarrow \frac{y}{-1} = \frac{-z}{-2} = \frac{1}{-1} \Rightarrow \frac{y}{-1} = \frac{z}{2} = \frac{1}{-1} \Rightarrow \boldsymbol{y = 1}\ , \boldsymbol{z = -2}$$

So $Q(0,1,-2)$ is the second point on the line (1).

Now equation of straight line in symmetric form will be written as

$$\frac{x-2}{0-2} = \frac{y+3}{1+3} = \frac{z-0}{-2-0} \qquad \Rightarrow \frac{x-2}{-2} = \frac{y+3}{4} = \frac{z}{-2}$$

Multiplying by '-2'

$$\Rightarrow \frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1}$$

Now put $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} \quad = t(say)$

$$\left.\begin{aligned} x &= 2 + 1t \\ y &= -3 - 2t \\ z &= 0 + 1t \end{aligned}\right\}$$

Let a point $A(2+1t, -3-2t, 0+1t)$ is on line (1) then

$$1(2+t) - 2(-3-2t) + 1(t) = 0$$

$$2 + t + 6 + 4t + t = 0 \qquad \Rightarrow 3t + 4 = 0 \qquad \Rightarrow \boldsymbol{t = -\frac{4}{3}}$$

So coordinates of foot of perpendicular are $A\left(2 - \frac{4}{3}, -3 - 2\left(-\frac{4}{3}\right), -\frac{4}{3}\right)$

$$= A\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$$

Now required perpendicular distance $= |OA| = \sqrt{\left(\frac{2}{3} - 0\right)^2 + \left(-\frac{1}{3} - 0\right)^2 + \left(-\frac{4}{3} - 0\right)^2}$

$$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{16}{9}}$$

$$= \sqrt{\frac{4+1+16}{9}}$$

$$= \sqrt{\frac{21}{9}}$$

$$|OA| = \sqrt{\frac{7}{3}}$$

Q#12: A variable plane is at a distance p from the origin and meets the axes in A, B, C.Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $\mathbf{x^{-2}+y^{-2}+z^{-2}=p^{-2}}$.

Solutiom:

Let the equation of plane in normal form is

$$lx+my+nz=p \text{ ----------- } (1)$$

Where l, m, n are the direction cosines of normal vector of the plane (1)

Now eq. (1) can be re written as

$$\frac{lx}{p}+\frac{my}{p}+\frac{nz}{p}=1$$

$$\frac{x}{\left(\frac{p}{l}\right)}+\frac{y}{\left(\frac{p}{m}\right)}+\frac{z}{\left(\frac{p}{n}\right)}=1$$

This is the equation of plane (1) in intercept form

Hence coordinates of points A, B & C are A$\left(\frac{p}{l},0,0\right)$, B$\left(0,\frac{p}{m},0\right)$ & C$\left(0,0,\frac{p}{n}\right)$

Now equation of plane through the point A$\left(\frac{p}{l},0,0\right)$ & parallel to yz-plane is $\frac{x}{\left(\frac{p}{l}\right)}+0+0=1$

$$\frac{x}{\left(\frac{p}{l}\right)}=1 \quad \Longrightarrow x=\frac{p}{l} \quad \Longrightarrow l=\frac{p}{x}$$

Equation of plane through the point B$\left(0,\frac{p}{m},0\right)$ & parallel to xz-plane is $0+\frac{y}{\left(\frac{p}{m}\right)}+0=1$

$$\frac{y}{\left(\frac{p}{m}\right)}=1 \quad \Longrightarrow y=\frac{p}{m} \quad \Longrightarrow m=\frac{p}{y}$$

Similarly equation of plane through the point C$\left(0,0,\frac{p}{n}\right)$ & parallel to xy-plane is $0+0+\frac{z}{\left(\frac{p}{n}\right)}=1$

$$\frac{z}{\left(\frac{p}{n}\right)}=1 \quad \Longrightarrow z=\frac{p}{n} \quad \Longrightarrow n=\frac{p}{z}$$

As we know that

$$l^2+m^2+n^2=1$$

$$\Longrightarrow \quad \frac{p^2}{x^2}+\frac{p^2}{y^2}+\frac{p^2}{z^2}=1$$

$$\Longrightarrow \quad \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$$

$\Longrightarrow x^{-2}+y^{-2}+z^{-2}=p^{-2}$ hence proved.

Q#13: Let A, B, C be the points as in Problem 12.Prove that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$, O being the origin.

Solution: Let the equation of plane in normal form is

$$lx+my+nz=p \quad \text{-----------} (1)$$

Where l, m, n are the direction cosines of normal vector of the plane (1)

Now eq. (1) can be written as $$\frac{lx}{p}+\frac{my}{p}+\frac{nz}{p}=1$$

$$\frac{x}{\left(\frac{p}{l}\right)}+\frac{y}{\left(\frac{p}{m}\right)}+\frac{z}{\left(\frac{p}{n}\right)}=1$$

This is the equation of plane (1) in intercept form

Hence coordinates of points A, B & C are A$\left(\frac{p}{l},0,0\right)$, B$\left(0,\frac{p}{m},0\right)$ & C $\left(0,0,\frac{p}{n}\right)$

Hence coordinates of point O are $O(0,0,0)$.

So the coordinates of the vertices of tetrahedron OABC are $O(0,0,0), A\left(\frac{p}{l},0,0\right), B\left(0,\frac{p}{m},0\right) \,\&\, C\left(0,0,\frac{p}{n}\right)$

Now centroid of the tetrahedron is $\left(\frac{p}{4l},\frac{p}{4m},\frac{p}{4n}\right)$.

$$x=\frac{p}{4l} \qquad \Rightarrow l=\frac{p}{4x}$$

$$y=\frac{p}{4m} \qquad \Rightarrow m=\frac{p}{4y}$$

$$z=\frac{p}{4n} \qquad \Rightarrow n=\frac{p}{4z}$$

As we know that

$$l^2+m^2+n^2=1$$

$$\Rightarrow \frac{p^2}{16x^2}+\frac{p^2}{16y^2}+\frac{p^2}{16z^2}=1$$

$$\Rightarrow \qquad \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$$

$$x^{-2}+y^{-2}+z^{-2}=16p^{-2}$$ hence proved.

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