Exercise 7.3

Types of Cusp.

Two branches of a curve has common tangent at a cusp. There are five different ways in which the two branches stand in relation to the common tangent and the common normal as illustrated in the following diagrams.

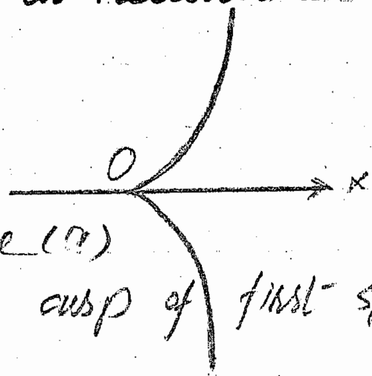

Figure (a)
Single cusp of first species

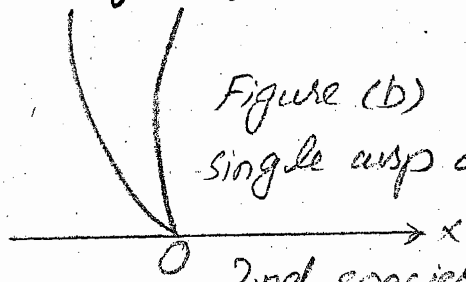

Figure (b)
Single cusp of 2nd species.

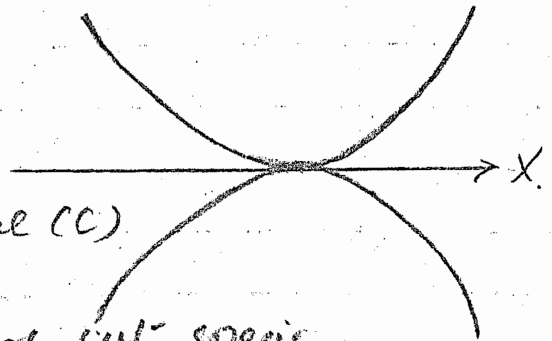

Figure (C)
Double cusp of first speic

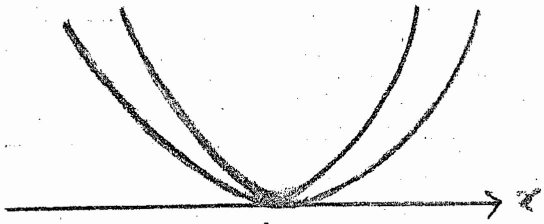

Figure (d)
Double cusp of second species

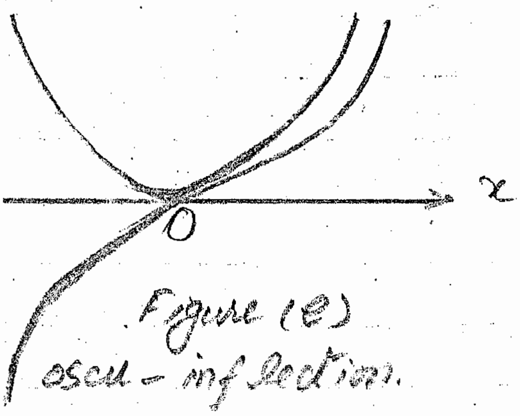

Figure (e)
Point of oscu - inflection.

<u>In figure (a)</u>, the two branches lie on the same side of the common normal and on different sides of the tangent. <u>In figure (b)</u> the two branches lie on the same side of the normal and on the same side of tangent. <u>In figure (c)</u> the two branches lie on different sides of the normal and on different sides of the tangent. <u>In figure (d)</u> the two branches lie on different sides of the normal and and on the same side of the tangent. <u>In figure (e)</u> the two branches lie on different sides of the normal but on one side they lie on the same and on the other on

opposite sides of the common tangent. one branch has an inflection point at O.

A cusp is single or double according as the two branches lie on the same or different sides of the common normal. Also, it is of the first or 2nd species according as the branches lie on different or on the same side of the common tangent.

Mathematically.

1) If cuspidal tangents are $y^2=0$ then solved the given eq. for y neglecting $y^3, y^4 \ldots\ldots$
2) If roots are real for one sign of x, then cusp is single.
3) If roots are real for both signs of x then cusp is double.
4) If the roots are of opposite signs, then cusp is called first species.
5) If the roots are of same signs, then cusp is called 2nd species.

How we can find out the tangent at the origin.

Arrange the given equation in descending powers of x and y and equate to zero. The lowest degree terms gives you the tangent at the origin.

How to search for Singular Points

Let an equation of a curve be $f(x,y)=0$ slope of the tangent at any point (x,y) on the curve is

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

For possible singular points put

$$f_x = 0 , \quad f_y = 0$$

Singular points are the common points if

$$f_x = 0 , \quad f_y = 0 , \quad f(x,y) = 0$$

Differentiating $f_x + f_y \frac{dy}{dx} = 0$ w.r.t. 'x' we have

$$f_{xx} + f_{xy}\frac{dy}{dx} + \left(f_{yx} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y \frac{d^2y}{dx^2} = 0$$

So that at a singular point the values of $\frac{dy}{dx}$ are the roots of quadratic equation

$$f_{yy}\left(\frac{dy}{dx}\right)^2 + 2f_{xy}\frac{dy}{dx} + f_{xx} = 0 \qquad (f_{yx} = f_{xy})$$

In case f_{xx}, f_{xy} and f_{yy} are not all zero, the point (x,y) will a double point. It will be a node or a cusp or an isolated point according as the values of $\frac{dy}{dx}$ are real and distinct, equal or imaginary, i.e. according as

$$(f_{xy})^2 - f_{xx}f_{yy} \gtreqless 0$$

i.e A point $p(x,y)$ on a curve is a nodal, a cusp or an isolated pt. according as

$$(f_{xy})^2 - f_{xx}f_{yy} \gtreqless 0$$

where

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

If $f_{xx} = f_{xy} = f_{yy} = 0$, the point (x,y) will be a multiple point of order higher than two.

Exercise 7.3

Determine the nature of the singular point (0,0) (1-4)

Q.1. $$(x^2+y^2)^2 = 4a^2xy$$

$$(x^2+y^2)^2 - 4a^2xy = 0$$

Equate to zero the lowest degree terms.

i.e $$-4a^2xy = 0$$

$$\Rightarrow \quad xy = 0$$

$$\Rightarrow \quad x = 0 \ , \ y = 0$$

$\because$ the tangents at (0,0) are $x=0, y=0$ real and distinct.

$\therefore$ Origin is a node.

Q.2. $$y^2(a^2-x^2) = x^2(b-x)^2$$

$$y^2a^2 - y^2x^2 = x^2(b^2+x^2-2bx)$$

$$a^2y^2 - x^2y^2 - b^2x^2 - x^4 + 2bx^3 = 0$$

For equations of tangents equate to zero the lowest degree terms.

i.e. $$a^2y^2 - b^2x^2 = 0$$

$$a^2y^2 = b^2x^2$$

$$ay = \pm bx$$

$$y = \pm \frac{b}{a}x$$

$\because$ There tangents are real disinct at the origin.

$\therefore$ Origin is a node.

Q.3. $$(x^2+y^2)(2a-x) = b^2x$$

$$2ax^2 + 2ay^2 - x^3 - xy^2 - b^2x = 0$$

For equations of tangents at the origin, equating the lowest degree terms to zero.

i.e. $$-b^2x = 0$$

$$\Rightarrow \quad x = 0$$

Origin is a cusp.

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Q.4.

$$a^2(x^2-y^2) = x^2y^2$$
$$a^2(x^2-y^2) - x^2y^2 = 0$$

For Equations of tangents at the origin, Equating the lowest degree terms to zero.

i.e
$$a^2(x^2-y^2)=0$$
$$x^2-y^2=0$$
$$x^2=y^2$$
$$y=\pm x$$

$\because$ These tangents are real and distinct.

$\therefore$ Origin is a node.

Find the position and nature of the Multiple points on the given curves. (5-10)

Q.5.

$$x^2(x-y)+y^2=0 \quad \text{——— ①}$$

Let $f(x,y) = x^3 - x^2y + y^2$

Now $f_x = 3x^2 - 2xy$

and $f_y = -x^2 + 2y$

For possible singular points put

$$f_x = 0 \ , \ f_y = 0$$
$$3x^2 - 2xy = 0$$
$$\Rightarrow \quad x(3x-2y)=0$$
$$\Rightarrow \quad x=0 \ , \ 3x-2y=0$$
$$x=0 \ , \ 3x=2y \quad \text{——— ②}$$

And

$$-x^2+2y = 0$$
$$2y = x^2 \quad \text{——— ③}$$

When $x=0$

$3) \Rightarrow \quad y=0$

possible Singular point : $(0,0)$

$\because$ The point $(0,0)$ lie on the curve

$\therefore$ $(0,0)$ is a singular point.

Now when $3x=2y$ put in 3

$$3x = x^2$$

$$x^2 - 3x = 0$$
$$x(x-3) = 0$$
$$x = 0, \quad x = 3$$

when $x = 0$ put in ②
$$y = 0$$
point $(0,0)$

when $x = 3$ put in ②
$$9 = 2y$$
$$y = 9/2$$
point. $(3, 9/2)$

Hence the possible singular points are
$$(0,0), \ (3, 9/2)$$

$\because$ Only $(0,0)$ lie on the given curve ~~and~~ or satisfy the given curve

$\therefore$ $(0,0)$ is a singular point.

Now
$$f_{xx} = 6x - 2y, \quad f_{xx}\big|_{(0,0)} = 0$$
$$f_{yy} = 2$$
$$f_{yy}\big|_{(0,0)} = 2$$

$$f_{xy} = -2x$$
$$f_{xy}\big|_{(0,0)} = 0$$

Now
$$(f_{xy})^2 - f_{xx} f_{yy} = (0)^2 - (0)(2)$$
$$= 0$$

$\Rightarrow$ $(0,0)$ is a cusp.

Q.6.
$$y^3 = x^3 + ax^2 \quad \text{———— ①}$$

Let $f(x,y) = y^3 - x^3 - ax^2$
$$f_x = -3x^2 - 2ax \quad \text{———— ②}$$
$$f_y = 3y^2 \quad \text{———— ③}$$

For possible singular points put $f_x = 0$, $f_y = 0$

2) $\Rightarrow$
$$-3x^2 - 2ax = 0$$
$$x(-3x - 2a) = 0$$
$$\Rightarrow \quad x = 0, \quad -3x - 2a = 0$$
$$x = 0, \quad x = -\frac{2a}{3}$$

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3) $\Rightarrow$ $3y^2 = 0$

$y = 0$

Hence the possible singular points are : $(0,0)$, $(-\frac{2a}{3}, 0)$

$\because$ $(0,0)$ only lie on the given curve

$\therefore$ $(0,0)$ is a singular point.

Now 2) $\Rightarrow$ $f_{xx} = -6x - 2a$, $f_{xx}\big|_{(0,0)} = -2a$

3) $\Rightarrow$ $f_{yy} = 6y$ $\quad f_{yy}\big|_{(0,0)} = 0$

2) $\Rightarrow$ $f_{xy} = 0$

$f_{xy}\big|_{(0,0)} = 0$.

So $(f_{xy})^2 - f_{xx} \cdot f_{yy} = 0 - (-2a)(0)$

$= 0$

$\Rightarrow$ $(0,0)$ is a cusp.

Q.7 $\quad x^4 + y^3 - 2x^3 + 3y^2 = 0$

Let $f(x,y) = x^4 + y^3 - 2x^3 + 3y^2$

$f_x = 4x^3 - 6x^2$ ———— (I)

$f_y = 3y^2 + 6y$ ———— (II)

For possible singular points put $f_x = 0$ and $f_y = 0$

I) $\Rightarrow$ $4x^3 - 6x^2 = 0$

$x^2(4x - 6) = 0$

$\Rightarrow$ $x^2 = 0$, $4x - 6 = 0$

$x = 0$, $4x = 6$

$x = 0$, $x = 3/2$

II) $\Rightarrow$ $3y^2 + 6y = 0$

$3y(y + 2) = 0$

$\Rightarrow$ $3y = 0$, $y + 2 = 0$

$y = 0$, $y = -2$

Hence the possible singular points are : $(0,0)$, $(0,-2)$, $(3/2, 0)$, $(3/2, -2)$

$\because$ $(0,0)$ only satisfy the given curve

$\therefore$ $(0,0)$ is a singular point.

Now I) $\Rightarrow$ $f_{xx} = 12x^2 - 12x$, $f_{xx}\big|_{(0,0)} = 0$

$f_{xy} = 0$

$\text{II}) \Rightarrow \quad f_{yy} = 6y + 6 \quad , \quad f_{yy}\big|_{(0,0)} = 6$

So $\quad (f_{xy})^2 - f_{xx} f_{yy} = 0 - (0)(6)$
$$= 0$$

$\Rightarrow (0,0)$ is a cusp.

Q. No. 8

$$x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$$

Let $\quad f(x,y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y$

$$f_x = 3x^2 + 4x + 2y + 5 \quad \text{———— (1)}$$
$$f_y = 2x - 2y - 2 \quad \text{———— (2)}$$

For possible singular points put $f_x = 0$ and $f_y = 0$

So 1) $\Rightarrow \quad 3x^2 + 4x + 2y + 5 = 0$ ———— (3)

also 2) $\Rightarrow \quad 2x - 2y - 2 = 0$

$$\Rightarrow \quad x - y - 1 = 0$$
$$\Rightarrow \quad y = x - 1 \quad \text{———— (4)}$$

(4) in (3) $\Rightarrow$

$$3x^2 + 4x + 2(x-1) + 5 = 0$$
$$3x^2 + 4x + 2x - 2 + 5 = 0$$
$$3x^2 + 6x + 3 = 0$$
$$x^2 + 2x + 1 = 0$$
$$x^2 + x + x + 1 = 0$$
$$x(x+1) + 1(x+1) = 0$$
$$\Rightarrow (x+1)(x+1) = 0$$
$$(x+1)^2 = 0$$
$$\Rightarrow x + 1 = 0$$
$$\Rightarrow x = -1$$

Put $x = -1$ in (4), we have

$$y = -1 - 1$$
$$y = -2$$

So the possible singular pt. is

$$(-1, -2)$$

So $(-1,-2)$ is only the singular pt.

Now (1) $\Rightarrow$ $f_{xx} = 6x+4$

$f_{yy} = -2$

$f_{xy} = 2$

$(f_{xy})^2 - f_{xx} f_{yy} \lesseqgtr 0$

At $(-1,-2)$

$(2)^2 - (6(-1)+4)(-2)$

$= (2)^2 - (-6+4)(-2)$

$= (2)^2 - (-2)(-2)$

$= 4 - 4 = 0$

Hence $(-1,-2)$ is a cusp.

Q. No. 9: $(2y+x+1)^2 - 4(1-x)^5 = 0$

Let $f(x,y) = (2y+x+1)^2 - 4(1-x)^5$

$f_x = 2(2y+x+1)(1) - 4 \cdot 5(1-x)^4(0-1)$

$f_x = 2(2y+x+1) + 20(1-x)^4$

and $f_y = 2(2y+x+1) \cdot 2 - 4(5)(1-x)^4(0-0)$

$f_y = 4(2y+x+1)$

Now put $f_x = 0$

$\Rightarrow$ $2(2y+x+1) + 20(1-x)^4 = 0$

$(2y+x+1) + 10(1-x)^4 = 0$ ——— (1)

Now put $f_y = 0$

$\Rightarrow$ $4(2y+x+1) = 0$

$\Rightarrow$ $2y+x+1 = 0$ ——— (2)

Put (2) in (1), we have

$0 + 10(1-x)^4 = 0$

$\Rightarrow$ $10(1-x)^4 = 0$

$1 - x^4 = 0$

$x^4 = 1$

$x = 1$

Put $x = 1$ in (2)

$\Rightarrow$ $2y+1+1 = 0$

$2y = -2$

$y = -1$

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So the possible Singular Point is
$(1,-1)$

$\because$ $(1,-1)$ satisfies the given eq.

$\therefore$ $(1,-1)$ is a singular point.

Now

$$f_{xy} = 4$$
$$f_{xx} = 2+80(1-x)^3$$
$$f_{yy} = 8$$

So $\left(f_{xy}(1,-1)\right)^2 - \left[f_{xx}(1,-1)\right]\left[f_{yy}(1,-1)\right]$

$$= 4^2 - [2+80(1-1)^3][8]$$
$$= 4^2 - [2+0][8]$$
$$= 4^2 - (2)(8)$$
$$= 16 - 16$$
$$= 0$$

Hence $(1,-1)$ is a cusp.

Q. No. 10. $(y^2-a^2)^3 + x^4(2x+3a)^2 = 0$

let $f(x,y) = (y^2-a^2)^3 + x^4(2x+3a)^2$

$$f_x = 0 + 4x^3(2x+3a)^2 + x^4(2)(2x+3a)(2)$$
$$f_x = 4x^3(2x+3a)^2 + 4x^4(2x+3a)$$

and $f_y = 3(y^2-a^2)^2(2y) + 0$

$$f_y = 6y(y^2-a^2)^2$$

For possible singular pts.

Put $f_x = 0$

$$\Rightarrow 4x^3(2x+3a)^2 + 4x^4(2x+3a) = 0$$
$$4x^3[(2x+3a)^2 + x(2x+3a)] = 0$$
$$\Rightarrow 4x^3 = 0 \quad , \quad (2x+3a)^2 + x(2x+3a) = 0$$
$$x^3 = 0 \quad , \quad (2x+3a)[2x+3a+x] = 0$$
$$x = 0 \quad , \quad 2x+3a = 0 \quad , \quad 3x+3a = 0$$
$$x = 0 \quad , \quad x = -\frac{3a}{2} \quad , \quad x = -a$$

and put $f_y = 0$

$\Rightarrow \quad 6y(y^2-a^2)^2 = 0$

$\Rightarrow \quad y = 0 \quad , (y^2-a^2)^2 = 0$

$y^2 - a^2 = 0$

$y^2 = a^2$

$y = \pm a$

$y = 0 \, , \, y = a \, , \, y = -a$

Hence the possible singular pts. are

$(0,0), (0,a), (0,-a), (-\frac{3a}{2},0), (-\frac{3a}{2},a), (-\frac{3a}{2},-a),$
$(-a,0), (-a,a), (-a,-a).$

$\because$ $(0,a), (0,-a), (-\frac{3a}{2},a), (-\frac{3a}{2},-a)$ and $(-a,0)$ satisfy the given eq.

$\therefore$ The pts. $(0,a), (0-a), (-\frac{3a}{2},a), (-\frac{3a}{2},-a)$ and $(-a,0)$ are the singular pts.

Now $\quad f_{xy} = 0$

$f_{xx} = 12x^2 + 4x^3 \cdot 2(2x+3a)2 + 16x^3(2x+3a) + 4x^4 \cdot 2$

$= 12x^2 + 16x^3(2x+3a) + 16x^3(2x+3a) + 8x^4$

$= 12x^2 + 32(x)^3(2x+3a) + 8x^4$

$= 8x^4 + 32x^3(2x+3a) + 12x^2$

$= 8x^4 + 64x^4 + 96x^3a + 12x^2$

$= 72x^4 + 96x^3a + 12x^2.$

$f_{yy} = 6(y^2-a^2)^2 + 2 \cdot 6y(y^2-a^2)2y$

$= 6(y^2-a^2)^2 + 24y^2(y^2-a^2)$

$(f_{xy})^2 - f_{xx}f_{yy}$

<u>At $(0,a)$.</u>

$= 0 - (0)(0)$

$\because f_{xy} = f_{xx} = f_{yy} = 0$

$\therefore (0,a)$ is a Multiple pt. of order higher than 2.

<u>At $(0,-a)$.</u> $\quad (f_{xy})^2 - f_{xx}f_{yy}$

$= 0 - (0)\,0$

$\because f_{xy} = f_{xx} = f_{yy} = 0$

$\therefore (0,-a)$ is singular pt. of order higher than 2.

At $\left(-\frac{3a}{2}, \pm a\right)$: $(f_{xy})^2 - f_{xx}f_{yy}$

$$= 0 - \left[72\left(\frac{81}{16}a^4\right) + 96\left(-\frac{3a}{2}\right)^3 a + 12\left(-\frac{3a}{2}\right)^2\right]\left[6(a^2-a^2)^2 + 24a^2(a^2-a^2)\right]$$

$$= 0 - \quad \text{''} \, (0)$$

$$= 0$$

$\Rightarrow \left(-\frac{3a}{2}, \pm a\right)$ is a cusp.

At $(-a,0)$. $(f_{xy})^2 - f_{xx}f_{yy}$

$$= 0 - \left[72(-a)^4 + 96(-a)^3 a + 12(-a)^2\right]\left[6(0-a^2)^2 + 24(0)(0-a^2)\right]$$

$$= 0 - \left[72a^4 - 96a^4 + 12a^2\right]\left[6a^4 - 24a^2\right]$$

$$= -\left[-24a^4 + 12a^2\right]\left[-18a^2\right]$$

$$= \left[24a^4 + 12a^2\right]\left[18a^2\right]$$

$$= > 0$$

thus $(-a,0)$ is a node.

Q. No: 11 Show that the origin is a node, a cusp or an isolated point on the curve

$y^2 = ax^2 + ax^3$ according as 'a' is +ve, zero or negative respectively.

Soln: For tangents at origin,

$$y^2 = ax^2$$

$$\Rightarrow y = \pm\sqrt{a}\, x$$

$\because$ The tangents at $(0,0)$ are real and distinct.

$\therefore$ The origion is a node when a is +ve.

The tanents will be coincident and real if $a = 0$

and the tangents will be imaginary if if a is $-$ve.

Find equations of the tangents at the multiple points of the given curves (Prob: 12-13):

Q. No. 12: $x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0$ —— (1)

Let $f(x,y) = x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4$

$f_x = 4x^3 - 12ax^2 + 8a^2x$

$f_y = -6ay^2 + 6a^2y$

For possible Multiple pts.

Put $f_x = 0$

$\Rightarrow \quad 4x^3 - 12ax^2 + 8a^2x = 0$

$x(4x^2 - 12ax + 8a^2) = 0$

$\Rightarrow x = 0, \quad 4x^2 - 12ax + 8a^2 = 0$

$x^2 - 3ax + 2a^2 = 0$

$x^2 - 2ax - ax + 2a^2 = 0$

$x(x-2a) - a(x-2a) = 0$

$(x-a)(x-2a) = 0$

$\Rightarrow x = a, 2a$

So $x = 0, a, 2a$.

Now put $f_y = 0$

$\Rightarrow \quad -6ay^2 + 6a^2y = 0$

$-ay^2 + a^2y = 0$

$-y^2 + ay = 0$

$y(-y+a) = 0$

$\Rightarrow \quad y = 0, \quad -y + a = 0$

$y = 0, \quad \Rightarrow y = a$

So $y = 0, y = a$.

Hence the possible Multiple Points are

$(0,0), (0,a), (a,0), (a,a), (2a,0), (2a,a)$

$\because$ Only $(0,a), (a,0), (2a,a)$ satisfying the given eq.

$\therefore (0,a), (a,0), (2a,a)$ are the Multiple pts.

Now for tangents at $(0,a)$ shifting the origin at $(0,a)$

For this $x = X + h$ and $y = Y + k$.

$$x = X+0 \quad , \quad y = Y+a$$

Put in (1), we have

$$X^4 - 4aX^3 - 2a(Y+a)^3 + 4a^2X^2 + 3a^2(Y+a)^2 - a^4 = 0$$

$$X^4 - 4aX^3 - 2a(Y^3+a^3+3Y^2a+3Ya^2) + 4a^2X^2 + 3a^2(Y^2+a^2+2aY) - a^4 = 0$$

$$X^4 - 4aX^3 - 2aY^3 - 2a^4 - 6Y^2a^2 - \cancel{6Ya^3} + 4a^2X^2 + 3a^2Y^2 + 3a^4 + \cancel{6a^3Y} - a^4 = 0$$

$$X^4 - 4aX^3 - 2aY^3 + 4a^2X^2 - 3a^2Y^2 = 0$$

$$X^4 - 4aX^3 - 2aY^3 + a^2(4X^2 - 3Y^2) = 0$$

For tangents at origin equating the lowest degree terms to zero, we have

$$a^2(4X^2 - 3Y^2) = 0$$

$$\Rightarrow \quad 4X^2 - 3Y^2 = 0$$

$$\Rightarrow \quad 3Y^2 = 4X^2$$

$$Y^2 = \frac{4}{3}X^2$$

$$Y = \pm \frac{2}{\sqrt{3}} X$$

$x = X + h \Rightarrow X = x - h$

$y = Y + k \Rightarrow Y = y - k$

which are the tangents at new origin.

Now the tangents at (0, a) are

$$(y - a) = \pm \frac{2}{\sqrt{3}} (x - 0)$$

$$y - a = \pm \frac{2}{\sqrt{3}} x.$$

Now for the tangents at (a, 0) on the given curve shifting the origin of the curve at (a, 0).

For this put $x = X + h$, $y = Y + k$

$$\Rightarrow \quad x = X + a \quad , \quad y = Y + 0$$

$$x = X + a \quad , \quad y = Y.$$

Put these values in (1), we have.

$$(X+a)^4 - 4a(X+a)^3 - 2aY^3 + 4a^2(X+a)^2 + 3a^2Y^2 - a^4 = 0$$

$$X^4 + 4X^3a + 6X^2a^2 + 4Xa^3 + a^4 - 4a(X^3+a^3+3X^2a+3Xa^2) - 2aY^3 +$$
$$4a^2(X^2+a^2+2aX) + 3a^2Y^2 - a^4 = 0$$

$$X^4 + \cancel{4X^3a} + 6X^2a^2 + \cancel{4Xa^3} + \cancel{a^4} - \cancel{4aX^3} - \cancel{4a^4} - 12X^2a^2 - \cancel{12Xa^3} - 2aY^3 + 4a^2X^2$$
$$+ \cancel{4a^4} + \cancel{8a^3X} + 3a^2Y^2 - \cancel{a^4} = 0$$

$$X^4 - 2X^2a^2 - 2aY^3 + 3aY^2 = 0$$

$$X^4 - 2aY^3 - 2X^2a^2 + 3Y^2a = 0$$

For tangents at new origin.

Equating the lowest degree terms to zero

$$-2X^2a^2 + 3Y^2a = 0$$

$$3Y^2 = 2X^2a$$

$$Y^2 = \frac{2a}{3}X^2$$

$$Y = \pm\sqrt{\frac{2a}{3}}\,X$$

Hence the equations of the tangents at the Multiple point $(a,0)$ are

$$(y-0) = \pm\sqrt{\frac{2a}{3}}(x-a)$$

$$y = \pm\sqrt{\frac{2a}{3}}(x-a)$$

Now For tangents at $(2a,a)$ Shifting the origin at $(2a,a)$.

So put $x = X+h \qquad y = Y+k$

$x = X+2a \qquad y = Y+a$

So (1) $\Rightarrow$

$$(X+2a)^4 - 4a(X+2a)^3 - 2a(Y+a)^3 + 4a^2(X+2a)^2 + 3a^2(Y+a)^2 - a^4 = 0$$

after Simplification.

$$X^4 + a(4X^3 - 2Y^3) + a^2(4X^2 - 3Y^2) = 0$$

For tangents equating the lowest degree terms to zero, we get.

$$a^2(4X^2 - 3Y^2) = 0$$

$$4X^2 - 3Y^2 = 0$$

$$3Y^2 = 4X^2$$

$$Y^2 = \frac{4}{3}X^2$$

$$Y = \pm\frac{2}{\sqrt{3}}X$$

Hence the tangents at $(2a,a)$ are

$$(y-a) = \pm\frac{2}{\sqrt{3}}(x-2a)$$

Q.No. 13. $(y-2)^2 = x(x-1)^2$ ——— (1)

Let $f(x,y) = x(x-1)^2 - (y-2)^2$

Then $f_x = 2x(x-1) + (x-1)^2$

$f_y = -2(y-2)$

For Multiple points put $f_x=0$ and $f_y=0$

$f_x = 0 \Rightarrow \quad 2x(x-1) + (x-1)^2 = 0$

$2x^2 - 2x + x^2 + 1 - 2x = 0$

$3x^2 - 4x + 1 = 0$

$\Rightarrow \quad 3x^2 - 3x - x + 1 = 0$

$3x(x-1) - 1(x-1) = 0$

$(3x-1)(x-1) = 0$

$\Rightarrow \quad x = 1, \ 1/3$

$f_y = 0 \Rightarrow \quad -2(y-2) = 0$

$y - 2 = 0$

$y = 2.$

So the possible Multiple Points are

$(1,2)$ and $(1/3, 2)$ and $(1,2)$ is the Only Singular pt.

<u>Tangents at (1,2):</u>

Shifting the origin at $(1,2)$

Put $x = X+h, \ y = Y+k$

$x = X+1, \ y = Y+2$

So (1) $\Rightarrow$

$(Y+2-2)^2 = (X+1)(X+1-1)^2$

$Y^2 = (X+1)X^2$

$Y^2 = X^3 + X^2$

$\Rightarrow \quad X^3 + X^2 - Y^2 = 0$

Equating to zero the lowest degree terms

$X^2 - Y^2 = 0$

$Y^2 = X^2$

$\Rightarrow \quad Y = \pm X$

$(y-2) = \pm(x-1)$

$y-2 = x-1, \quad y-2 = -(x-1)$

$\Rightarrow \quad \boxed{x - y + 1 = 0}, \quad \boxed{x + y - 3 = 0}$

Find the nature of the cusps on the given curves (Problems 14-17):

Q.No. 14 $x^2(x-y)+y^2=0$

The curve has concident tangents
$$y^2=0$$
at the origin. Hence the origin is a cusp and the branches of the curve through it are real. The equation of the curve can be written as
$$y^2-x^2y+x^3=0$$
$$y=\frac{x^2\pm\sqrt{x^4-4x^3}}{2}$$
$$=\frac{x^2\pm x\sqrt{x^2-4x}}{2}$$
$$=\frac{x^2\pm x\sqrt{x(x-4)}}{2}$$

The values of y are real only for negative values of x near origin.

Hence the origin is a cusp.

Also for any particular -ve value of x, y has opposite signs,

i.e. the curve exists on both sides of the X-axis, the cuspidal tangent.

The cusp is of the first species.

Hence the origin is a single cusp of the first species.

Q. No. 15 $x^3+y^3-2ay^2=0$
$$y^3=x^3+ax^2 \qquad \text{------ } 1)$$

Tangent at the origin are $x^2=0$
i.e. the curve has two coincident tangent
$$x^2=0$$
at the origin.

$1) \Rightarrow \quad ax^2=y^3 \quad$ (Neglecting x^3)

or $x^2 = \frac{y^3}{a}$ or $x = \pm y\sqrt{\frac{y}{a}}$

The values of x are real only for one sign of y, viz +ve.

Hence the origin is a single cusp.

Also, for any particular +ve value of x, y has opposite signs

i.e. the curve exists on both sides of the y-axis,

i.e. the cuspidal tangent.

So The cusp is of the first species.

Hence the origin is a single cusp of the first species.

<u>Q. No. 16</u>: $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$

<u>Soln</u>: $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$ ——— (1)

The tangents at the origin are

$$y^2 = 0$$

Hence the origin is a cusp.

1) ⟹ $a^4y^2 - a(x^4 + a^2x^2)y + x^6 = 0$

$$y = \frac{a(x^4+a^2x^2) \pm \sqrt{a^2(x^4+a^2x^2)^2 - 4a^4x^6}}{2a^4}$$

$$= \frac{a(x^4+a^2x^2) \pm \sqrt{a^2\{(x^4+a^2x^2)^2 - 4a^2x^6\}}}{2a^4}$$

$$= \frac{ax^2(x^2+a^2) \pm \sqrt{a^2\{x^8+2a^2x^6+a^4x^4-4a^2x^6\}}}{2a^4}$$

$$= \frac{ax^2(x^2+a^2) \pm \sqrt{a^2(x^4-a^2x^2)^2}}{2a^4}$$

$$= \frac{ax^2(x^2+a^2) \pm ax^2(x^2-a^2)}{2a^4}$$

$$= \frac{x^2[(x^2+a^2) \pm (x^2-a^2)]}{2a^3}$$

$$= \frac{x^4}{a^3} \text{ or } \frac{x^2}{a}$$

The values of y are real and positive for both positive and negative values of x.
The curve exists on one side of x-axis.
This shows that the cusp is double of the second species.

Q. No. 17 $y^3 = (x-a)^2(2x-a)$

Soln: $y^3 = (x-a)^2(2x-a)$ ——— (1)

Shifting the origin $(a, 0)$, equation (1) becomes
$$y^3 = x^2(2x+a) \quad \text{———} \quad (2)$$
Tangents to (2) at the new origin are $x^2=0$
(equating to zero the lowest degree terms)
Since the tangents are coincident, the new origin is a cusp.
The branches of the curve through it being real as shown below

From (2), neglecting $2x^3$, we get
$$ax^2 = y^3 \quad \text{or} \quad x = \pm\sqrt{\frac{y^3}{a}}$$
The values of x are real only for one sign of y viz +ve.
Hence the new origin is a single cusp.
Also for any particular positive value of y, x has opposite signs i.e., the curve exists on both sides of the new y-axis, the cuspidal tangent.
The cusp is of the first species
Hence the point $(a,0)$ is a single cusp of the first species.
