Find an equation of the plane through the three given points:

**Q#1: (2, 1, 1), (6, 3, 1), (−2, 1, 2).**

**Solution:**

Given points are \(P(2,1,1), Q(6,3,1)\) & \(R(−2,1,2)\)

Let required equation of the plane is

\[ ax + by + cz + d = 0 \] \(--------- (1)\)

Since it passes through the points \(P(2,1,1), Q(6,3,1)\) & \(R(−2,1,2)\)

So \[ 2a + b + c + d = 0 \] \(--------- (2)\)

\[ 6a + 3b + c + d = 0 \] \(--------- (3)\)

\[ −2a + b + 2c + d = 0 \] \(--------- (4)\)

Subtracting eq . (2),(3) & (3),(4)

\[ \Rightarrow −4a − 2b + 0c = 0 \]

\[ 8a + 2b − c = 0 \]

\[ \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ −1 \end{bmatrix} = \begin{bmatrix} −b \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} c \\ 1/2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a/b \\ −b/2 \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow a = −1, \ b = 2 \ & \ c = −4 \]

Putting these values in equation (2)

\[ 2(−1) + 2 + (−4) + d = 0 \Rightarrow −2 + 2 − 4 + d = 0 \Rightarrow d = 4 \]

Now putting all values in equation (1)(−1)x + 2y + (−4)z + 4 = 0 \[ \Rightarrow −x + 2y − 4z − 4 = 0 \]

\[ x − 2y + 4z − 4 = 0 \] is required equation of plane

**Method II**

Given points are \(P(2,1,1), Q(6,3,1)\) & \(R(−2,1,2)\)

Let required equation of the plane is

\[ ax + by + cz + d = 0 \] \(--------- (1)\)

For point \(P(2,1,1)\) \[ \Rightarrow 2a + b + c + d = 0 \] \(--------- (2)\)

For point \(Q(6,3,1)\) \[ \Rightarrow 6a + 3b + c + d = 0 \] \(--------- (3)\)

For point \(R(−2,1,2)\) \[ \Rightarrow −2a + b + 2c + d = 0 \] \(--------- (4)\)

Eliminating \(a, b, c\) and \(d\) from equation (1), (2), (3) & (4)

we get an equation of the required plane as

\[ \begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 1 & 1 \\ 6 & 3 & 1 & 1 \\ −2 & 1 & 2 & 1 \end{vmatrix} = 0 \]

By using row operations
Now expanding by $C_4$

\[
\begin{vmatrix}
4 & 0 & -1 & 0 \\
8 & 2 & -1 & 0 \\
-2 & 1 & 2 & 1 \\
\end{vmatrix}
= 0,
\]

by $R_1 - R_4, R_2 - R_4$

and $R_3 - R_4$

\[
\begin{vmatrix}
3 & y - 1 & z - 2 \\
4 & 0 & -1 \\
8 & 2 & -1 \\
\end{vmatrix} = 0
\]

\[
-4 \begin{vmatrix}
y - 1 & z - 2 \\
2 & -1 \\
\end{vmatrix} + 1 \begin{vmatrix}
x + 2 & y - 1 \\
8 & 2 \\
\end{vmatrix} = 0
\]

\[
\Rightarrow -4( -1(y - 1) - 2(z - 2)) + 1(2x + 2y - 8(1 - 1)) = 0
\]

\[
\Rightarrow -4( -y + 1 - 2z + 4) + 1(2x + 4 - 8y + 8) = 0
\]

\[
\Rightarrow 4y - 4 + 8z - 16 + 2x + 4 - 8y + 8 = 0
\]

\[
\Rightarrow 2x - 4y + 8z - 8 = 0
\]

\[
\Rightarrow x - 2y + 4z - 4 = 0 \quad \text{is required equation of plane.}
\]

Q#2: $(1, -1, 2), (-3, -2, 6), (6, 0, 1)$. \textbf{DO YOURSELF AS ABOVE}

Q#3: $(-1, 1, 1), (2, -8, -2), (4, 1, 0)$. \textbf{DO YOURSELF AS ABOVE}

Q#4: Find equations of the plane bisecting the angles between the planes $3x + 2y - 6z + 1 = 0$ and $2x + y + 2z - 5 = 0$.

Solution: Given equations of the plane are

\[3x + 2y - 6z + 1 = 0 \quad \text{--------- (P_1)}\]

\[2x + y + 2z - 5 = 0 \quad \text{--------- (P_2)}\]

Let a point $A$ on the plane $(P_1)$ and point $B$ on the plane $(P_2)$ and point $P(x_1, y_1, z_1)$ on the required plane bisecting the angles between given planes.

Then the distance of point $P(x_1, y_1, z_1)$ from both planes should be equal.

From the figure \[|\overrightarrow{AP}| = |\overrightarrow{BP}| \]

\[
\frac{|3x_1 + 2y_1 - 6z_1 + 1|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{|2x_1 + y_1 + 2z_1 - 5|}{\sqrt{2^2 + 1^2 + 2^2}}
\]

\[
3x_1 + 2y_1 - 6z_1 + 1 = \pm \frac{2x_1 + y_1 + 2z_1 - 5}{3}
\]

\[
\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = 7(2x_1 + y_1 + 2z_1 - 5)
\]

\[
\Rightarrow 9x_1 + 6y_1 - 18z_1 + 3 = 14x_1 + 7y_1 + 14z_1 - 35
\]

\[
9x_1 + 6y_1 - 18z_1 + 3 - 14x_1 - 7y_1 - 14z_1 + 35 = 0
\]

\[
\Rightarrow -5x_1 - y_1 - 32z_1 + 38 = 0
\]

\[
5x_1 + y_1 + 32z_1 - 38 = 0
\]

\[
\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = -\frac{2x_1 + y_1 + 2z_1 - 5}{7}
\]

\[
\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = -\frac{2x_1 + y_1 + 2z_1 - 5}{7}
\]

\[
\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = -\frac{2x_1 + y_1 + 2z_1 - 5}{3}
\]

\[
\Rightarrow 9x_1 + 6y_1 - 18z_1 + 3 = -14x_1 - 7y_1 - 14z_1 + 35
\]

\[
9x_1 + 6y_1 - 18z_1 + 3 + 14x_1 + 7y_1 + 14z_1 - 35 = 0
\]

\[
\Rightarrow 23x_1 + 13y_1 - 4z_1 - 32 = 0
\]
Q#5: Transform the equations of the planes \(3x - 4y + 5z = 0\) and \(2x - y - 2z = 5\) to normal form and hence find measure of the angle between them.

Solution: Given equations of the plane are

\[
3x - 4y + 5z = 0 \quad \text{------------------ } (P_1) \quad \text{&} \quad 2x - y - 2z = 5 \quad \text{------------------ } (P_2)
\]

Let \(\vec{n}_1 = 3\hat{i} - 4\hat{j} + 5\hat{k}\) is a normal vector of plane \((P_1)\)

\[
|\vec{n}_1| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \quad (P_1) \Rightarrow \quad \frac{3}{\sqrt{50}}x - \frac{4}{\sqrt{50}}y + \frac{5}{\sqrt{50}}z = 0
\]

Let \(\theta\) be the angle between \((P_1)\) & \((P_2)\) then by

\[
\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{(3\hat{i} - 4\hat{j} + 5\hat{k}) \cdot (2\hat{i} - \hat{j} - 2\hat{k})}{(3\sqrt{50})(3\sqrt{50})} = \frac{6 - 4 - 10}{3\sqrt{50}} = \frac{-10}{3\sqrt{50}} = 0 \quad \Rightarrow \quad \theta = \cos^{-1}(0) \quad \Rightarrow \quad \theta = 90^\circ
\]

Q#6: Find equations to the plane through the points \((4, -5, 3), (2, 3, 1)\) and parallel to the coordinate axis.

Solution: Let required equation of the plane is

\[
a x + b y + c z + d = 0 \quad \text{------------------ } (1)
\]

Since it passes through the points \((4, -5, 3), (2, 3, 1)\)

So

\[
4a - 5b + 3c + d = 0 \quad \text{------------------ } (2) \quad \text{&} \quad 2a + 3b + c + d = 0 \quad \text{------------------ } (3)
\]

Subtracting eq. (2) & (3)

\[
2a - 8b + 2c = 0 \quad \text{------------------ } (4)
\]

Case (I): Since required plane is parallel to x-axis whose direction ratios are 1,0,0

So (1) \Rightarrow \quad a + 0b + 0c = 0 \quad \text{------------------ } (5)

Now using eq. (4) & (5)

\[
\begin{vmatrix}
-8 & 2 & 1 \\
0 & 2 & 1 \\
0 & 1 & 0
\end{vmatrix} = -\begin{vmatrix}
b & 2 & 1 \\
0 & 2 & 1 \\
0 & 1 & 0
\end{vmatrix} = -b \quad \Rightarrow \quad a \quad 0 \quad 0 -2 -8 0 \quad \Rightarrow \quad a \quad 0 \quad 0 -2 -8 0 \quad \Rightarrow \quad a \quad 0 \quad 0 \quad b \quad 0 \quad c \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = b \quad \Rightarrow \quad c \quad 8 \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = c \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = -1
\]

Putting these values in eq. (3)

\[
0 + 3 + 4 + d = 0 \quad \Rightarrow \quad d = -7
\]

Put all values in eq. (1)

\[
0x + y + 4z - 7 = 0
\]

\[
\Rightarrow \quad y + 4z - 7 = 0 \quad \text{is required equation of plane}
\]

Case (II): Since required plane is parallel to y-axis whose direction ratios are 0,1,0

So (1) \Rightarrow \quad 0a + b + 0c = 0 \quad \text{------------------ } (6)

Now using eq. (4) & (6)

\[
\begin{vmatrix}
-8 & 2 & 1 \\
0 & 2 & 1 \\
1 & 0 & 0
\end{vmatrix} = -\begin{vmatrix}
b & 2 & 1 \\
0 & 2 & 1 \\
1 & 0 & 0
\end{vmatrix} = -b \quad \Rightarrow \quad a \quad 0 \quad 0 -2 -8 0 \quad \Rightarrow \quad a \quad 0 \quad 0 -2 -8 0 \quad \Rightarrow \quad a \quad 0 \quad 0 \quad b \quad 0 \quad c \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = b \quad \Rightarrow \quad c \quad 8 \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = c \\
\begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
0 & 2 & 1 \\
1 & 2 & 1 \\
0 & 0 & 1
\end{vmatrix} = -1
\]
Putting these values in eq. (3)

\[ 2 + 1 + 0 - 0 - 1 + d = 0 \implies d = -1 \]

Putt all values in eq. (1)

\[ x + 0y - z - 1 = 0 \implies x - z - 1 = 0 \] is required equation of plane

Case (III): Since required plane is parallel to z- axis whose direction ratios are 0,0,1

So (1) \[ 0a + 0b + c = 0 \] -------------- (7)

Now using eq. (4) & (7)

\[
\begin{bmatrix}
-8 \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
-a \\
0 \\
-c
\end{bmatrix} \quad \begin{bmatrix}
-1 \\
0 \\
-1
\end{bmatrix} = \begin{bmatrix}
-b \\
0 \\
-c
\end{bmatrix} \quad \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
-c \\
0 \\
0
\end{bmatrix}
\]

Putting these values in eq. (3)

\[ 2(4) + 3 + 0 + d = 0 \implies d = -11 \]

Putt all values in eq. (1)

\[ 4x + y - 0z - 11 = 0 \implies 4x + y - 11 = 0 \] is required equation of plane.

**Q#7: Find an equation of the plane through the points (1, 0, 1) and (2, 2, 1) and perpendicular to the plane \( x - y - z + 4 = 0 \).**

**Solution:** Let required equation of the plane is

\[ ax + by + cz + d = 0 \] -------------- (1)

It passes through the points (1,0,1) and (2,2,1)

(1,0,1) \[ a + 0b + c + d = 0 \] -------------- (2)

(2,2,1) \[ 2a + 2b + c + d = 0 \] -------------- (3)

Subtracting eq. (2) & (3)

\[ -a - 2b + 0c = 0 \] -------------- (4)

Given equation of the plane is

\[ x - y - z + 4 = 0 \] -------------- (5)

Let for plane (1) \[ \vec{n}_1 = ai + bj + ck \]

Let for plane (5) \[ \vec{n}_2 = i - j - k \]

As plane (1) is perpendicular to given plane . So by condition of perpendicularity

\[ \vec{n}_1 \perp \vec{n}_2 \implies \vec{n}_1 \cdot \vec{n}_2 = 0 \]

\[ (ai + bj + ck) \cdot (i - j - k) = 0 \]

\[ a - b - c = 0 \] -------------- (6)

Using eq. (4) & (6)

\[
\begin{bmatrix}
-2 \\
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
-a \\
-b \\
-c
\end{bmatrix} \quad \begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix} = \begin{bmatrix}
-b \\
0 \\
-c
\end{bmatrix} \quad \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
-c \\
0 \\
0
\end{bmatrix}
\]

Putting these proportional values of a, b & c in eq. (2)

\[ 2 + 0 + 3 + d = 0 \implies d = -5 \]

Now putting all values in equation (1)

\[ 2x - y + 3z - 5 = 0 \] required equation of plane.
Q#8: Find an equation of the plane which is perpendicular bisector of the line segment joining the points (3, 4, −1) and (5, 2, 7).

Solution: Let AB is a line segment.

Coordinates of given points are A(3, 4, −1) & B(5, 2, 7).

Now direction ratios of line AB are \( \overrightarrow{AB} = (5, 2, 7) - (3, 4, −1) \)

\( \overrightarrow{AB} = 2\hat{i} - 2\hat{j} + 8\hat{k} \) it is also normal vector of the required plane.

As line \( \overrightarrow{AB} \) is perpendicular to required plane so midpoint M of line \( \overrightarrow{AB} \) is

\[ M = \left( \frac{3+5}{2}, \frac{4+2}{2}, \frac{-1+7}{2} \right) = (4,3,3) \]

As plane is perpendicular bisector of line \( \overrightarrow{AB} \) so point \( M = (4,3,3) \) lies on required plane

Hence required equation of the plane through the point \( M = (4,3,3) \)

having normal vector \( \overrightarrow{AB} = 2\hat{i} - 2\hat{j} + 8\hat{k} \).

\[ 2(x-4) - 2(y-3) + 8(z-3) = 0 \]

\[ 2x - 2y + 8z - 8 + 6 - 24 = 0 \]

\[ 2x - 2y + 8z = 26 = 0 \]

\[ x - y + 4z = 13 = 0 \] required plane.

Q#9: Show that the join of \((0, -1, 0) \) and \((2, 4, -1) \) intersects the join of \((1, 1, 1) \) and \((3, 3, 9) \).

Solution: First we will show that the four given points are coplanar.

Now we find equation of the plane through three points.

Let the equation of required plane is

\[ ax + by + cz + d = 0 \] (1)

As it passes through \((0, -1,0), (2,4,-1) \) & \((1,1,1) \)

So

\[ 0a - b + 0c + d = 0 \] (2)

\[ 2a + 4b - c + d = 0 \] (3)

\[ a + b + c + d = 0 \] (4)

Subtracting eq. (3) from (2) & eq. (4) from (3)

\[ \Rightarrow \begin{bmatrix} a \\ -5 \end{bmatrix} = \begin{bmatrix} -b \\ 1 \end{bmatrix} \]

\[ \begin{bmatrix} a \\ -5 \end{bmatrix} = \begin{bmatrix} -b \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ -5 \end{bmatrix} = \begin{bmatrix} -b \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ 10-3 \end{bmatrix} = \begin{bmatrix} -b \\ 4-1 \end{bmatrix} = \begin{bmatrix} c \\ -6+5 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ 7 \end{bmatrix} = \begin{bmatrix} b \\ -3 \end{bmatrix} = \begin{bmatrix} c \\ -1 \end{bmatrix} \]

Now putting these proportional values of \( a, b \) & \( c \) in eq. (2)

\[ 0 + 3 + 0 + d = 0 \] \[ \Rightarrow d = -3 \]

Put all values in eq. (1)

\[ 7x - 3y - z - 3 = 0 \]
Put the fourth point (3,3,9) in above equation.

\[
7(3) - 3(3) - 9 - 3 = 0 \\
21 - 9 - 9 - 3 = 0 \\
21 - 21 = 0 \quad \Rightarrow 0 = 0
\]

As the equation of plane is satisfied, hence the four points are coplanar. Hence the two joins are coplanar.

Now direction ratios of join of (0,−1,0) and (2,4,−1) are \( 2 - 0, 4 + 1, -1 - 0 \) \( \Rightarrow 2,5,-1 \)

& direction ratios of join of (1,1,1) and (3,3,9) are \( 3 - 1, 3 - 1, 9 - 1 \) \( \Rightarrow 2,2,8 \)

As direction ratios of both joins are not proportional so the two joins are not parallel & so being coplanar they intersect each other.

Q#10: The vertices of tetrahedron are \((0,0,0),(3,0,0),(0,-4,0)\) and \((0,0,5)\). Find equations of planes of its faces.

Solution: Let the vertices of given tetrahedron are \( A(0,0,0), B(3,0,0), C(0,-4,0) \) and \( D(0,0,5) \).

Then we want to find the equations of the plane faces \( ABC, ABD, ACD \) & \( BCD \).

(I) Equation of plane for face \( ABC \)

Let the equation of required plane is

\[
ax + by + cz + d = 0 \quad \text{----------- (1)}
\]

As it passes through \( A(0,0,0), B(3,0,0) \) & \( C(0,-4,0) \)

So \( 0a + 0b + 0c + d = 0 \quad \text{----------- (2)} \)

\[
3a + 0b + 0c + d = 0 \quad \text{----------- (3)}
\]

\[
0a - 4b + 0c + d = 0 \quad \text{----------- (4)}
\]

Subtracting eq. (3) from (2) and (4) from (3)

\[
\begin{align*}
-3a + 0b + 0c & = 0 \\
3a + 4b + 0c & = 0
\end{align*}
\]

\[
\begin{align*}
\frac{a}{0} = \frac{b}{0} = \frac{c}{0} &= 0 \\
0 - 0 &= 12 - 0 \\
\Rightarrow a, b, c &= 0
\end{align*}
\]

Putting these proportional values of \( a, b \) & \( c \) in eq. (2)

\( 0 + 0 + 0 + d = 0 \quad \Rightarrow d = 0 \)

Put all values in eq. (1)

\( 0x + 0y + z + 0 = 0 \quad \Rightarrow z = 0 \)

(II) Equation of plane for face \( ABD \)

Let the equation of required plane is

\[
ax + by + cz + d = 0 \quad \text{----------- (1)}
\]

As it passes through \( A(0,0,0), B(3,0,0) \) & \( D(0,0,5) \)

So \( 0a + 0b + 0c + d = 0 \quad \text{----------- (2)} \)

\[
3a + 0b + 0c + d = 0 \quad \text{----------- (3)}
\]

\[
0a - 0b + 5c + d = 0 \quad \text{----------- (4)}
\]
Subtracting eq. (3) from (2) and (4) from (3)

\[
\begin{align*}
-3a + 0b + 0c &= 0 \\
3a + 0b - 5c &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{a}{0-0} &= \frac{-b}{15-0} = \frac{c}{0-0} &\Rightarrow a &= \frac{-b}{15} = c \quad \Rightarrow a &= \frac{-b}{1} = c
\end{align*}
\]

Putting these proportional values of \(a, b \& c\) in eq. (2)

\[
0 + 0 + 0 + d = 0 \quad \Rightarrow d = 0
\]

Put all values in eq. (1)

\[
x + y + 0z + 0 = 0
\]

\[
y = 0
\]

(III) Equation of plane face \(ACD \& BCD\) DO YOURSELF AS ABOVE

Q#11: Find an equation of the plane through \((5, -1, 4)\) and perpendicular to each of the planes \(x + y - 2z - 3 = 0\) and \(2x - 3y + z = 0\).

Solution:

Let the equation of required plane is

\[
ax + by + cz + d = 0 \quad ----- (1)
\]

The normal vector of required plane is \(\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}\)

As plane (1) passes through \((5, -1, 4)\), so

\[
5a - b + 4c + d = 0 \quad ----- (2)
\]

As plane (1) is perpendicular to given

planes \(x + y - 2z - 3 = 0\) and \(2x - 3y + z = 0\).

Normal vector of given plane (1) is \(\vec{n_1} = \hat{i} + \hat{j} - 2\hat{k}\)

Normal vector of given plane (2) is \(\vec{n_2} = 2\hat{i} - 3\hat{j} + \hat{k}\)

By using given condition

\[
\begin{align*}
\vec{n} \perp \vec{n_1} &\Rightarrow \vec{n}.\vec{n_1} = 0 \\
(a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} + \hat{j} - 2\hat{k}) &= 0 \\
a + b - 2c &= 0 \quad ----- (3)
\end{align*}
\]

\[
\begin{align*}
\vec{n} \perp \vec{n_2} &\Rightarrow \vec{n}.\vec{n_2} = 0 \\
(a\hat{i} + b\hat{j} + c\hat{k}).(2\hat{i} - 3\hat{j} + \hat{k}) &= 0 \\
2a - 3b + c &= 0 \quad ----- (4)
\end{align*}
\]

Now using eq. (3) & (4)

\[
\begin{align*}
\begin{vmatrix}
1 & -2 \\
-3 & 1
\end{vmatrix} &= \begin{vmatrix}
-2 \\
1
\end{vmatrix} = \begin{vmatrix}
1 & 1 \\
2 & -3
\end{vmatrix} \\
\Rightarrow \frac{a}{1-6} = \frac{-b}{1+4} = \frac{c}{-3-2} \Rightarrow a &= \frac{-b}{5} = \frac{c}{-5} \Rightarrow a = \frac{b}{1} = \frac{c}{1}
\end{align*}
\]

Using these proportional values of \(a, b \& c\) in eq. (2)

\[
5 - 1 + 4 + d = 0 \quad \Rightarrow d = -8
\]

Put all values in eq. (1) we get

\[
x + y + z - 8 = 0\] is required equation of plane.
Q#12: Find an equation of the plane each of whose point is equidistant from the points $A(2, -1, 1)$ and $B(3, 1, 5)$.

Solution:

As the required plane is equidistant from the points $A(2, -1, 1)$ and $B(3, 1, 5)$, so it should be a perpendicular bisector of the line $AB$ and line $AB$ will be a normal vector of the required plane.

Now direction ratios of line $AB$ are $AB = B(3, 1, 5) - A(2, -1, 1) = \hat{i} + 2\hat{j} + 4\hat{k}$, it is also normal vector of the required plane.

Midpoint of $AB$ is $M = \left(\frac{2+3}{2}, \frac{-1+1}{2}, \frac{1+5}{2}\right) = \left(\frac{5}{2}, 0, 3\right)$

Now equation of plane through the point $M\left(\frac{5}{2}, 0, 3\right)$ having normal with direction ratios $1, 2, 4$

$1\left(x - \frac{5}{2}\right) + 2(y - 0) + 4(z - 3) = 0$

$x + 2y + 4z - \frac{5}{2} - 12 = 0$

$2x + 4y + 8z - 5 - 24 = 0$

$2x + 4y + 8z - 29 = 0$ is required equation of plane.

Q#13: Find an equation of the plane through the point $(3, -2, 5)$ and perpendicular to the line $x = 2 + 3t, y = 1 - 6t, z = -2 + 2t$.

Solution:

given that the required plane passes through the point $(3, -2, 5)$ & it is perpendicular to the given line $x = 2 + 3t, y = 1 - 6t, z = -2 + 2t$.

Here direction ratios of given line are $3, -6, 2$

Normal vector of the plane is parallel to the given line, therefore direction ratios of the normal vector of the plane are $a = 3, b = -6, c = 2$

Hence required equation of the plane through $(3, -2, 5)$.

$3(x - 3) - 6(y + 2) + 2(z - 5) = 0$

$3x - 9 - 6y - 12 + 2z - 10 = 0$

$3x - 6y + 2z - 31 = 0$ is required equation of plane.

Q#14: Find parametric equations of the line containing the point $(2, 4, -3)$ and perpendicular to the plane $3x + 3y - 7z = 9$.

Solution:

As given that required line is perpendicular to the given plane $3x + 3y - 7z = 9$, we see that the direction ratios of given planes are $3, 3, -7$.

As required line is parallel to the normal vector of the plane so direction ratios of required line will $3, 3, -7$.
Now equation of required line through (2,4, -3) having
direction ratios 3,3, -7.
\[
\frac{x - 2}{3} = \frac{y - 4}{3} = \frac{z + 3}{-7} = t \quad \text{(say)}
\]
Now parametric equations of the above line are
\[
\Rightarrow \begin{cases} 
  x = 2 + 3t \\
  y = 4 + 3t \\
  z = -3 - 7t 
\end{cases}
\]

**Q#15:** Write equation of the family of all planes whose distance from the origin is 7. Find those members of the family which are parallel to the plane \(x + y + z + 5 = 0\).

**Solution:**

The equation of family of all planes in normal form is
\[
lx + my + nz = p \quad \text{------------ (1)}
\]
As given \(p = 7\) \(p\) is the distance between plane and origin
Then above equation becomes \(lx + my + nz = 7\)
Here \(l, m, n\) are the direction cosines of normal vector of the plane.
Equation of given plane is
\[
x + y + z + 5 = 0
\]
\(\text{or} \ x + y + z = -5\)
Dividing both sides by \(\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}\)
\[
\pm \frac{x}{\sqrt{3}} \pm \frac{y}{\sqrt{3}} \pm \frac{z}{\sqrt{3}} = \pm \left(-\frac{5}{\sqrt{3}}\right) \\
\frac{x}{\sqrt{3}} + \frac{y}{\sqrt{3}} + \frac{z}{\sqrt{3}} = \frac{-5}{\sqrt{3}} \quad \text{and} \quad \frac{-x}{\sqrt{3}} + \frac{y}{\sqrt{3}} - \frac{z}{\sqrt{3}} = \frac{5}{\sqrt{3}} \quad \text{------ (2)}
\]
A plane parallel to (1) has normal vector with direction cosines \(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\) \(\text{or} \ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\)
Here two sets of family of plane parallel to plane (1) so these members are
\[
\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z = 7 \quad \text{&} \quad \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = 7
\]

**Q#16:** Find an equation of the plane which passes through the point (3, 4, 5) has an x-intercept equal to \(-5\) and is perpendicular to the plane \(2x + 3y - z = 8\).

**Solution:**

Equation of a plane in intercept form is
\[
\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1
\]
As given x-intercept \(a = -5\)
Putting in above equation
\[
\frac{x}{-5} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{------ (1)}
\]
As this plane is perpendicular to $2x + 3y - z = 8$

So by condition of perpendicularity.

$$-\frac{1}{5}(2) + \frac{1}{b}(3) + \frac{1}{c}(-1) = 0 \quad \Rightarrow \quad -\frac{2}{5} + \frac{3}{b} - \frac{1}{c} = 0 \quad \Rightarrow \quad \frac{3}{b} - \frac{1}{c} = \frac{2}{5}$$

Also since plane (1) passes through point (3,4,5)

so

$$\frac{3}{-5} + \frac{4}{b} + \frac{5}{c} = 1 \Rightarrow \frac{4}{b} + \frac{5}{c} = 1 + \frac{3}{5}$$

$$\frac{4}{b} + \frac{5}{c} = \frac{8}{5} \quad \Rightarrow \quad (3)$$

$$\frac{15}{b} - \frac{5}{c} = 2 \quad \Rightarrow \quad (4)$$

mutiplying eq. (2) by 5

Adding eq. (3) & (5)

$$\frac{4}{b} + \frac{15}{b} = \frac{8}{5} + 2 \quad \Rightarrow \quad \frac{19}{b} = \frac{18}{5} \quad \Rightarrow \quad 18b = 95 \quad \Rightarrow \quad b = \frac{95}{18}$$

Putting in eq. (2)

$$\frac{3}{95/18} - \frac{1}{c} = \frac{2}{5} \quad \Rightarrow \quad \frac{54}{95} - \frac{1}{c} = \frac{2}{5} \quad \Rightarrow \quad \frac{54}{95} - \frac{2}{5} = \frac{1}{c} \quad \Rightarrow \quad \frac{1}{c} = \frac{16}{95} \quad \Rightarrow \quad c = \frac{95}{16}$$

Putting values in eq.(1)

$$\frac{x}{-5} + \frac{y}{95/18} + \frac{z}{95/16} = 1$$

$$\frac{x}{-5} + \frac{18y}{95} + \frac{16z}{95} = 1$$

Multiplying both sides by 95

$$-19x + 8y + 16z = 95$$

$$19x - 8y - 16z = -95$$

$$19x - 8y - 16z + 95 = 0$$
Q#17: Show that the distance of the point $P(3,-4,5)$ from the plane $2x + 5y - 6z = 16$ measured parallel to the line

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \quad \text{is} \quad \frac{60}{7}. $$

Solution:

As the line through $P(3,-4,5)$ is parallel to given line.

So direction ratios of the line through $P$ are $2,1,-2$.

Hence equation of line through $P(3,-4,5)$ & parallel to given line is $\frac{x-3}{2} = \frac{y+4}{1} = \frac{z-5}{-2} = t$ (say)

$$x = 3 + 2t \quad \Rightarrow y = -4 + t \quad z = 5 - 2t$$

Any point on this line is $Q(3 + 2t, -4 + t, 5 - 2t)$.

If $Q(3 + 2t, -4 + t, 5 - 2t)$ lies on plane $2x + 5y - 6z = 16$, then it satisfied the eq. of plane:

$$2(3 + 2t) + 5(-4 + t) - 6(5 - 2t) = 16$$

$$\Rightarrow 6 + 4t - 20 + 5t - 30 + 12t = 16 \quad \Rightarrow 21t - 44 = 16 \quad \Rightarrow t = \frac{20}{7}$$

So coordinates of point are $Q\left(3 + 2\left(\frac{20}{7}\right), -4 + \frac{20}{7}, 5 - 2\left(\frac{20}{7}\right)\right)$

$$= Q\left(\frac{21 + 40}{7}, \frac{-28 + 20}{7}, \frac{35 - 40}{7}\right) = Q\left(\frac{61}{7}, \frac{-8}{7}, \frac{-5}{7}\right)$$

Now the required distance $PQ = \sqrt{\left(\frac{61}{7} - 3\right)^2 + \left(-\frac{8}{7} + 4\right)^2 + \left(-\frac{5}{7} - 5\right)^2}$

$$= \sqrt{\left(\frac{61 - 21}{7}\right)^2 + \left(-\frac{8 + 28}{7}\right)^2 + \left(-\frac{5 - 35}{7}\right)^2}$$

$$= \sqrt{\left(\frac{40}{7}\right)^2 + \left(\frac{20}{7}\right)^2 + \left(-\frac{40}{7}\right)^2}$$

$$= \sqrt{\frac{1600}{49} + \frac{400}{49} + \frac{1600}{49}}$$

$$= \sqrt{\frac{1600 + 400 + 1600}{49}}$$

$$= \sqrt{\frac{3600}{49}}$$

Distance $= \frac{60}{7}$
Q#18: Show that the lines 
\[ L : x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t \]
\[ M : x = -3 + 4s, \quad y = 5 - 4s, \quad z = 6 - 5s \] intersect. Find an equation of the plane containing these lines.

Solution:

Given equations of lines are
\[ L : x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t \]
\[ M : x = -3 + 4s, \quad y = 5 - 4s, \quad z = 6 - 5s \]

Let a point \( P(x_0, y_0, z_0) \) is a point of intersection of lines \( L \) & \( M \). So this point will lie on both lines
\[ L : x = x_0 = 3 + 2t \]
\[ y = y_0 = 2 + t \]
\[ z = z_0 = -2 - 3t \]
\[ M : x = x_0 = -3 + 4s \]
\[ y = y_0 = 5 - 4s \]
\[ z = z_0 = 6 - 5s \]

Comparing \( L \) & \( M \)
\[ 3 + 2t = -3 + 4s \quad \Rightarrow \quad 2t - 4s = -6 \quad \text{------- (1)} \]
\[ 2 + t = 5 - 4s \quad \Rightarrow \quad t + 4s = 3 \quad \text{------- (2)} \]
\[ -2 - 3t = 6 - 5s \quad \Rightarrow \quad -3t + 5s = 8 \quad \text{------- (3)} \]

Adding eq. (1) & (2)
\[ 3t = -3 \quad \Rightarrow \quad t = -1 \]

Putting in eq. (1)
\[ 2(-1) - 4s = -6 \quad \Rightarrow \quad -2 - 4s = -6 \quad \Rightarrow \quad -4s = -4 \quad s = 1 \]

Using value of \( t \) & \( s \) in eq. (3)
\[ -3(-1) + 5(1) = 8 \quad \Rightarrow \quad 8 = 8 \]

We see that these values of \( t \) & \( s \) satisfy eq. (3) So given lines intersect.

Now we find eq. of plane containing given lines \( L \) & \( M \)

Equations of lines \( L \) & \( M \) in symmetric form are
\[ L : \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{-3} \]
\[ M : \frac{x+3}{4} = \frac{y-5}{-4} = \frac{z-6}{-5} \]

As the required plane contains both lines, So it could contain every point of both lines.

A point on the line \( L \) is \((3,2,-2)\)

If \( a, b, c \) are direction ratios of required plane then eq. of plane through \((3,2,-2)\) is
\[ a(x - 3) + b(y - 2) + c(z + 2) = 0 \quad \text{------- (A)} \]

As this plane contain both lines, So the normal vector of the plane is perpendicular to both the lines
\[ \text{Hence \quad} \frac{2a + \underline{a} - 3c}{4a - 4b - 5c} = 0 \]
\[ \Rightarrow \quad \frac{a}{-5-12} = \frac{-b}{-10+12} = \frac{c}{-8-4} \quad \Rightarrow \quad \frac{a}{-17} = \frac{-b}{-2} = \frac{c}{-12} \quad \Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{12} \]

Putting these values of \( a, b, c \) in eq. (A)
\[ 17(x - 3) + 2(y - 2) + 12(z + 2) = 0 \]
\[ 17x - 51 + 2y - 4 + 12z + 24 = 0 \]
\[ 17x + 2y + 12z - 31 = 0 \] is required equation of plane.
Q#19: If \(a, b, c\) are the intercepts of a plane on the coordinate axes and \(r\) is the distance of the origin from the plane, prove that 
\[
\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}
\]

Solution: Let equation of plane in slope intercept form is
\[
\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \Rightarrow \frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z - 1 = 0 \quad \text{--- (A)}
\]

Now \(r\) is the distance between Origin and plane then By using formula , we get
\[
r = \frac{|\frac{1}{a}(0) + \frac{1}{b}(0) + \frac{1}{c}(0) - 1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}
\]

Inverting on both sides 
\[
\frac{1}{r} = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}
\]

Now squaring on both sides 
\[
\frac{1}{r^2} = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \quad \text{hence proved.}
\]

Q#20: Find equations of two planes whose distances from the origin are 3 units each and which are perpendicular to the line through the point \(A(7,3,1)\) and \(B(6,4,-1)\).

Solution: Let equation of set of planes in normal form is 
\[
lx + my + nz = p
\]

As given \(p = 3\) units
\[
(Distance \ between \ plane \ & \ Origin)
\]
\[
lx + my + nz = \pm 3 \quad \text{---------- (1)}
\]

Given points are \(A(7,3,1)\) and \(B(6,4,-1)\).
\[
\vec{u} = \vec{AB} = B(6,4,-1) - A(7,3,1) = -\vec{l} + \vec{j} - 2\vec{k}
\]

Here \(\vec{AB}\) is also normal vector of both required planes
\[
|\vec{u}| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}
\]

Now we can find direction cosines of \(\vec{AB}\)
\[
l = -\frac{1}{\sqrt{6}} , m = \frac{1}{\sqrt{6}} , n = -\frac{2}{\sqrt{6}}
\]

Now using values of \(l, m, n\) in eq. (1)
\[
-\frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = \pm 3 \quad \Rightarrow -x + y - 2z = \pm 3\sqrt{6} \quad \text{are required planes.}
\]

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