Exercise #8.2

In each of Problem $1-4$, find parametric equations, direction ratios, direction cosines and measures of the direction angles of the straight line through P and Q:

Q#1: $P(1,-2,0), Q(5,-10,1)$

Solution:

Given points are $P(1,-2,0)$ and $Q(5,-10,1)$

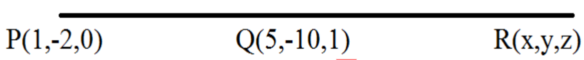
Let $R(x,y,z)$ is another point on the straight line then equation of straight line will be written as

$$\frac{x-1}{5-1}=\frac{y+2}{-10+2}=\frac{z-0}{1-0}\Rightarrow \quad \frac{x-1}{4}=\frac{y+2}{-8}=\frac{z}{1}=t$$

Parametric equations of given straight line can be written as

$\frac{x-1}{4}=t \qquad \Rightarrow x=1+4t$

$\frac{y+2}{-8}=t \qquad \Rightarrow y=-2-8t$



$\frac{z}{1}=t \qquad \Rightarrow z=0+1t$

Direction ratios of straight line PQ are $4,-8,1$.

Let $\vec{d}$ is the direction vector , $\vec{d}=4\hat{\imath}-8\hat{\jmath}+\hat{k}$ $\Rightarrow |\vec{d}|=\sqrt{4^2+(-8)^2+1^2}=\sqrt{81}=9$

Direction cosines of line through P & Q are

$$\cos\alpha=\frac{4}{9}, \cos\beta=\frac{-8}{9}, \cos\gamma=\frac{1}{9}$$

Directional angles of line PQ are

$$\alpha=\cos^{-1}\frac{4}{9}, \beta=\cos^{-1}\frac{-8}{9}, \gamma=\cos^{-1}\frac{1}{9}$$

$$\Rightarrow \alpha=63^o 37', \beta=152^o 44', \gamma=83^o 37'$$

Q#2: $P(6,5,-3), Q(4,1,1)$

Now do yourself as above

Q#3: $P(1,-5,1), Q(4,-5,4)$

Do yourself as above

Q#4: $P(3,5,7), Q(6,-8,10)$

Solution:

Given points are $P(3,5,7)$ & $Q(6,-8,10)$

Let $R(x,y,z)$ is another point on the straight line then equation of straight line will be written as

$$\frac{x-3}{6-3}=\frac{y-5}{-8-5}=\frac{z-7}{10-7}$$

$$\Rightarrow \frac{x-3}{3}=\frac{y-5}{-13}=\frac{z-7}{3}=t$$

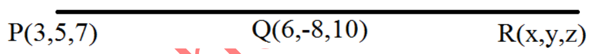
Parametric equations of given straight line can be written as

$$\frac{x-3}{3}=t \qquad \Rightarrow x=3+3t$$

$$\frac{y-5}{-13}=t \qquad \Rightarrow y=5-13t$$

$$\frac{z-7}{3}=t \qquad \Rightarrow z=7+3t$$

Direction ratios of straight line PQ are $3,-13,3$.



Let $\vec{d}$ is the direction vector

$$\vec{d}=3\hat{\imath}-13\hat{\jmath}+3\hat{k} \qquad \Rightarrow |\vec{d}|=\sqrt{3^2+(-13)^2+3^2}=\sqrt{187}$$

Direction cosines of line through P & Q are

$$\cos\alpha=\frac{3}{\sqrt{187}}, \cos\beta=\frac{-13}{\sqrt{187}}, \cos\gamma=\frac{3}{\sqrt{187}}$$

Directional angles of line PQ are

$$\alpha=\cos^{-1}\frac{3}{\sqrt{187}}, \beta=\cos^{-1}\frac{-13}{\sqrt{187}}, \gamma=\cos^{-1}\frac{3}{\sqrt{187}}$$

$$\Rightarrow \alpha=77^o19', \beta=159^o19', \gamma=77^o19'$$

Q#5: Find the direction cosines the coordinate axis.

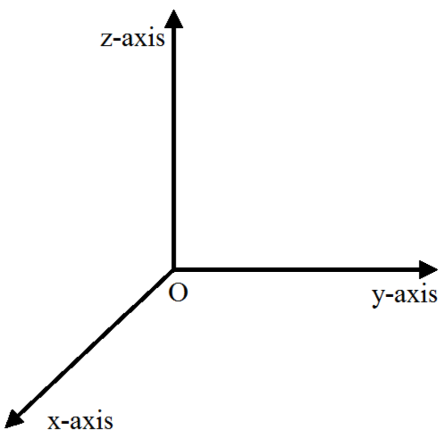
Solution:

We want to find the direction cosine of x-axis , y-axis and z- axis.

(I) Let x-axis makes angles $0^o, 90^o, 90^o$ with x-axis ,y-axis and z- axis

So direction cosines of x-axis are

$\cos 0^o=1, \cos 90^o=0, \cos 90^o=0$



(II) Let y-axis makes angles $90^o, 0^o, 90^o$ with x-axis ,y-axis and z- axis

So direction cosines of y-axis are

$\cos 90^o=0, \cos 0^o=1, \cos 90^o=0$

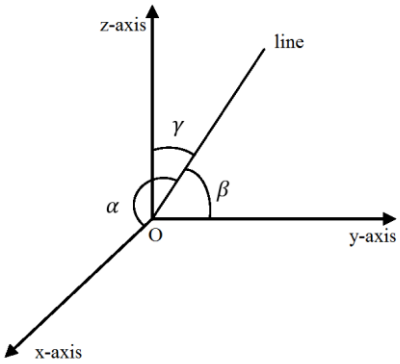
(III) Let z-axis makes angles $90^o, 90^o, 0^o$ with x-axis ,y-axis and z- axis

So direction cosines of z-axis are

$\cos 90^o=0, \cos 90^o=0, \cos 0^o=1$

Q#6: Prove that if measures of the direction angles of a straight line are α, β and γ, then
$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Solution: Let α, β and γ be the directional angles of a straight line then direction cosines of line are $\cos\alpha$, $\cos\beta$ & $\cos\gamma$.



As we know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$$

$$3 - 1 = (\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$ **Hence proved.**

Q#7: If measures of two of the direction angles of a straight line are 45^o and 60^o, find measure of the third direction angle.

Solution: Let α, β and γ be the directional angles of a straight line $\alpha = 45^o, \beta = 60^o, \gamma = ?$

As we know that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\cos^2 45^o + \cos^2 60^o + \cos^2\gamma = 1$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1 \quad \Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\cos^2\gamma = 1 - \frac{3}{4} \quad \Rightarrow \cos^2\gamma = \frac{1}{4} \quad \Rightarrow \cos\gamma = \frac{1}{\sqrt{4}} \Rightarrow \cos\gamma = \frac{1}{2} \quad \Rightarrow \quad \gamma = \cos^{-1}\left(\frac{1}{2}\right)$$

$\Rightarrow \gamma = 60^o$ is required direction angle.

Q#8: The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$. Find measure of the angle between them.

Solution: We have to find the angle between two straight lines.

Let l_1, m_1, n_1 be the direction cosines of line L and l_2, m_2, n_2 be the direction cosines of line M.

Let θ be the angle between line L and M then it is described as

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \text{ -------- (A)}$$

$$l + m + n = 0 \text{ -------------------------------- (1)}$$

$$l^2 + m^2 - n^2 = 0 \text{ ------------------------------ (2)}$$

$$(1) \Rightarrow \quad n = -(l + m) \text{ -------------------- (3)}$$

Putting in eq.(2)

$$l^2 + m^2 - [-(l + m)]^2 = 0$$

$$l^2 + m^2 - [l^2 + m^2 + 2lm] = 0$$

$$l^2 + m^2 - l^2 - m^2 - 2lm = 0$$

$$-2lm = 0 \quad \Rightarrow lm = 0$$

Either $l = 0$ or $m = 0$

Put $l = 0$ in eq. (3)

$\Rightarrow n = -m$

$\Rightarrow \frac{n}{1} = \frac{m}{-1}$

$\frac{l}{0} = \frac{n}{1} = \frac{m}{-1} \quad \Rightarrow \frac{l^2 + m^2 + n^2}{\sqrt{0^2 + 1^2 + (-1)^2}} = \frac{1}{\sqrt{2}}$

So direction cosines of line L are

$l_1 = 0\,, \quad m_1 = \frac{-1}{\sqrt{2}}\,, \quad n_1 = \frac{1}{\sqrt{2}}$

Put $m = 0$ in eq. (3)

$\Rightarrow n = -l$

$\Rightarrow \frac{n}{1} = \frac{l}{-1}$

$\frac{l}{-1} = \frac{n}{1} = \frac{m}{0} \quad \Rightarrow \frac{l^2 + m^2 + n^2}{\sqrt{(-1)^2 + 1^2 + 0^2}} = \frac{1}{\sqrt{2}}$

So direction cosines of line M are

$l_2 = \frac{-1}{\sqrt{2}}\,, \quad m_2 = 0\,, \quad n_2 = \frac{1}{\sqrt{2}}$

Using in equation (A)

$\cos\theta = (0)\left(\frac{-1}{\sqrt{2}}\right) + \left(\frac{-1}{\sqrt{2}}\right)(0) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \Rightarrow \cos\theta = 0 + 0 + \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2} \quad \Rightarrow \theta = \cos^{-1}\frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{3}$

Q#9: The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $2lm + 2ln - mn = 0$. Find measure of the angle between them.

Solution: We have to find the angle between two straight lines.

Let l_1, m_1, n_1 be the direction cosines of line L and l_2, m_2, n_2 be the direction cosines of line M.

Let θ be the angle between line L and M then it is described as $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ -------- (A)

$l + m + n = 0$ -------------------------------- (1)

$2lm + 2ln - mn = 0$ -------------------------------- (2)

(1) $\Rightarrow$ $n = -(l + m)$------------------------------ (3)

Putting in eq. (2) $\quad 2lm + 2l[-(l+m)] - m[-(l+m)] = 0$

$2lm - 2l(l+m) + m(l+m) = 0$

$2lm - 2l^2 - 2lm + lm + m^2 = 0$

$-2l^2 + lm + m^2 = 0$

$2l^2 - lm - m^2 = 0$

$2l^2 - 2lm + lm - m^2 = 0$

$2l(l - m) + m(l - m) = 0 \quad \Rightarrow (l - m)(2l + m) = 0$

$l - m = 0$

$or \quad l = m \quad \Rightarrow \frac{l}{1} = \frac{m}{1}$

Putting in eq. (3)

$n = -2l$

$\frac{n}{-2} = \frac{l}{1}$

$So \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{-2} \quad \Rightarrow \frac{l^2 + m^2 + n^2}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{6}}$

So direction cosines of line L are

$l_1 = \frac{1}{\sqrt{6}}\,, m_1 = \frac{1}{\sqrt{6}}\,, n_1 = \frac{-2}{\sqrt{6}}$

$2l + m = 0$

$or \quad m = -2l \quad \Rightarrow \frac{l}{1} = \frac{m}{-2}$

Putting in eq. (3)

$n = l$

$\frac{l}{1} = \frac{n}{1}$

$So \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1} \quad \Rightarrow \frac{l^2 + m^2 + n^2}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt{6}}$

So direction cosines of line M are

$l_2 = \frac{1}{\sqrt{6}}\,, m_2 = \frac{-2}{\sqrt{6}}\,, n_2 = \frac{1}{\sqrt{6}}$

Using in equation (A)

$\cos\theta = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{-2}{\sqrt{6}}\right) + \left(\frac{-2}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right) = \frac{1}{6} - \frac{2}{6} - \frac{2}{6} \quad = \frac{1-2-2}{6} = -\frac{3}{6} \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right) \quad \Rightarrow \theta = 60^o$

Find equations of the straight line L and M in symmetric forms. Determine whether the pairs of lines intersect. Find the point of intersection if it exists.

Q#10: L : through $\mathbf{A(2,1,3), B(-1,2,-4)}$
M : through $\mathbf{P(5,1,-2), Q(0,4,3)}$

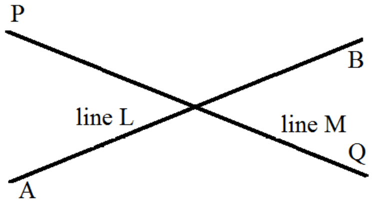
Solution:

The equation of the straight line L through $A(2,1,3)$ & $B(-1,2,-4)$

$$\frac{x-2}{2+1}=\frac{y-1}{1-2}=\frac{z-3}{3+4} \quad \Rightarrow \frac{x-2}{3}=\frac{y-1}{-1}=\frac{z-3}{7}$$

The equation of the straight line M through $P(5,1,-2)$ & $Q(0,4,3)$

$$\frac{x-5}{5-0}=\frac{y-1}{1-4}=\frac{z+2}{-2-3} \quad \Rightarrow \frac{x-5}{5}=\frac{y-1}{-3}=\frac{z+2}{-5}$$



Which are the required equations in symmetric form of L & M.

Now we write the equations of L & M in parametric forms.

$$Let \quad \frac{x-2}{3}=\frac{y-1}{-1}=\frac{z-3}{7} \quad =t \qquad (say)$$

$$let \quad \frac{x-5}{5}=\frac{y-1}{-3}=\frac{z+2}{-5} \quad =s \qquad (say)$$

Now parametric equations of lines L & M are

$$\frac{x-2}{3}=t, \quad \frac{y-1}{-1}=t, \quad \frac{z-3}{7}=t \quad \Rightarrow \quad L:\begin{cases}x=2+3t\\ y=1-1t\\ z=3+7t\end{cases}$$

$$\frac{x-5}{5}=s, \quad \frac{y-1}{-3}=s, \quad \frac{z+2}{-5}=s \quad \Rightarrow \quad M:\begin{cases}x=5+5s\\ y=1-3s\\ z=-2-5s\end{cases}$$

Let the lines L & M intersect at $\mathrm{P}(x,y,z)$ so this point will lie on both lines L & M.

Comparing above equations

$\Rightarrow 2+3t=5+5s \qquad \Rightarrow 3t-5s=3$ ------------ (1)

$1-t=1-3s \qquad \Rightarrow t-3s=0$ ------------ (2)

$3+7t=-2-5s \qquad \Rightarrow 7t+5s=-5$ ----------- (3)

Now multiply eq. (2) by 3 and subtracting

$$4s=3 \quad \Rightarrow \boldsymbol{s}=\frac{\mathbf{3}}{\mathbf{4}}$$

Put in eq. (2)

$$t-3\left(\frac{3}{4}\right)=0 \quad \Rightarrow \boldsymbol{t}=\frac{\mathbf{9}}{\mathbf{4}}$$

Now putting these values in equation (3) we have

$$7\left(\frac{9}{4}\right)+5\left(\frac{3}{4}\right)=-5 \quad \Rightarrow \frac{63}{4}+\frac{15}{4}=-5 \quad \Rightarrow \frac{78}{4}\neq-5$$

We see that these values of t & s do not satisfy equation (3)

Hence, given straight lines L & M do not intersect. So point of intersection doesn't exist.

Q#11: **L** : $\vec{r} = (3\hat{i} + 2\hat{j} - \hat{k}) + t(6\hat{i} - 4\hat{j} - 3\hat{k})$
M : $\vec{r} = (5\hat{i} + 4\hat{j} + 7\hat{k}) + s(14\hat{i} - 6\hat{j} + 2\hat{k})$

Solution:

For given straight line L

$\vec{r} = (3 + 6t)\hat{\imath} + (2 - 4t)\hat{\jmath} + (-1 - 3t)\hat{k}$

For given straight line M

$\vec{r} = (5 + 14s)\hat{\imath} + (4 - 6s)\hat{\jmath} + (-7 + 2s)\hat{k}$

Now parametric equations for line L are

$$\left.\begin{aligned} x &= 3 + 6t \\ y &= 2 - 4t \\ z &= -1 - 3t \end{aligned}\right\} \quad \text{---------(A)}$$

Parametric equations for given line M are

$$\left.\begin{aligned} x &= 5 + 14s \\ y &= 4 - 6s \\ z &= 7 + 2s \end{aligned}\right\} \quad \text{--------- (B)}$$

Now equations of straight lines L & M are in symmetric form

$$\frac{x-3}{6} = \frac{y-2}{-4} = \frac{z+1}{-3} \quad ----(L)$$

$$\frac{x-5}{14} = \frac{y-4}{-6} = \frac{z-7}{2} \quad ----(M)$$

Let the lines L & M intersect at (x, y, z) so this point will lie on both lines L & M.

Comparing equations (A) & (B)

$\Rightarrow 3 + 6t = 5 + 14s \quad \Rightarrow 6t - 14s = 2 \quad \Rightarrow 3t - 7s = 1$ ------------- (1)

$2 - 4t = 4 - 6s \quad \Rightarrow 4t - 6s = -2$ ------------ (2)

$-1 - 3t = 7 + 2s \quad \Rightarrow 3t + 2s = -8$ ------------- (3)

Subtracting eq. (1) & (3)

$$-9s = 9 \quad \Rightarrow \boldsymbol{s = -1}$$

Putting value of s in equation (1)

$$3t - 7(-1) = 1 \quad \Rightarrow 3t = 1 - 7 \quad \Rightarrow 3t = -6 \quad \Rightarrow \boldsymbol{t = -2}$$

Now putting values of s & t in equation (2)

$$4(-2) - 6(-1) = -2 \quad \Rightarrow -8 + 6 = -2 \quad \Rightarrow -2 = -2$$

We see that these values of s & t satisfy equation (2)

Hence given straight lines L & M intersect.

For point of intersection we put $s = -1$ in equation (B)

Hence point of intersection of given straight line is $(x, y, z) = (-9,10,5)$.

Q#12: **L : through $A(2,-1,0)$ and parallel to $\vec{b} = [4,3,2]$**
M : through $P(-1,3,5)$ and parallel to $\vec{c} = [1,7,3]$

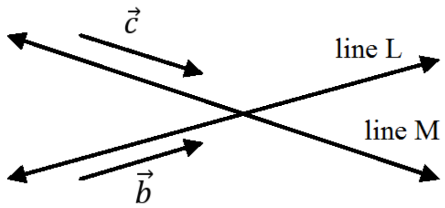
Solution:

Symmetric form of given straight line L through the point $A(2,-1,0)$ and parallel to $\vec{b} = [4,3,-2]$

$$\frac{x-2}{4} = \frac{y+1}{3} = \frac{z-0}{2}$$

let $$\frac{x-2}{4} = \frac{y+1}{3} = \frac{z}{-2} = t$$

Symmetric form of given straight line M through the point $P(-1,3,5)$ and parallel to $\vec{c} = [1,7,3]$.



$$\frac{x+1}{1} = \frac{y-3}{7} = \frac{z-5}{3}$$

let $$\frac{x+1}{1} = \frac{y-3}{7} = \frac{z-5}{3} = s$$

Now parametric equations of lines L & M are

for line L
$$\frac{x-2}{4} = t \implies x = 2+4t$$
$$\frac{y+1}{3} = t \implies y = -1+3t$$
$$\frac{z}{-2} = t \implies z = 0-2t$$

for line M
$$\frac{x+1}{1} = s \implies x = -1+1s$$
$$\frac{y-3}{7} = s \implies y = 3+7s$$
$$\frac{z-5}{3} = s \implies z = 5+3s$$

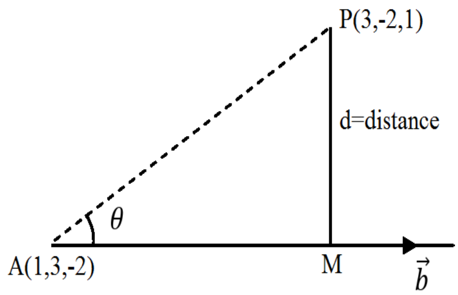
NOW DO YOURSELF AS ABOVE

- **Find the distance of the given point P from the given line L.**

Q#13: $P = (3,-2,1)$, L : $\begin{cases} x = 1+t \\ y = 3-2t \\ z = -2+2t \end{cases}$

Solution: Given point and given line are

$$P = (3,-2,1), \quad L : \begin{cases} x = 1+t \\ y = 3-2t \\ z = -2+2t \end{cases}$$



A point on the line L is $A = (1,3,-2)$

Now $\overrightarrow{AP} = P(3,-2,1) - A(1,3,-2) = [3-1,-2-3,1+2]$

$\overrightarrow{AP} = 2\hat{\imath} - 5\hat{\jmath} + 3\hat{k}$

Now direction vector of the given line L is $\vec{b} = 1\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$

Let d be the required distance of a point from a line L then by using formula

$$d = \frac{|\overrightarrow{AP}\times\vec{b}|}{|\vec{b}|} \quad ----(A)$$

$$\therefore \overrightarrow{AP}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -5 & 3 \\ 1 & -2 & 2 \end{vmatrix} = (-10+6)\hat{\imath} - (4-3)\hat{\jmath} + (-4+5)\hat{k} \implies \overrightarrow{AP}\times\vec{b} = -4\hat{\imath} - \hat{\jmath} + \hat{k}$$

$$|\overrightarrow{AP}\times\vec{b}| = \sqrt{(-4)^2+(-1)^2+1^2} = \sqrt{16+1+1} = \sqrt{18} \quad \& \quad |\vec{b}| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

Putting in equation (A) $\quad d = \frac{\sqrt{18}}{3}$ *is required distance*

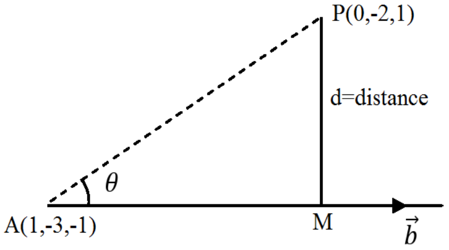
Q#14: $\mathbf{P = (0,-2,1)}$, $\quad \mathbf{L} \ : \ \dfrac{\mathbf{x-1}}{\mathbf{4}} = \dfrac{\mathbf{y+3}}{\mathbf{-2}} = \dfrac{\mathbf{z+1}}{\mathbf{5}}$

Solution: A point on the line L is $A = (1,-3,-1)$

$$\overrightarrow{AP} = P(0,-2,1) - A(1,-3,-1) = [0-1,-2+3,1+1]$$

$$\overrightarrow{AP} = -\hat{\imath} + \hat{\jmath} + 2\hat{k}$$

Now direction vector of the given line L is $\vec{b} = 4\hat{\imath} - 2\hat{\jmath} + 5\hat{k}$



DO YOURSELF AS ABOVE

Q#15: If the edges of a rectangular parallelepiped are a, b, c; show that the angles between the four diagonals are given by

$$\arccos\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

Solution:

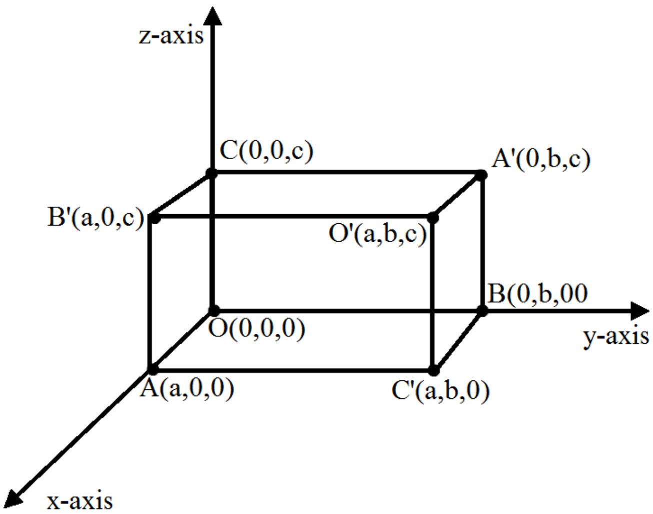
Consider a parallelepiped as shown in the figure.

Here the lengths of the edges OA, OB & OC are a, b, c respectively.

The coordinates of the vertices of parallelepiped are

$O = (0,0,0), O' = (a,b,c), A = (a,0,0), A' = (0,b,c)$

$B = (0,b,0), B' = (a,0,c), C = (0,0,c), C' = (a,b,0)$



The four diagonals of given parallelepiped are

$\overrightarrow{OO'}, \overrightarrow{AA'}, \overrightarrow{BB'}, \overrightarrow{CC'}$

Now

$$\overrightarrow{OO'} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

$$\overrightarrow{AA'} = -a\hat{\imath} + b\hat{\jmath} + c\hat{k}$$

$$\overrightarrow{BB'} = a\hat{\imath} - b\hat{\jmath} + c\hat{k}$$

$$\overrightarrow{CC'} = a\hat{\imath} + b\hat{\jmath} - c\hat{k}$$

So direction cosines of

$$\overrightarrow{OO'} \quad \Rightarrow l_1 = \frac{a}{\sqrt{a^2+b^2+c^2}}, m_1 = \frac{b}{\sqrt{a^2+b^2+c^2}}, n_1 = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\overrightarrow{AA'} \quad \Rightarrow l_2 = \frac{-a}{\sqrt{a^2+b^2+c^2}}, m_2 = \frac{b}{\sqrt{a^2+b^2+c^2}}, n_2 = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\overrightarrow{BB'} \quad \Rightarrow l_3 = \frac{a}{\sqrt{a^2+b^2+c^2}}, m_3 = \frac{-b}{\sqrt{a^2+b^2+c^2}}, n_3 = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$\overrightarrow{CC'} \quad \Rightarrow l_4 = \frac{a}{\sqrt{a^2+b^2+c^2}}, m_4 = \frac{b}{\sqrt{a^2+b^2+c^2}}, n_4 = \frac{-c}{\sqrt{a^2+b^2+c^2}}$$

Let α be the angle between $\overrightarrow{OO'}$ and $\overrightarrow{AA'}$

$\cos\alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \left(\frac{a}{\sqrt{a^2+b^2+c^2}}\right)\left(\frac{-a}{\sqrt{a^2+b^2+c^2}}\right) + \left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right)\left(\frac{b}{\sqrt{a^2+b^2+c^2}}\right) + \left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)\left(\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

$\cos\alpha = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$ − − − − − − − − (1)

Let β be the angle between $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$

$\cos\beta = l_2 l_3 + m_2 m_3 + n_2 n_3$

$\cos\beta = \frac{-a^2 - b^2 + c^2}{a^2 + b^2 + c^2}$ − − − − − − (2)

Let γ be the angle between $\overrightarrow{BB'}$ and $\overrightarrow{CC'}$

$\cos\gamma = l_3 l_4 + m_3 m_4 + n_3 n_4$

$\cos\gamma = \frac{a^2 - b^2 - c^2}{a^2 + b^2 + c^2}$ − − − − − − −(3)

Let θ be the angle between $\overrightarrow{OO'}$ and $\overrightarrow{CC'}$

$\cos\theta = l_1 l_4 + m_1 m_4 + n_1 n_4$

$\cos\theta = \frac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2}$ − − − − − (4)

From (1),(2),(3) & (4) we see that the angles between four diagonals are

$\cos(angle) = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2} \Rightarrow angle = \cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$ *hence proved.*

Q#16: A straight line makes angles of measure $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

Solution:

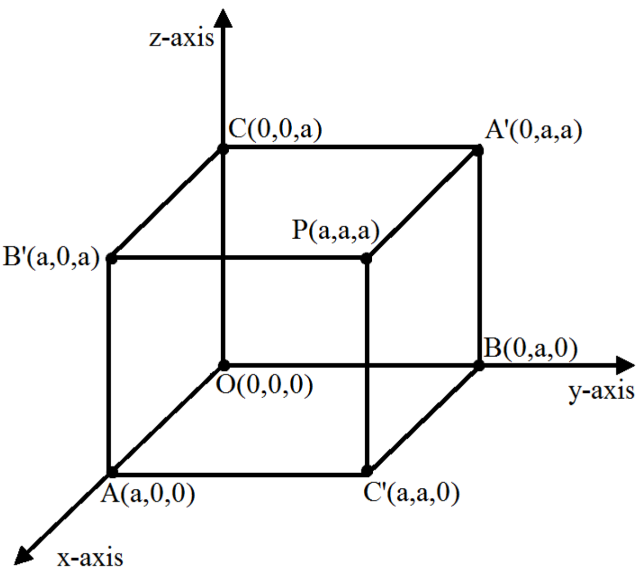
Let "a" be the length of each side of a cube

Points of the each corner of the cube are

$O = (0,0,0), P = (a,a,a), A = (a,0,0), B = (0,a,0), C = (0,0,a)$

$A' = (0,a,a), B' = (a,0,a), C' = (a,a,0)$

Now $\overrightarrow{OP}$, $\overrightarrow{AA'}$, $\overrightarrow{BB'}$, $\overrightarrow{CC'}$ are the diagonals of a cube



$\overrightarrow{OP} = (a,a,a) - (0,0,0) = a\hat{i} + a\hat{j} + a\hat{k}$

$\overrightarrow{AA'} = (0,a,a) - (a,0,0) = -a\hat{i} + a\hat{j} + a\hat{k}$

$\overrightarrow{BB'} = (a,0,a) - (0,a,0) = a\hat{i} - a\hat{j} + a\hat{k}$

$\overrightarrow{CC'} = (a,a,0) - (0,0,a) = a\hat{i} + a\hat{j} - a\hat{k}$

Length of each diagonal is $\sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = \sqrt{3}a$

Now direction cosines of each diagonals are

$\overrightarrow{OP} \implies l_1 = \frac{a}{\sqrt{3}a}, m_1 = \frac{a}{\sqrt{3}a}, n_1 = \frac{a}{\sqrt{3}a}$

$\overrightarrow{OP} \implies l_1 = \frac{1}{\sqrt{3}}, m_1 = \frac{1}{\sqrt{3}}, n_1 = \frac{1}{\sqrt{3}}$

$\overrightarrow{AA'} \implies l_2 = \frac{-1}{\sqrt{3}}, m_2 = \frac{1}{\sqrt{3}}, n_2 = \frac{1}{\sqrt{3}}$

$\overrightarrow{BB'} \implies l_3 = \frac{1}{\sqrt{3}}, m_3 = \frac{-1}{\sqrt{3}}, n_3 = \frac{1}{\sqrt{3}}$

$\overrightarrow{CC'} \implies l_4 = \frac{1}{\sqrt{3}}, m_4 = \frac{1}{\sqrt{3}}, n_4 = \frac{-1}{\sqrt{3}}$

Let l, m, n be the direction cosines of line L which makes angles α, β, γ & δ with each diagonals

α is the angle between line L and $\overrightarrow{OP}$

$$\cos\alpha = ll_1 + mm_1 + nn_1 = l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) = \frac{l+m+n}{\sqrt{3}} \qquad ---(1)$$

β is the angle between line L and $\overrightarrow{AA'}$

$$\cos\beta = ll_2 + mm_2 + nn_2 = l\left(\frac{-1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) = \frac{-l+m+n}{\sqrt{3}} \qquad ---(2)$$

γ is the angle between line L and $\overrightarrow{BB'}$

$$\cos\gamma = ll_3 + mm_3 + nn_3 = l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{-1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) = \frac{l-m+n}{\sqrt{3}} \qquad ---(3)$$

δ is the angle between line L and $\overrightarrow{CC'}$

$$\cos\delta = ll_4 + mm_4 + nn_4 = l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{-1}{\sqrt{3}}\right) = \frac{l+m-n}{\sqrt{3}} \qquad ---(4)$$

Squaring eq. (1),(2),(3) & (4) and adding

$$cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \left(\frac{l+m+n}{\sqrt{3}}\right)^2 + \left(\frac{-l+m+n}{\sqrt{3}}\right)^2 + \left(\frac{l-m+n}{\sqrt{3}}\right)^2 + \left(\frac{l+m-n}{\sqrt{3}}\right)^2$$

$$= \frac{l^2+m^2+n^2+2lm+2mn+2nl}{3} + \frac{l^2+m^2+n^2-2lm+2mn-2nl}{3}$$

$$+ \frac{l^2+m^2+n^2-2lm-2mn+2nl}{3} + \frac{l^2+m^2+n^2+2lm-2mn-2nl}{3}$$

$$cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4l^2+4m^2+4n^2}{3} = \frac{4(l^2+m^2+n^2)}{3}$$

$$cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3} \qquad \textit{hence proved}$$

Q#17: Find equations of the straight line passing through the point $\mathbf{P(0,-3,2)}$ and parallel to the straight line joining the points $\mathbf{A(3,4,7)}$ and $\mathbf{B(2,7,5)}$.

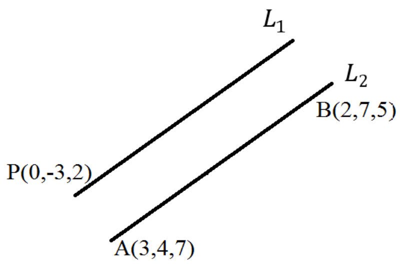
Solution: Consider two lines L_1 & L_2

Let L_1 be the required equation of the straight line passing through the point $P(0,-3,2)$

The points $A(3,4,7)$ and $B(2,7,5)$ on line L_2. So direction ratios from $A(3,4,7)$ to $B(2,7,5)$ are

$$\overrightarrow{AB} = (2,7,5) - (3,4,7) = -\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$$

Then $-1,3,-2$ are direction ratios of line L_2.



By given condition both lines are parallel so both lines having same direction ratios.

Now required equation through the point $P(0,-3,2)$

$$\frac{x-0}{-1} = \frac{y+3}{3} = \frac{z-2}{-2} \Rightarrow \frac{x}{1} = \frac{y+3}{-3} = \frac{z-2}{2} \quad \textit{required equation.}$$

Q#18: Find equations of the straight line passing through the point $\mathbf{P(2,0,-2)}$ and perpendicular to each of straight lines

$$\frac{\mathbf{x-3}}{\mathbf{2}} = \frac{\mathbf{y}}{\mathbf{2}} = \frac{\mathbf{z+1}}{\mathbf{2}} \quad \textbf{and} \quad \frac{\mathbf{x}}{\mathbf{3}} = \frac{\mathbf{y+1}}{\mathbf{-1}} = \frac{\mathbf{z+2}}{\mathbf{2}}$$

Solution:

Given equations of lines are

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \quad ----(L_1)$$

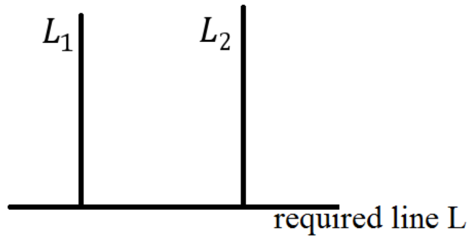
$$\frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2} \quad ----(L_2)$$

Direction ratios of line L_1 are 2,2,2 $\Rightarrow \vec{d}_1 = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Direction ratios of line L_2 are $3,-1,2 \Rightarrow \vec{d}_2 = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$

Suppose L be the required line through the point $P(2,0,-2)$ with

direction ratios $c_1, c_2, c_3 \quad \Rightarrow \vec{d}_3 = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$



Since $L \perp L_1$

So by condition of perpendicularity

$$\vec{d}_1.\vec{d}_3 = 0 \quad \Rightarrow (2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}).(c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}) = 0$$

$$2c_1 + 2c_2 + 2c_3 = 0 \text{ --------------- } (1)$$

$$\vec{d}_2.\vec{d}_3 = 0 \quad \Rightarrow \quad (3\hat{\imath} - \hat{\jmath} + 2\hat{k}).(c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}) = 0$$

$$3c_1 - c_2 - 2c_3 = 0 \text{ -------------- } (2)$$

Now

$$\frac{c_1}{\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}} = \frac{-c_2}{\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{c_3}{\begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix}} \quad \Rightarrow \frac{c_1}{6} = \frac{-c_2}{-2} = \frac{c_3}{-8} \quad \Rightarrow \frac{c_1}{3} = \frac{c_2}{1} = \frac{c_3}{-4}$$

Hence $(3,1,-4)$ be the direction ratios of required line L

Now equation of required line L passing through the point $P(2,0,-2)$ is $\frac{x-2}{3} = \frac{y-0}{1} = \frac{z+2}{-4}$

Find equations of straight line through the given point A and intersecting at right angles the given straight line:

Q#19: $A = (11, 4, -6)$ and $x = 4 - t, y = 7 + 2t, z = -1 + t$.

Solution: Let L be the required line passing through $A = (11,4,-6)$ and perpendicular to given line.

Suppose it meets the given line at point B.

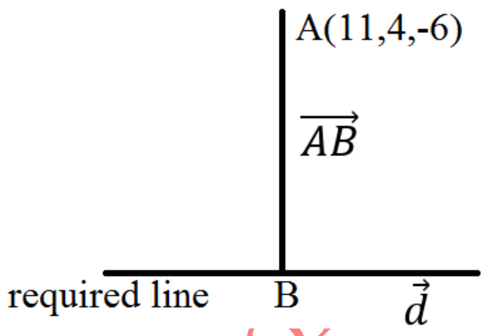
Now a point on given line is B $(4 - t, 7 + 2t, -1 + t)$

$$\overrightarrow{AB} = (4 - t - 11)\hat{\imath} + (7 + 2t - 4)\hat{\jmath} + (-1 + t + 6)\hat{k}$$

$$\overrightarrow{AB} = (-t - 7)\hat{\imath} + (2t + 3)\hat{\jmath} + (t + 5)\hat{k}$$

Hence direction vector of the given straight line is $\vec{d} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$

Since $\overrightarrow{AB}$ is perpendicular to given line.



By perpendicular condition

$$\overrightarrow{AB} \perp \vec{d} = 0 \quad \Rightarrow \overrightarrow{AB}.\vec{d} = 0$$

$$[(-t - 7)\hat{\imath} + (2t + 3)\hat{\jmath} + (t + 5)\hat{k}].[-\hat{\imath} + 2\hat{\jmath} + \hat{k}] = 0$$

$$(-1)(-t - 7) + (2)(2t + 3) + (1)(t + 5) = 0 \quad \Rightarrow 7 + t + 4t + 6 + t + 5 = 0 \quad \Rightarrow \quad \boldsymbol{t = -3}$$

Direction vector of required line will become

$$\overrightarrow{AB} = (-(-3) - 7)\hat{\imath} + (2(-3) + 3)\hat{\jmath} + (-3 + 5)\hat{k} \quad \Rightarrow \overrightarrow{AB} = -4\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$$

Now required equation passing through the point $A = (11,4,-6)$ having direction ratios $-4, -3, 2$

$$\frac{x-11}{-4} = \frac{y-4}{-3} = \frac{z+6}{2} \qquad \text{OR} \qquad \frac{x-11}{4} = \frac{y-4}{3} = \frac{z+6}{-2}$$

Q#20: $A = (5, -4, 4)$ and $\frac{x}{-1} = \frac{y-1}{1} = \frac{z}{-2}$

Solution: Given point and line are $A = (5,-4,4)$ *and* $\frac{x}{-1} = \frac{y-1}{1} = \frac{z}{-2} = t \quad (say)$

The parametric equations of given line are $x = -t, y = 1 + t, z = -2t$

DO YOURSELF AS ABOVE

Q#21: Find the length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the straight line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where $l^2 + m^2 + n^2 = 1$

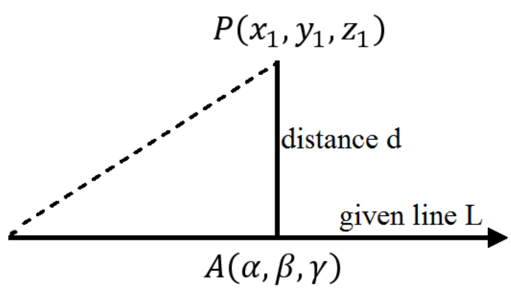
Solution: Given point and line are

$$P(x_1, y_1, z_1) \quad \& \quad \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Hence $A = (\alpha, \beta, \gamma)$ is a point of given line

Direction vector of given line is $\vec{b} = l\hat{\imath} + m\hat{\jmath} + n\hat{k}$

Now $\overrightarrow{AP} = (x_1 - \alpha)\hat{\imath} + (y_1 - \beta)\hat{\jmath} + (z_1 - \gamma)\hat{k}$



Let d be the required distance then by using formula

$$d = \frac{|\overrightarrow{AP} \times \vec{b}|}{|\vec{b}|} \quad ----(A)$$

$$\therefore\ \overrightarrow{AP}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ x_1-\alpha & y_1-\beta & z_1-\gamma\\ l & m & n\end{vmatrix}$$

$$=[n(y_1-\beta)-m(z_1-\gamma)]\hat{\imath}-[n(x_1-\alpha)-l(z_1-\gamma)]\hat{\jmath}+[m(x_1-\alpha)-l(y_1-\beta)]\hat{k}$$

$$\overrightarrow{AP}\times\vec{b}=[n(y_1-\beta)-m(z_1-\gamma)]\hat{\imath}+[l(z_1-\gamma)-n(x_1-\alpha)]\hat{\jmath}+[m(x_1-\alpha)-l(y_1-\beta)]\hat{k}$$

$$|\overrightarrow{AP}\times\vec{b}|=\sqrt{[n(y_1-\beta)-m(z_1-\gamma)]^2+[l(z_1-\gamma)-n(x_1-\alpha)]^2+[m(x_1-\alpha)-l(y_1-\beta)]^2}$$

$$|\overrightarrow{AP}\times\vec{b}|=\sqrt{\Sigma[n(y_1-\beta)-m(z_1-\gamma)]^2} \qquad \& \qquad |\vec{b}|=\sqrt{l^2+m^2+n^2}=\sqrt{1}=1$$

Putting in equation (A)

$$d=\frac{\sqrt{\Sigma[n(y_1-\beta)-m(z_1-\gamma)]^2}}{1} \quad \Rightarrow d=\sqrt{\sum[n(y_1-\beta)-m(z_1-\gamma)]^2} \quad \textit{required distance}$$

Q#22: Find equations of the perpendicular from the point P(1, 6, 3) to the straight line

$$\frac{\mathbf{x}}{\mathbf{1}}=\frac{\mathbf{y-1}}{\mathbf{2}}=\frac{\mathbf{z-2}}{\mathbf{3}}$$

Also obtain its length and coordinates of the foot of the perpendicular.

Solution:

Given point $P(1,6,3)$ and equation of straight line is

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$$

Let AP be the length of perpendicular from point A to given line

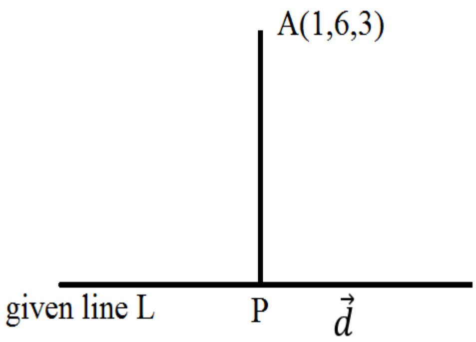
$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}=t$$

So parametric equations of given line are

$$\left.\begin{aligned} x&=t\\ y&=1+2t\\ z&=2+3t\end{aligned}\right\}$$

Any point on this line is $(t,1+2t,2+3t)$

So coordinates of point P are $P(t,1+2t,2+3t)$



Direction ratios of line $\overrightarrow{AP}$ are $=(t,1+2t,2+3t)-(1,6,3)$

$$\overrightarrow{AP}=(t-1)\hat{\imath}+(2t-5)\hat{\jmath}+(3t-1)\hat{k}$$

& direction vector of given line is $\vec{d}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$

Since $\overrightarrow{AP}\perp\vec{d}$ $\quad\Rightarrow \overrightarrow{AP}.\vec{d}=0$

$$[(t-1)\hat{\imath}+(2t-5)\hat{\jmath}+(3t-1)\hat{k}].(\hat{\imath}+2\hat{\jmath}+3\hat{k})=0$$

$$(1)(t-1)+(2)(2t-5)+(3)(3t-1)=0 \quad\Rightarrow 14t-14=0 \quad\Rightarrow \boldsymbol{t=1}$$

So coordinates of point P are $P(1,3,5)$

Length of perpendicular $=|AP|=\sqrt{(1-1)^2+(3-1)^2+(5-3)^2}=\sqrt{0+9+4}=\sqrt{13}$

Now equation of perpendicular $\overrightarrow{AP}$ is

$$\frac{x-1}{1-1}=\frac{y-6}{3-6}=\frac{z-3}{5-3}\Rightarrow\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2} \quad \textit{required equation of line}$$

Q#23: Find necessary and sufficient condition that the point $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ and $R(x_3,y_3,z_3)$ are collinear.

Solution: Given points are $P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$ & $R(x_3,y_3,z_3)$

Suppose that the points P,Q &R are collinear.

Now equation of line through $P(x_1,y_1,z_1)$ & $Q(x_2,y_2,z_2) is$ $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

Since points P,Q &R are collinear. So point $R(x_3,y_3,z_3)$ lies on line.

$$\frac{x_3-x_1}{x_2-x_1}=\frac{y_3-y_1}{y_2-y_1}=\frac{z_3-z_1}{z_2-z_1}=t\ (say)$$

$$\Rightarrow \left.\begin{aligned} x_3-x_1&=t(x_2-x_1)\\ y_3-y_1&=t(y_2-y_1)\\ z_3-z_1&=t(z_2-z_1)\end{aligned}\right] \quad or \quad \left.\begin{aligned} x_3-x_1-tx_2+tx_1&=0\\ y_3-y_1-ty_2+ty_1&=0\\ z_3-z_1-tz_2+tz_1&=0\end{aligned}\right] \Rightarrow \left.\begin{aligned}(t-1)x_1-tx_2+x_3&=0\\(t-1)y_1-ty_2+y_3&=0\\(t-1)z_1-tz_2+z_3&=0\end{aligned}\right\}$$

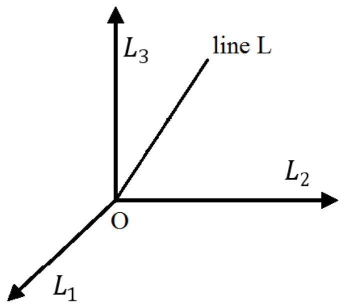
Eliminating $(t-1),-t,1$ from above equations

$$\begin{vmatrix} x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{vmatrix}=0 \quad or \quad \begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}=0$$

Which is necessary condition for three points P,Q &R to be collinear.

Q#24: If $l_1,m_1,n_1;\ l_2,m_2,n_2$ and l_3,m_3,n_3 are direction cosines of three mutual perpendicular lines, prove that the lines whose direction cosines are proportional to $l_1+l_2+l_3, m_1+m_2+m_3, n_1+n_2+n_3$ makes congruent angles with them.

Solution: Suppose that L_1,L_2 & L_3 are given line such that



Direction cosines of L_1 are l_1,m_1,n_1

Direction cosines of L_2 are l_2,m_2,n_2

Direction cosines of L_3 are l_3,m_3,n_3

Since lines L_1,L_2 & L_3 are mutually perpendicular

$$\text{So}\quad \left.\begin{aligned} l_1l_2+m_1m_2+n_1n_2&=0\\ l_2l_3+m_2m_3+n_2n_3&=0\\ l_1l_3+m_1m_3+n_1n_3&=0\end{aligned}\right\}$$

Let L be the line having direction ratios $l_1+l_2+l_3$, $m_1+m_2+m_3$ & $n_1+n_2+n_3$

Let α be the angle between L & L_1 then

$$\cos\alpha=\frac{l_1(l_1+l_2+l_3)+m_1(m_1+m_2+m_3)+n_1(n_1+n_2+n_3)}{\sqrt{{l_1}^2+{m_1}^2+{n_1}^2}.\ \sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$=\frac{({l_1}^2+{m_1}^2+{n_1}^2)+(l_1l_2+m_1m_2+n_1n_2)+(l_1l_3+m_1m_3+n_1n_3)}{\sqrt{({l_1}^2+{m_1}^2+{n_1}^2)+({l_2}^2+{m_2}^2+{n_2}^2)+({l_3}^2+{m_3}^2+{n_3}^2)+2(l_1l_2+m_1m_2+n_1n_2)}}$$

$$\cos\alpha=\frac{1+0+0}{\sqrt{1+1+1+0+0+0}} \Rightarrow \cos\alpha=\frac{1}{\sqrt{3}} \Rightarrow \alpha=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad -----(1)$$

Let β be the angle between L & L_2 then

$$\cos\beta = \frac{l_2(l_1+l_2+l_3)+m_2(m_1+m_2+m_3)+n_2(n_1+n_2+n_3)}{\sqrt{{l_2}^2+{m_2}^2+{n_2}^2}.\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{(l_1l_2+m_1m_2+n_1n_2)+\left({l_2}^2+{m_2}^2+{n_2}^2\right)++(l_2l_3+m_2m_3+n_2n_3)}{\sqrt{\left({l_1}^2+{m_1}^2+{n_1}^2\right)+\left({l_2}^2+{m_2}^2+{n_2}^2\right)+\left({l_3}^2+{m_3}^2+{n_3}^2\right)+2(l_1l_2+m_1m_2+n_1n_2)}}$$

$$\cos\beta = \frac{0+1+0}{\sqrt{1+1+1+0+0+0}} \Rightarrow \cos\beta = \frac{1}{\sqrt{3}} \quad \Rightarrow \beta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad -----(2)$$

Let γ be the angle between L & L_1 then

$$\cos\gamma = \frac{l_3(l_1+l_2+l_3)+m_3(m_1+m_2+m_3)+n_3(n_1+n_2+n_3)}{\sqrt{{l_3}^2+{m_3}^2+{n_3}^2}.\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{(l_1l_3+m_1m_3+n_1n_3)+(l_2l_3+m_2m_3+n_2n_3)+\left({l_3}^2+{m_3}^2+{n_3}^2\right)}{\sqrt{\left({l_1}^2+{m_1}^2+{n_1}^2\right)+\left({l_2}^2+{m_2}^2+{n_2}^2\right)+\left({l_3}^2+{m_3}^2+{n_3}^2\right)+2(l_1l_2+m_1m_2+n_1n_2)}}$$

$$\cos\gamma = \frac{1+0+0}{\sqrt{1+1+1+0+0+0}} \Rightarrow \cos\gamma = \frac{1}{\sqrt{3}} \quad \Rightarrow \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad ----(3)$$

From (1) (2) & (3) it is proved that $\Rightarrow \alpha \cong \beta \cong \gamma$ *required result*

Q#25: A variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l, m+\delta m, n+\delta n$. Show that measure of the small angle δθ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

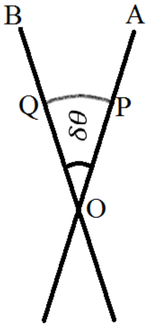
Solution:

Let OA and OB be the two adjacent positions of the line. Let PQ be the arc of the circle with centre at O and radius 1.

Then the coordinates of the points.

P & Q are $P(l,m,n)$& $Q(l+\delta l, m+\delta m, n+\delta n)$.

Let $\delta\theta$ be the angle between two positions of line.



Now $\delta\theta = \text{chord } PQ$

So $\delta\theta = |PQ|$

$$\delta\theta = \sqrt{(l+\delta l-l)^2+(m+\delta m-m)^2+(n+\delta n-n)^2}$$

$$\delta\theta = \sqrt{\delta l^2+\delta m^2+\delta n^2}$$

$$\Rightarrow \quad \delta\theta^2 = \delta l^2+\delta m^2+\delta n^2$$

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