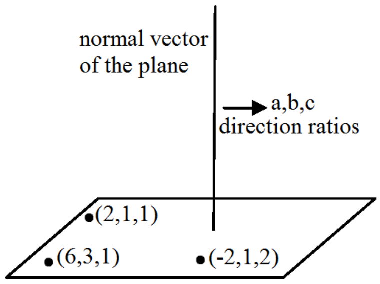
Exercise #8.3

Find an equation of the plane through the three given points:

Q#1: $(\mathbf{2}, \mathbf{1}, \mathbf{1}), (\mathbf{6}, \mathbf{3}, \mathbf{1}), (-\mathbf{2}, \mathbf{1}, \mathbf{2})$.

Solution:

Given points are $P(2,1,1), Q(6,3,1)$ & $R(-2,1,2)$



Let required equation of the plane is

$$ax + by + cz + d = 0 \quad \text{------------ (1)}$$

Since it passes through the points $P(2,1,1), Q(6,3,1)$ & $R(-2,1,2)$

So $\quad 2a + b + c + d = 0 \quad$ ----------- (2)

$$6a + 3b + c + d = 0 \quad \text{----------- (3)}$$

$$-2a + b + 2c + d = 0 \quad \text{----------- (4)}$$

Subtracting eq . (2),(3) & (3),(4)

$\Longrightarrow -4a - 2b + 0c = 0 \qquad \left.\begin{matrix} 2a + b + 0c = 0 \\ 8a + 2b - c = 0 \end{matrix}\right]$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 2 & 0 \\ 8 & -1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 2 & 1 \\ 8 & 2 \end{vmatrix}} \Longrightarrow \frac{a}{-1-0} = \frac{-b}{-2-0} = \frac{c}{4-8} \quad \Longrightarrow \frac{a}{-1} = \frac{b}{2} = \frac{c}{-4} \Longrightarrow \boldsymbol{a = -1}, \quad \boldsymbol{b = 2} \ \& \quad c = -4$$

Putting these values in equation (2)

$2(-1) + 2 + (-4) + d = 0 \qquad \Longrightarrow -2 + 2 - 4 + d = 0 \qquad \Longrightarrow \boldsymbol{d = 4}$

Now putting all values in equation (1)$(-1)x + 2y + (-4)z + 4 = 0 \qquad \Longrightarrow -x + 2y - 4z - 4 = 0$

$x - 2y + 4z - 4 = 0 \quad$ is required equation of plane

Method II.

Given points are $P(2,1,1), Q(6,3,1)$ & $R(-2,1,2)$

Let required equation of the plane is

$$ax + by + cz + d = 0 \quad \text{------------- (1)}$$

For point $P(2,1,1) \quad \Longrightarrow \qquad 2a + b + c + d = 0 \quad$ ------------- (2)

For point $Q(6,3,1) \quad \Longrightarrow \qquad 6a + 3b + c + d = 0 \quad$ ------------- (3)

For point $R(-2,1,2) \quad \Longrightarrow \qquad -2a + b + 2c + d = 0 \quad$ ------------- (4)

Eliminating a, b, c *and* d from *equation* (1), (2), (3)& (4)

we get an equation of the required plane as

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 1 & 1 & 1 \\ 6 & 3 & 1 & 1 \\ -2 & 1 & 2 & 1 \end{vmatrix} = 0$$

By using row operations

$$\begin{vmatrix} x+2 & y-1 & z-2 & 0 \\ 4 & 0 & -1 & 0 \\ 8 & 2 & -1 & 0 \\ -2 & 1 & 2 & 1 \end{vmatrix} = 0, \qquad \begin{matrix} by\ R_1 - R_4, R_2 - R_4 \\ and\ R_3 - R_4 \end{matrix}$$

Now expanding by C_4

$$\begin{vmatrix} x+2 & y-1 & z-2 \\ 4 & 0 & -1 \\ 8 & 2 & -1 \end{vmatrix} = 0$$

$$-4\begin{vmatrix} y-1 & z-2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} x+2 & y-1 \\ 8 & 2 \end{vmatrix} = 0$$

$\Rightarrow -4\big(-1(y-1) - 2(z-2)\big) + 1\big(2(x+2) - 8(y-1)\big) = 0$

$\Rightarrow \quad -4(-y+1-2z+4) + 1(2x+4-8y+8) = 0$

$\Rightarrow \quad 4y - 4 + 8z - 16 + 2x + 4 - 8y + 8 = 0$

$\Rightarrow \quad 2x - 4y + 8z - 8 = 0$

$\Rightarrow \quad x - 2y + 4z - 4 = 0$ is required equation of plane.

Q#2: $(1,-1,2), (-3,-2,6), (6,0,1)$. DO YOURSELF AS ABOVE

Q#3: $(-1,1,1), (2,-8,-2), (4,1,0)$. DO YOURSELF AS ABOVE

Q#4: Find equations of the plane bisecting the angles between the planes $3x + 2y - 6z + 1 = 0$ and $2x + y + 2z - 5 = 0$.

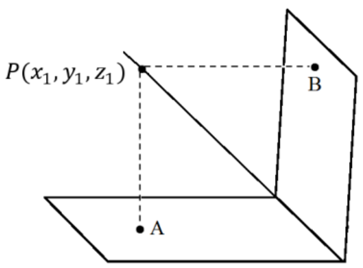
Solution: Given equations of the plane are

$3x + 2y - 6z + 1 = 0$ ----------- (P_1)

$2x + y + 2z - 5 = 0$ -------------(P_2)

Let a point A on the plane (P_1) and point B on the plane (P_2) and point $P(x_1, y_1, z_1)$ on the required plane bisecting the angles between given planes.

Then the distance of point $P(x_1, y_1, z_1)$ from both planes should be equal.



From the figure $|\overrightarrow{AP}| = |\overrightarrow{BP}|$

$$\frac{|3x_1 + 2y_1 - 6z_1 + 1|}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{|2x_1 + y_1 + 2z_1 - 5|}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$\frac{3x_1 + 2y_1 - 6z_1 + 1}{7} = \pm\frac{2x_1 + y_1 + 2z_1 - 5}{3}$$

$$\frac{3x_1 + 2y_1 - 6z_1 + 1}{7} = \frac{2x_1 + y_1 + 2z_1 - 5}{3}$$

$\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = 7(2x_1 + y_1 + 2z_1 - 5)$

$\Rightarrow 9x_1 + 6y_1 - 18z_1 + 3 = 14x_1 + 7y_1 + 14z_1 - 35$

$9x_1 + 6y_1 - 18z_1 + 3 - 14x_1 - 7y_1 - 14z_1 + 35 = 0$

$\Rightarrow -5x_1 - y_1 - 32z_1 + 38 = 0$

$5x_1 + y_1 + 32z_1 - 38 = 0$

$$\frac{3x_1 + 2y_1 - 6z_1 + 1}{7} = -\frac{2x_1 + y_1 + 2z_1 - 5}{3}$$

$\Rightarrow 3(3x_1 + 2y_1 - 6z_1 + 1) = -7(2x_1 + y_1 + 2z_1 - 5)$

$\Rightarrow 9x_1 + 6y_1 - 18z_1 + 3 = -14x_1 - 7y_1 - 14z_1 + 35$

$9x_1 + 6y_1 - 18z_1 + 3 + 14x_1 + 7y_1 + 14z_1 - 35 = 0$

$23x_1 + 13y_1 - 4z_1 - 32 = 0$

Q#5: Transform the equations of the planes $3x-4y+5z=0$ and $2x-y-2z=5$ to normal form and hence find measure of the angle between them.

Solution: Given equations of the plane are

$3x-4y+5z=0$ ------------- (P_1) & $2x-y-2z=5$ ------------- (P_2)

Let $\overrightarrow{n_1}=3\hat{\imath}-4\hat{\jmath}+5\hat{k}$ is a normal vector of plane (P_1)

& $\overrightarrow{n_2}=2\hat{\imath}-\hat{\jmath}-2\hat{k}$ is a normal vector of plane (P_2)

$|\overrightarrow{n_1}|=\sqrt{3^2+4^2+5^2}=\sqrt{50}$ $\qquad (P_1)\Rightarrow \qquad \frac{3}{\sqrt{50}}x-\frac{4}{\sqrt{50}}y+\frac{5}{\sqrt{50}}z=0$

$|\overrightarrow{n_2}|=\sqrt{2^2+1^2+2^2}=\sqrt{9}=3$ $\qquad (P_2)\Rightarrow \qquad \frac{2}{3}x-\frac{1}{3}y-\frac{2}{3}z=\frac{5}{3}$

Let θ be the angle between (P_1) & (P_2) then by

$$\cos\theta=\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}=\frac{(3\hat{\imath}-4\hat{\jmath}+5\hat{k}).(2\hat{\imath}-\hat{\jmath}-2\hat{k})}{(3)\sqrt{50}}=\frac{6+4-10}{3\sqrt{50}}=\frac{10-10}{3\sqrt{50}}=0 \quad\Rightarrow\quad \theta=\cos^{-1}(0) \quad\Rightarrow \theta=90^o$$

Q#6: Find equations to the plane through the points $(4,-5,3),(2,3,1)$ and parallel to the coordinate axis.

Solution: Let required equation of the plane is

$ax+by+cz+d=0$ ------------- (1)

Since it passes through the points $(4,-5,3),(2,3,1)$

So $\qquad 4a-5b+3c+d=0$ -------------- (2)

$2a+3b+c+d=0$ -------------- (3)

Subtracting eq. (2) & (3)

$2a-8b+2c=0$ ------------- (4)

Case (I): Since required plane is parallel to x- axis whose direction ratios are 1,0,0

So (1) $\Rightarrow$ $\quad a+0b+0c=0$ ------------ (5)

Now using eq. (4) & (5)

$$\frac{a}{\begin{vmatrix}-8 & 2\\ 0 & 0\end{vmatrix}}=\frac{-b}{\begin{vmatrix}2 & 2\\ 1 & 0\end{vmatrix}}=\frac{c}{\begin{vmatrix}2 & -8\\ 1 & 0\end{vmatrix}}\Rightarrow\frac{a}{0-0}=\frac{-b}{0-2}=\frac{c}{0+8}\quad\Rightarrow\frac{a}{0}=\frac{b}{2}=\frac{c}{8}\quad\Rightarrow\frac{a}{0}=\frac{b}{1}=\frac{c}{4}$$

Putting these values in eq. (3)

$0+3+4+d=0 \quad \Rightarrow \boldsymbol{d=-7}$

Putt all values in eq. (1)

$0x+y+4z-7=0$

$\Rightarrow y+4z-7=0$ is required equation of plane

Case (II): Since required plane is parallel to y- axis whose direction ratios are 0,1,0

So (1)$\Rightarrow$ $\quad 0a+b+0c=0$ ------------ (6)

Now using eq. (4) & (6)

$$\frac{a}{\begin{vmatrix}-8 & 2\\ 1 & 0\end{vmatrix}}=\frac{-b}{\begin{vmatrix}2 & 2\\ 0 & 0\end{vmatrix}}=\frac{c}{\begin{vmatrix}2 & -8\\ 0 & 1\end{vmatrix}}\Rightarrow\frac{a}{0-2}=\frac{-b}{0-0}=\frac{c}{2-0}\Rightarrow\frac{a}{-2}=\frac{b}{0}=\frac{c}{2}\quad\Rightarrow\frac{a}{1}=\frac{b}{0}=\frac{c}{-1}$$

Putting these values in eq. (3) $2(1)+0-1+d=0 \quad \Rightarrow \boldsymbol{d=-1}$

Putt all values in eq. (1)

$x+0y-z-1=0 \quad \Rightarrow x-z-1=0$ is required equation of plane

Case (III): Since required plane is parallel to z- axis whose direction ratios are 0,0,1

So (1)⟹ $0a+0b+c=0$ ------------ (7)

Now using eq. (4) & (7)

$$\frac{a}{\begin{vmatrix}-8 & 2\\ 0 & 1\end{vmatrix}}=\frac{-b}{\begin{vmatrix}2 & 2\\ 0 & 1\end{vmatrix}}=\frac{c}{\begin{vmatrix}2 & -8\\ 0 & 0\end{vmatrix}} \Rightarrow \frac{a}{-8-0}=\frac{-b}{2-0}=\frac{c}{0-0} \Rightarrow \frac{a}{-8}=\frac{-b}{2}=\frac{c}{0} \Rightarrow \frac{a}{4}=\frac{b}{1}=\frac{c}{0}$$

Putting these values in eq. (3)

$2(4)+3+0+d=0 \quad \Rightarrow \boldsymbol{d=-11}$

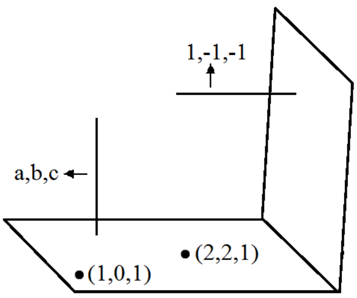
Putt all values in eq. (1)

$4x+y-0z-11=0 \quad \Rightarrow \quad 4x+y-11=0$ is required equation of plane.

Q#7: Find an equation of the plane through the points $(1,0,1)$ and $(2,2,1)$ and perpendicular to the plane $\boldsymbol{x-y-z+4=0}$.

Solution: Let required equation of the plane is

$ax+by+cz+d=0$ ------------------ (1)



It passes through the points (1,0,1) and (2,2,1)

(1,0,1) ⟹ $a+0b+c+d=0$ --------------- (2)

(2,2,1) ⟹ $2a+2b+c+d=0$ --------------- (3)

Subtracting eq. (2) & (3)

$-a-2b+0c=0$ --------------- (4)

Given equation of the plane is

$x-y-z+4=0$ ---------------- (5)

Let for plane (1) $\overrightarrow{n_1}=a\hat{\imath}+b\hat{\jmath}+c\hat{k}$

Let for plane (5) $\overrightarrow{n_2}=\hat{\imath}-\hat{\jmath}-\hat{k}$

As plane (1) is perpendicular to given plane . So by condition of perpendicularity

$\overrightarrow{n_1}\perp\overrightarrow{n_2} \quad \Rightarrow \overrightarrow{n_1}.\overrightarrow{n_2}=0$

$(a\hat{\imath}+b\hat{\jmath}+c\hat{k}).(\hat{\imath}-\hat{\jmath}-\hat{k})=0$

$a-b-c=0$ -------------------------- (6)

Using eq. (4) & (6)

$$\frac{a}{\begin{vmatrix}-2 & 0\\ -1 & -1\end{vmatrix}}=\frac{-b}{\begin{vmatrix}-1 & 0\\ 1 & -1\end{vmatrix}}=\frac{c}{\begin{vmatrix}-1 & -2\\ 1 & -1\end{vmatrix}} \Rightarrow \frac{a}{2+0}=\frac{-b}{1-0}=\frac{c}{1+2} \Rightarrow \frac{a}{2}=\frac{b}{-1}=\frac{c}{3}$$

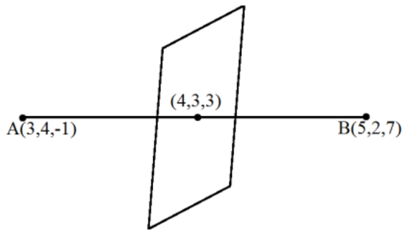
Putting these proportional values of a ,b & c in eq. (2) $2+0+3+d=0 \quad \Rightarrow \boldsymbol{d=-5}$

Now putting all values in equation (1)

$2x-y+3z-5=0$ required equation of plane.

Q#8: Find an equation of the plane which is perpendicular bisector of the line segment joining the points $(3,4,-1)$ and $(5,2,7)$.

Solution: Let AB is a line segment.



Coordinates of given points are $A(3,4,-1)$ & $B(5,2,7)$.

Now direction ratios of line AB are $\overrightarrow{AB} = B(5,2,7) - A(3,4,-1)$

$\overrightarrow{AB} = 2\hat{\imath} - 2\hat{\jmath} + 8\hat{k}$ it is also normal vector of the required plane.

As line $\overrightarrow{AB}$ is perpendicular to required plane so midpoint M of line $\overrightarrow{AB}$ is

$$M = \left(\frac{3+5}{2}, \frac{4+2}{2}, \frac{-1+7}{2}\right) = (4,3,3)$$

As plane is perpendicular bisector of line $\overrightarrow{AB}$ so point $M = (4,3,3)$ lies on required plane

Hence required equation of the

plane through the point $M = (4,3,3)$

having normal vector $\overrightarrow{AB} = 2\hat{\imath} - 2\hat{\jmath} + 8\hat{k}$.

$$2(x-4) - 2(y-3) + 8(z-3) = 0$$

$$2x - 2y + 8z - 8 + 6 - 24 = 0$$

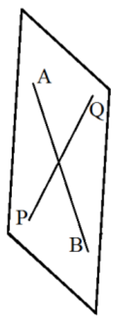
$$2x - 2y + 8z - 26 = 0$$

$$x - y + 4z - 13 = 0$$ required plane.

Q#9: Show that the join of $(0,-1,0)$ and $(2,4,-1)$ intersects the join of $(1,1,1)$ and $(3,3,9)$.

Solution: First we will show that the four given points are coplanar.

Now we find equation of the plane through three points.



Let the equation of required plane is

$ax + by + cz + d = 0$ -------------- (1)

As it passes through $(0,-1,0)$, $(2,4,-1)$ &$(1,1,1)$

So $0a - b + 0c + d = 0$ --------------- (2)

$2a + 4b - c + d = 0$ --------------- (3)

$a + b + c + d = 0$ ------------------ (4)

Subtracting eq. (3) from (2) & eq. (4) from (3)

$$\Rightarrow \begin{matrix} -2a - 5b + c = 0 \\ a + 3b - 2c = 0 \end{matrix}$$

$$\frac{a}{\begin{vmatrix} -5 & 1 \\ 3 & -2 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & -5 \\ 1 & 3 \end{vmatrix}} \quad \Rightarrow \quad \frac{a}{10-3} = \frac{-b}{4-1} = \frac{c}{-6+5} \Rightarrow \frac{a}{7} = \frac{b}{-3} = \frac{c}{-1}$$

Now putting these proportional values of a,b & c in eq. (2)

$$0 + 3 + 0 + d = 0 \quad \Rightarrow \boldsymbol{d = -3}$$

Put all values in eq. (1)

$$7x - 3y - z - 3 = 0$$

Put the fourth point (3,3,9) in above equation.

$$7(3)-3(3)-9-3=0$$

$$21-9-9-3=0$$

$$21-21=0 \qquad \Rightarrow 0=0$$

As the equation of plane is satisfied , hence the four points are coplanar. Hence the two joins are coplanar.

Now direction ratios of join of $(0,-1,0)$ and $(2,4,-1)$ are $2-0,4+1,-1-0 \qquad \Rightarrow 2,5,-1$

& direction ratios of join of $(1,1,1)$ and $(3,3,9)$ are $\qquad 3-1,3-1,9-1 \qquad \Rightarrow 2,2,8$

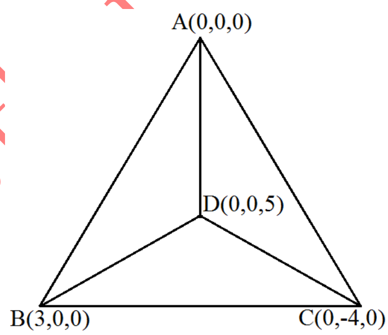
As direction ratios of both joins are not proportional so the two joins are not parallel & so being coplanar they intersect each other.

Q#10: The vertices of tetrahedron are $(0,0,0),(3,0,0),(0,-4,0)$ and $(0,0,5)$.Find equations of planes of its faces.

Solution: Let the vertices of given tetrahedron are $A(0,0,0), B(3,0,0), C(0,-4,0)$ and $D(0,0,5)$.

Then we want to find the equations of the plane faces ABC, ABD, ACD & BCD.

(I) Equation of plane for face ABC



Let the equation of required plane is

$ax+by+cz+d=0$ -------------- (1)

As it passes through $A(0,0,0), B(3,0,0)$ & $C(0,-4,0)$

So $0a+0b+0c+d=0$ --------------- (2)

$3a+0b+0c+d=0$ --------------- (3)

$0a-4b+0c+d=0$ -------------- (4)

Subtracting eq. (3) from (2) and (4) from (3)

$$\left.\begin{array}{r}-3a+0b+0c=0\\ 3a+4b+0c=0\end{array}\right]$$

$$\frac{a}{0-0}=\frac{-b}{0-0}=\frac{c}{-12-0} \qquad \Rightarrow \frac{a}{0}=\frac{b}{0}=\frac{c}{-12} \qquad \Rightarrow \frac{a}{0}=\frac{b}{0}=\frac{c}{1}$$

Putting these proportional values of a, b & c in eq. (2)

$0+0+0+d=0 \qquad \Rightarrow d=0$

Put all values in eq. (1)

$0x+0y+z+0=0 \qquad \Rightarrow z=0$

(II) Equation of plane for face ABD

Let the equation of required plane is

$ax+by+cz+d=0$ -------------- (1)

As it passes through $A(0,0,0), B(3,0,0)$ & $D(0,0,5)$

So $0a+0b+0c+d=0$ --------- (2)

$3a+0b+0c+d=0$ -------------- (3)

$0a-0b+5c+d=0$ -------------- (4)

Subtracting eq. (3) from (2) and (4) from (3)

$$\left.\begin{aligned}-3a+0b+0c&=0\\3a+0b-5c&=0\end{aligned}\right]$$

$$\frac{a}{0-0}=\frac{-b}{15-0}=\frac{c}{0-0} \quad \Rightarrow \frac{a}{0}=\frac{b}{-15}=\frac{c}{0} \quad \Rightarrow \frac{a}{0}=\frac{b}{1}=\frac{c}{0}$$

Putting these proportional values of a, b & c in eq. (2)

$0+0+0+d=0 \qquad \Rightarrow d=0$

Put all values in eq. (1)

$0x+y+0z+0=0$

$y=0$

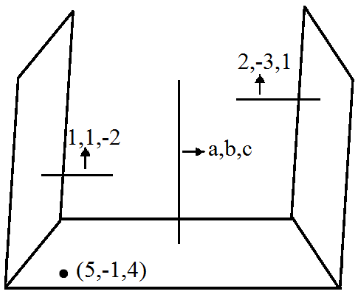
(III) Equation of plane face ACD & BCD DO YOURSELF AS ABOVE

Q#11: Find an equation of the plane through $(5,-1,4)$ and perpendicular to each of the planes $x+y-2z-3=0$ and $2x-3y+z=0$.

Solution:

Let the equation of required plane is

$ax+by+cz+d=0$ ---------- (1)



The normal vector of required plane is $\vec{n}=a\hat{\imath}+b\hat{\jmath}+c\hat{k}$

As plane (1) passes through $(5,-1,4)$, so

$5a-b+4c+d=0$ ---------- (2)

As plane (1) is perpendicular to given

planes a $x+y-2z-3=0$ and $2x-3y+z=0$

Normal vector of given plane (1) is $\vec{n_1}=\hat{\imath}+\hat{\jmath}-2\hat{k}$

Normal vector of given plane (2) is $\vec{n_2}=2\hat{\imath}-3\hat{\jmath}+\hat{k}$

By using given condition

$\vec{n}\perp\vec{n_1}$ Then $\vec{n}.\vec{n_1}=0$

$(a\hat{\imath}+b\hat{\jmath}+c\hat{k}).(\hat{\imath}+\hat{\jmath}-2\hat{k})=0$

$a+b-2c=0$ ---------------- (3)

$\vec{n}\perp\vec{n_2}$ Then $\vec{n}.\vec{n_2}=0$

$(a\hat{\imath}+b\hat{\jmath}+c\hat{k}).(2\hat{\imath}-3\hat{\jmath}+\hat{k})=0$

$2a-3b+c=0$ ----------------- (4)

Now using eq. (3) & (4)

$$\frac{a}{\begin{vmatrix}1&-2\\-3&1\end{vmatrix}}=\frac{-b}{\begin{vmatrix}1&-2\\2&1\end{vmatrix}}=\frac{c}{\begin{vmatrix}1&1\\2&-3\end{vmatrix}} \quad \Rightarrow \frac{a}{1-6}=\frac{-b}{1+4}=\frac{c}{-3-2} \Rightarrow \frac{a}{-5}=\frac{-b}{5}=\frac{c}{-5} \quad \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{1}$$

Using these proportional values of a, b & c in eq. (2)

$5-1+4+d=0 \qquad \Rightarrow d=-8$

Put all values in eq. (1) we get $\quad x+y+z-8=0 \quad$ is required equation of plane.

Q#12: Find an equation of the plane each of whose point is equidistant from the points $A(2,-1,1)$ and $B(3,1,5)$.

Solution:

As the required plane is equidistant from the points $A(2,-1,1)$ and $B(3,1,5)$,

so it should be a perpendicular bisector of the line AB & line AB

will be a normal vector of the required plane.

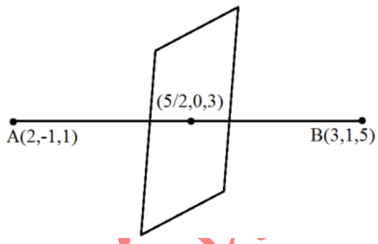
Now direction ratios of line AB are

$\overrightarrow{AB} = B(3,1,5) - A(2,-1,1) \Rightarrow \hat{\imath} + 2\hat{\jmath} + 4\hat{k}$ *it is also normal vector of the required plane*

midpoint of AB is $M = \left(\frac{2+3}{2}, \frac{1-1}{2}, \frac{1+5}{2}\right) = \left(\frac{5}{2}, 0, 3\right)$

Now equation of plane through the point $M\left(\frac{5}{2}, 0, 3\right)$

having normal with direction ratios 1,2,4



$$1\left(x - \frac{5}{2}\right) + 2(y - 0) + 4(z - 3) = 0$$

$$x + 2y + 4z - \frac{5}{2} - 12 = 0$$

$$2x + 4y + 8z - 5 - 24 = 0$$

$2x + 4y + 8z - 29 = 0$ is required equation of plane.

Q#13: Find an equation of the plane through the point $(3,-2,5)$ and perpendicular to the line $x = 2 + 3t, y = 1 - 6t, z = -2 + 2t$.

Solution:

given that the required plane passes through the point $(3,-2,5)$ & it is perpendicular to the given line

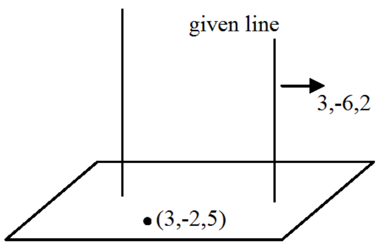
$x = 2 + 3t, \quad y = 1 - 6t, \quad z = -2 + 2t.$

Here direction ratios of given line are $3, -6, 2$

Normal vector of the plane is parallel to the given line,

therefore direction ratios of the normal vector of the plane are

$a = 3, \quad b = -6, \quad c = 2$



Hence required equation of the plane through $(3,-2,5)$.

$$3(x - 3) - 6(y + 2) + 2(z - 5) = 0$$

$$3x - 9 - 6y - 12 + 2z - 10 = 0$$

$3x - 6y + 2z - 31 = 0$ required equation of plane

Q#14: Find parametric equations of the line containing the point $(2,4,-3)$ and perpendicular to the plane $3x + 3y - 7z = 9$.

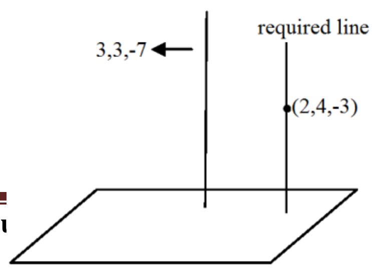
Solution:

As given that required line is perpendicular to the given plane $3x + 3y - 7z = 9$

we see that the direction ratios of given planes are $3,3,-7$.

As required line is parallel to the normal vector

of the plane so direction ratios of required line will $3,3,-7$.



Now equation of required line through $(2,4,-3)$ having

direction ratios $3,3,-7$.

$$\frac{x-2}{3}=\frac{y-4}{3}=\frac{z+3}{-7}\quad=t\ (say)$$

Now parametric equations of the above line are

$$\Rightarrow \left.\begin{array}{r} x=2+3t \\ y=4+3t \\ z=-3-7t \end{array}\right\}$$

Q#15: Write equation of the family of all planes whose distance from the origin is 7. Find those members of the family which are parallel to the plane $x+y+z+5=0$.

Solution:

The equation of family of all planes in normal form is

$lx+my+nz=p$ ------------- (1)

As given $p=7$ p is the distance between plane and origin

Then above equation becomes $lx+my+nz=7$

Here l, m & n are the direction cosines of normal vector of the plane.

Equation of given plane is

$$x+y+z+5=0$$

$$or\ \ x+y+z=-5$$

Dividing both sides by $\sqrt{1^1+1^1+1^1}=\pm\sqrt{3}$

$$\pm\frac{x}{\sqrt{3}}\pm\frac{y}{\sqrt{3}}\pm\frac{z}{\sqrt{3}}=\pm\left(-\frac{5}{\sqrt{3}}\right)$$

$$\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}=-\frac{5}{\sqrt{3}}\quad and\ -\frac{x}{\sqrt{3}}-\frac{y}{\sqrt{3}}-\frac{z}{\sqrt{3}}=\frac{5}{\sqrt{3}}\qquad ------(2)$$

A plane parallel to (1) has normal vector with direction cosines $-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\quad or\ \ \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$

Here two sets of family of plane parallel to plane (1) so these members are

$$-\frac{1}{\sqrt{3}}x-\frac{1}{\sqrt{3}}y-\frac{1}{\sqrt{3}}z=7\ \&\ \frac{1}{\sqrt{3}}x+\frac{1}{\sqrt{3}}y+\frac{1}{\sqrt{3}}z=7$$

Q#16: Find an equation of the plane which passes through the point $(3,4,5)$ has an x-intercept equal to -5 and is perpendicular to the plane $2x+3y-z=8$.

Solution:

Equation of a plane in intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

As given x-intercept $a=-5$

Putting in above equation $\frac{x}{-5}+\frac{y}{b}+\frac{z}{c}=1\quad -----(1)$

As this plane is perpendicular to $2x+3y-z=8$

So by condition of perpendicularity.

$-\frac{1}{5}(2)+\frac{1}{b}(3)+\frac{1}{c}(-1)=0 \quad \Rightarrow -\frac{2}{5}+\frac{3}{b}-\frac{1}{c}=0 \quad \Rightarrow \frac{3}{b}-\frac{1}{c}=\frac{2}{5}$

Also since plane (1) passes through point (3,4,5)

so $\frac{3}{-5}+\frac{4}{b}+\frac{5}{c}=1 \Rightarrow \frac{4}{b}+\frac{5}{c}=1+\frac{3}{5}$

$$\frac{4}{b}+\frac{5}{c}=\frac{8}{5} \qquad ----(3)$$

$$\frac{15}{b}-\frac{5}{c}=2 \qquad ----(4) \qquad \text{mutiplying eq. (2) by 5}$$

Adding eq. (3) & (5)

$$\frac{4}{b}+\frac{15}{b}=\frac{8}{5}+2 \quad \Rightarrow \frac{19}{b}=\frac{18}{5} \quad \Rightarrow 18b=95 \quad \Rightarrow \mathbf{b}=\mathbf{\frac{95}{18}}$$

Putting in eq. (2)

$$\frac{3}{95/18}-\frac{1}{c}=\frac{2}{5} \quad \Rightarrow \frac{54}{95}-\frac{1}{c}=\frac{2}{5} \quad \Rightarrow \frac{54}{95}-\frac{2}{5}=\frac{1}{c} \quad \Rightarrow \frac{1}{c}=\frac{16}{95} \quad \Rightarrow \mathbf{c}=\mathbf{\frac{95}{16}}$$

Putting values in eq.(1)

$$\frac{x}{-5}+\frac{y}{95/18}+\frac{z}{95/16}=1$$

$$\frac{x}{-5}+\frac{18y}{95}+\frac{16z}{95}=1$$

Multiplying both sides by 95

$$-19x+8y+16z=95$$

$$19x-8y-16z=-95$$

$$19x-8y-16z+95=0$$

Q#17: Show that the distance of the point $P(3,-4,5)$ from the plane $2x+5y-6z=16$ measured parallel to the line

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{-2} \quad is\ \frac{60}{7}.$$

Solution:

As the line through $P(3,-4,5)$ is parallel to given line.

So direction ratios of the line through P are $2,1,-2$.

Hence equation of line through $P(3,-4,5)$ & parallel to given line is $\frac{x-3}{2}=\frac{y+4}{1}=\frac{z-5}{-2}\ =t\ (say)$

$$\Rightarrow \left.\begin{aligned} x&=3+2t\\ y&=-4+t\\ z&=5-2t \end{aligned}\right]$$

Any point on this line is $Q(3+2t,-4+t,5-2t)$.

If $Q(3+2t,-4+t,5-2t)$ lies on plane $2x+5y-6z=16$,Then it satisfied the eq. of plane.

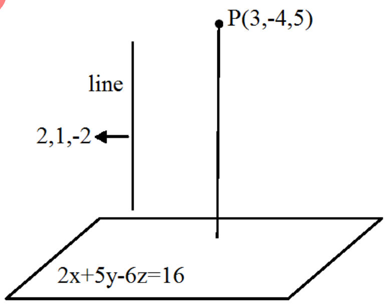
$$2(3+2t)+5(-4+t)-6(5-2t)=16$$

$$\Rightarrow 6+4t-20+5t-30+12t=16 \qquad \Rightarrow 21t-44=16 \qquad t=\frac{20}{7}$$

So coordinates of point are $Q\left(3+2\left(\frac{20}{7}\right),-4+\frac{20}{7},5-2\left(\frac{20}{7}\right)\right)$

$$=Q\left(\frac{21+40}{7},\frac{-28+20}{7},\frac{35-40}{7}\right)$$

$$=Q\left(\frac{61}{7},\frac{-8}{7},\frac{-5}{7}\right)$$



Now the required distance $=|PQ|=\sqrt{\left(\frac{61}{7}-3\right)^2+\left(\frac{-8}{7}+4\right)^2+\left(\frac{-5}{7}-5\right)^2}$

$$=\sqrt{\left(\frac{61-21}{7}\right)^2+\left(\frac{-8+28}{7}\right)^2+\left(\frac{-5-35}{7}\right)^2}$$

$$=\sqrt{\left(\frac{40}{7}\right)^2+\left(\frac{20}{7}\right)^2+\left(\frac{-40}{7}\right)^2}$$

$$=\sqrt{\frac{1600}{49}+\frac{400}{49}+\frac{1600}{49}}$$

$$=\sqrt{\frac{1600+400+1600}{49}}$$

$$=\sqrt{\frac{3600}{49}}$$

distance$=\frac{60}{7}$

Q#18: Show that the lines
$\boldsymbol{L} \ : \ \boldsymbol{x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t}$
$\boldsymbol{M} \ : \ \boldsymbol{x = -3 + 4s, \ \ y = 5 - 4s, \ \ z = 6 - 5s}$ **intersect.Find an equation of the plane containing these lines.**

Solution:

Given equations of lines are

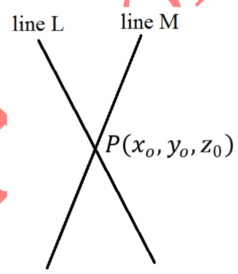
$$L \ : \ x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t$$

$$M \ : \ x = -3 + 4s, \ \ y = 5 - 4s, \ \ z = 6 - 5s$$

Let a point $P(x_o, y_o, z_o)$ is a point of intersection of lines L & M. So this point will lie on both lines

$$L = \begin{cases} x_o = 3 + 2t \\ y_o = 2 + t \\ z_o = -2 - 3t \end{cases} \qquad \& \qquad M = \begin{cases} x_o = -3 + 4s \\ y_o = 5 - 4s \\ z_o = 6 - 5s \end{cases}$$

Comparing L & M



$3 + 2t = -3 + 4s \qquad \Rightarrow \qquad 2t - 4s = -6$ ------------ (1)

$2 + t = 5 - 4s \qquad \Rightarrow \qquad t + 4s = 3$ ------------ (2)

$-2 - 3t = 6 - 5s \qquad \Rightarrow \qquad -3t + 5s = 8$ ------------- (3)

Adding eq. (1) & (2) $\qquad 3t = -3 \quad \Rightarrow \boldsymbol{t = -1}$

Putting in eq. (1)

$2(-1) - 4s = -6 \qquad \Rightarrow -2 - 4s = -6 \qquad \Rightarrow -4s = -4 \qquad \boldsymbol{s = 1}$

Using value of t & s in eq. (3) $\qquad -3(-1) + 5(1) = 8 \qquad \Rightarrow 8 = 8$

We see that these values of t & s satisfy eq. (3) So given lines intersect.

Now we find eq. of plane containing given lines L & M

Equations of lines L & M in symmetric form are

$$L \ : \ \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{-3} \qquad \& \qquad M \ : \ \frac{x+3}{4} = \frac{y-5}{-4} = \frac{z-6}{-5}$$

As the required plane contains both lines ,So it could contain every point of both lines.

A point on the line L is $(3,2,-2)$

If a, b, c are direction ratios of required plane then eq. of plane through $(3,2,-2)$ is

$a(x-3) + b(y-2) + c(z+2) = 0$ ------------------ (A)

As this plane contain both lines ,So the normal vector of the plane is perpendicular to both the lines

$$Hence \ \begin{matrix} 2a + 前 - 3c = 0 \\ 4a - 4b - 5c = 0 \end{matrix}$$

$$\frac{a}{-5-12} = \frac{-b}{-10+12} = \frac{c}{-8-4} \quad \Rightarrow \quad \frac{a}{-17} = \frac{-b}{-2} = \frac{c}{-12} \quad \Rightarrow \frac{a}{17} = \frac{b}{2} = \frac{c}{12}$$

Putting these values of a, b, c in eq. (A)

$$17(x-3) + 2(y-2) + 12(z+2) = 0$$

$$17x - 51 + 2y - 4 + 12z + 24 = 0$$

$17x + 2y + 12z - 31 = 0$ is required equation of plane.

Q#19: If a, b, c are the intercepts of a plane on the coordinate axes and r is the distance of the origin from the plane, prove that $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

Solution: Let equation of plane in slope intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \Rightarrow \frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z - 1 = 0 \quad ----(A)$$

Now r is the distance between Origin and plane then By using formula , we get

$$r = \frac{\left|\frac{1}{a}(0)+\frac{1}{b}(0)+\frac{1}{c}(0)-1\right|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = \frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow r = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

Inverting on both sides $\Rightarrow \frac{1}{r} = \sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$

Now squaring on both sides $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$ *hence proved.*

Q#20: Find equations of two planes whose distances from the origin are 3 units each and which are perpendicular to the line through the point $A(7,3,1)$ and $B(6,4,-1)$.

Solution: Let equation of set of planes in normal form is $lx + my + nz = p$

As given $p = 3$ units $(Distance\ betweeen\ plane\ \&\ Origin)$

$$lx + my + nz = \pm 3 \quad \text{---------------} (1)$$

Given points are $A(7,3,1)$ and $B(6,4,-1)$.

$\vec{u} = \overrightarrow{AB} = B(6,4,-1) - A(7,3,1) = -\hat{\imath} + \hat{\jmath} - 2\hat{k}$

Here $\overrightarrow{AB}$ is also normal vector of both required planes

$|\vec{u}| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$

Now we can find direction cosines of $\overrightarrow{AB}$

$$l = -\frac{1}{\sqrt{6}} \ , m = \frac{1}{\sqrt{6}} \ , n = -\frac{2}{\sqrt{6}}$$

Now using values of l, m, n in eq. (1)

$-\frac{1}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y - \frac{2}{\sqrt{6}}z = \pm 3 \Rightarrow -x + y - 2z = \pm 3\sqrt{6}$ are required planes.

Checked by: Sir Hameed ullah (*hameedmath2017 @ gmail.com)*

Specially thanks to my Respected Teachers

Prof. Muhammad Ali Malik (M.phill physics and Publisher of www.Houseofphy.blogspot.com)

Muhammad Umar Asghar sb (MSc Mathematics)

Hameed Ullah sb (MSc Mathematics)