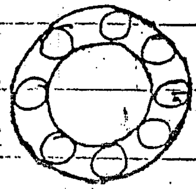
ENVELOPE

"The locus of the limiting position of the pts of X_n of any two curves of the family when one of them tends to coincide with the other which is kept fixed."



An annulus as the Envelope of the family of circles.

"A curve that is tangential to each of a family of curves"

Envelope of Normals to a curve is the Evolute of curve.

Ex 7.9

Working Rule.

curve: $f(x,y,\alpha)$ — ①

Find $\frac{\partial f}{\partial \alpha} = 0$ — ②

Eliminate α from ① & ② we get Envelope.

QNo1 $\quad y = mx + \frac{a}{m}$

$$f(x,y,m) = mx + \frac{a}{m} - y = 0 \quad \text{— ①}$$

$$\frac{\partial f}{\partial m} = x - \frac{a}{m^2} - 0 = 0$$

$$x - \frac{a}{m^2} = 0 \qquad x = \frac{a}{m^2}$$

$$m^2 = \frac{a}{x} \quad \text{— ②}$$

from ① $\quad mx + \frac{a}{m} - y = 0$

$$m^2x + a - my = 0 \qquad \text{LCM}$$

$$\frac{a}{\cancel{x}}\cancel{x} + a - \sqrt{\frac{a}{x}}\,y = 0$$

$$2a = \sqrt{\frac{a}{x}}\,y$$

$$4a^2 = \frac{a}{x}y^2$$

$$\frac{4a^{\cancel{2}}x}{\cancel{a}} = y^2 \quad \Rightarrow \quad y^2 = 4ax.$$

7.9-2

Q No 2 (i) $f(x,y,\alpha) = A\alpha^2 + B\alpha + C = 0$ —— (i)

$$\frac{\partial f}{\partial \alpha} = 2A\alpha + B = 0$$

$$2A\alpha + B = 0 \quad \Rightarrow \alpha = -\frac{B}{2A} \text{ —— (ii)}$$

using (ii) in (i)

$$A\left(\frac{B^2}{4A^2}\right) + B\left(\frac{-B}{2A}\right) + C = 0.$$

$$\frac{B^2}{4A} - \frac{B^2}{2A} + C = 0.$$

$$B^2 - 2B^2 + 4AC = 0 \quad \Rightarrow -B^2 + 4AC = 0.$$

$$\Rightarrow B^2 - 4AC = 0.$$

(b) $f(x,y,\alpha) = A\alpha^3 + 3B\alpha^2 + 3C\alpha + D = 0$ —— (1)

$$\frac{\partial f}{\partial \alpha} = 3A\alpha^2 + 6B\alpha + 3C = 0$$

$$3A\alpha^2 + 6B\alpha + 3C = 0.$$

$$A\alpha^2 + 2B\alpha + C = 0 \quad \Rightarrow A\alpha^2 = -2B\alpha - C$$

$\therefore$ from (1) $A\alpha^3 + 3C\alpha + 3B\alpha^2 + D = 0.$

$$\alpha(A\alpha^2 + 3C) + 3B\alpha^2 + D = 0$$

$$\alpha(-2B\alpha - C + 3C) + 3B\alpha^2 + D = 0$$

$$-2B\alpha^2 + 2C\alpha + 3B\alpha^2 + D = 0$$

$$B\alpha^2 + 2C\alpha + D = 0 \text{ —— (3)}$$

$$= \sec^2\theta\left(x - \frac{gx^2}{u^2}\tan\theta\right) = 0$$

$$x - g\frac{x^2}{u^2}\tan\theta = 0.$$

$$\Rightarrow \frac{u^2x - gx^2\tan\theta}{u^2} = 0 \Rightarrow u^2x - gx^2\tan\theta = 0$$

$$\Rightarrow \frac{u^2\cancel{x}}{gx^{\cancel{2}}} = \tan\theta \text{ --------- } (2)$$

Eliminate θ from (1) & (2)

$$x\tan\theta - y - \frac{gx^2}{2u^2}\sec^2\theta = 0 \qquad \because \sec^2\theta = 1 + \tan^2\theta$$

$$x\left(\frac{u^2}{gx}\right) - y - \frac{gx^2}{2u^2}\left(1 + \frac{u^4}{g^2x^2}\right)$$

$$\frac{u^2}{g} - y - \frac{gx^2}{2u^2} - \frac{\cancel{g}\cancel{x^2}}{2u^2}\frac{u^4}{g^{\cancel{2}}\cancel{x^2}}$$

$$\frac{u^2}{g} - y - \frac{gx^2}{2u^2} - \frac{u^2}{2g} = 0$$

$$y = -\frac{gx^2}{2u^2} - \frac{u^2}{2g} + \frac{u}{g} = \frac{-g^2x^2 - u^4 + 2u^4}{2u^2g}$$

$$y = \frac{u^4 - g^2x^2}{2u^2g} \qquad 2u^2gy = u^4 - g^2x^2 \text{ Ans.}$$

Solving ② & ③

$$A\alpha^2 + 2B\alpha + C = 0$$

$$B\alpha^2 + 2C\alpha + D = 0$$

$$\frac{\alpha^2}{2BD-2C^2} = \frac{-\alpha}{AD-BC} = \frac{1}{2AC-2B^2}$$

$$\alpha^2 = \frac{2BD-2C^2}{2AC-2B^2} = \frac{2(BD-C^2)}{2(AC-B^2)} \quad \text{—— ④}$$

$$-\alpha = \frac{AD-BC}{2(AC-B^2)} \Rightarrow \alpha = \frac{BC-AD}{2(AC-B^2)} \quad \text{—— ⑤}$$

Comparing α^2 ④ & ⑤

$$\frac{BD-C^2}{AC-B^2} = \frac{(BC-AD)^2}{4(AC-B^2)^2}$$

$$(BD-C^2)\,4(AC-B^2) = (BC-AD)^2$$

$$(BC-AD)^2 = 4(BD-C^2)(AC-B^2)$$

x———————x

Q3 $\quad y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta} \quad$ ①

$$f(x,y,\theta) = x\tan\theta - y - \frac{gx^2}{2u^2\cos^2\theta} = x\tan\theta - y - \frac{gx^2\sec^2\theta}{2u^2} = 0$$

$$\frac{\partial f}{\partial\theta} = x\sec^2\theta - 0 - \frac{gx^2}{\cancel{2}u^2}\cancel{2}\sec\theta(\sec\theta\tan\theta) = 0$$

Q No 4 $\quad y = mx + \sqrt{a^2m^2+b^2}$

$$f(x,y,m) = mx - y + \sqrt{a^2m^2+b^2} = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial m} = x - 0 + \frac{1}{2}\frac{2a^2m}{\sqrt{a^2m^2+b^2}} = 0$$

$$x = \frac{-a^2m}{\sqrt{a^2m^2+b^2}} \Rightarrow \sqrt{a^2m^2+b^2} = \frac{-a^2m}{x}$$

Squaring $\quad a^2m^2+b^2 = \frac{a^4m^2}{x^2}$

$$b^2 = \frac{a^4m^2}{x^2} - a^2m^2 \Rightarrow b^2 = a^2m^2\left(\frac{a^2}{x^2}-1\right)$$

$$b^2 = a^2m^2\left(\frac{a^2-x^2}{x^2}\right) \Rightarrow \frac{b^2x^2}{a^2-x^2} = a^2m^2$$

$$\frac{b^2x^2}{a^2(a^2-x^2)} = m^2 \quad \text{--- (2)}$$

Eliminating m from $f(x,y,m)=0$ for Envelope
using (2) in (1)

$$mx - y + \sqrt{a^2m^2+b^2} = 0$$

$$mx - y - \frac{a^2m}{x} = 0$$

$$\frac{mx^2 - a^2m}{x} = y \Rightarrow y = m\frac{(x^2-a^2)}{x}$$

$$\Rightarrow m = \frac{xy}{x^2-a^2} = \frac{-xy}{a^2-x^2} \Rightarrow m^2 = \frac{x^2y^2}{(a^2-x^2)^2} \quad \text{--- (3)}$$

Equating (2) & (3)

$$\frac{b^2x^2}{a^2(a^2-x^2)} = \frac{x^2y^2}{(a^2-x^2)^2}$$

$$\frac{b^2}{a^2} = \frac{y^2}{(a^2-x^2)} \Rightarrow b^2(a^2-x^2) = a^2y^2 \Rightarrow a^2b^2 - b^2x^2 = a^2y^2$$

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(6) $(x-\alpha)^2+y^2=\alpha^2$

$f(x,y,\alpha)=(x-\alpha)^2+y^2-\alpha^2=0$

$\frac{\partial f}{\partial \alpha}= 2(x-\alpha)(-1)+0-2\alpha=0$

$-2x+\cancel{2\alpha}-\cancel{2\alpha}=0$

$-2x=0 \Rightarrow x=0.$

Elimination of α is not possible

OR α is absent in $\frac{\partial f}{\partial \alpha}=0$ which is to be eliminated.

~~Hence No Envelope~~

(5) Let C be the centre of ellipse and CP & CD be its semi-conjugate diameters. If $P(a\cos\theta, b\sin\theta)$ & D is $(-a\sin\theta, b\cos\theta)$ then eq of PD is

$y-y_1=m(x-x_1)$

$m=\frac{y_2-y_1}{x_2-x_1}$

$=\frac{b\cos\theta-b\sin\theta}{-a\sin\theta-a\cos\theta}$

$=\frac{b\sin\theta-b\cos\theta}{a(\sin\theta+\cos\theta)}$

$y-b\sin\theta=\frac{b(\sin\theta-\cos\theta)}{a(\sin\theta+\cos\theta)}(x-a\cos\theta)$

$ay(\sin\theta+\cos\theta)-ab\sin\theta(\sin\theta+\cos\theta)=(b\sin\theta-b\cos\theta)(x-a\cos\theta)$

$ay\sin\theta+ay\cos\theta-ab\sin^2\theta-\cancel{ab\sin\theta\cos\theta}=xb\sin\theta-\cancel{ab\sin\theta\cos\theta}-bx\cos\theta+ab\cos^2\theta.$

$ay(\sin\theta+\cos\theta)=bx(\sin\theta-\cos\theta)+ab(\cos^2\theta+\sin^2\theta)$

$bx(\sin\theta-\cos\theta)-ay(\sin\theta+\cos\theta)+ab=0$

$f(x,y,\theta)=\frac{x}{a}(\sin\theta-\cos\theta)-\frac{y}{b}(\sin\theta+\cos\theta)+1=0$ ——— (1)

$\frac{\partial f}{\partial \theta} = 0$

$\frac{x}{a}(\cos\theta + \sin\theta) - \frac{y}{b}(\cos\theta - \sin\theta) = 0.$ ②

~~Eliminating θ from ① & ②~~

~~from ①~~ $\frac{x}{a}(\sin\theta - \cos\theta) - \frac{y}{b}(\sin\theta + \cos\theta) = -1$ ③

Squaring ② & ③ and adding

$\left[\frac{x}{a}(\cos\theta + \sin\theta) - \frac{y}{b}(\cos\theta - \sin\theta)\right]^2 + \left[\frac{x}{a}(\sin\theta - \cos\theta) - \frac{y}{b}(\sin\theta + \cos\theta)\right]^2 = (-1)^2$

$\Rightarrow \left[\frac{x^2}{a^2}(\cos\theta + \sin\theta)^2 + \frac{y^2}{b^2}(\cos\theta - \sin\theta)^2 - \frac{2xy}{ab}(\cos^2\theta - \sin^2\theta)\right] + \left[\frac{x^2}{a^2}(\sin\theta - \cos\theta)^2 + \frac{y^2}{b^2}(\sin\theta + \cos\theta)^2 - \frac{2xy}{ab}(\sin^2\theta - \cos^2\theta)\right] = 1$

$\Rightarrow \frac{x^2}{a^2}\left[(\cos\theta + \sin\theta)^2 + (\sin\theta - \cos\theta)^2\right] + \frac{y^2}{b^2}\left[(\cos\theta - \sin\theta)^2 + (\sin\theta + \cos\theta)^2\right] \cancel{- \frac{2xy}{ab}(\cos^2\theta - \sin^2\theta)} \cancel{+ \frac{2xy}{ab}(\cos^2\theta - \sin^2\theta)} = 1$

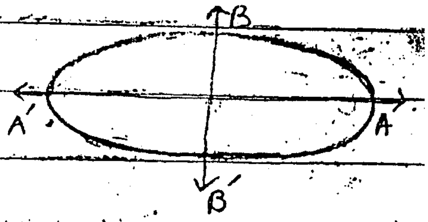
$\Rightarrow \frac{x^2}{a^2}\left[(\cos^2\theta + \sin^2\theta) \cancel{+ 2\sin\theta\cos\theta} + (\sin^2\theta + \cos^2\theta) \cancel{- 2\sin\theta\cos\theta}\right] + \frac{y^2}{b^2}\left[(\cos^2\theta + \sin^2\theta) \cancel{- 2\sin\theta\cos\theta} + (\sin^2\theta + \cos^2\theta) \cancel{+ 2\sin\theta\cos\theta}\right] = 1$

$\Rightarrow \frac{x^2}{a^2}\left[1+1\right] + \frac{y^2}{b^2}\left[1+1\right] = 1$

$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 1$; Envelope !

(8) Given that $AA' + BB' = 2a$

Consider the ellipse



$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1 \qquad AA' = 2\alpha$$

$$BB' = 2\beta$$

$2\alpha + 2\beta = 2a$ (from given)

$\alpha + \beta = a \Rightarrow \beta = a - \alpha.$

$\therefore$ $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$ becomes $\frac{x^2}{\alpha^2}+\frac{y^2}{(a-\alpha)^2}=1$ {One parameter family with parameter α, a is const}

$$f(x,y,\alpha) = \frac{x^2}{\alpha^2}+\frac{y^2}{(a-\alpha)^2}-1=0 \quad \text{(1)}$$

$$\frac{\partial f}{\partial \alpha} = -\frac{2x^2}{\alpha^3} - \frac{2y^2(-1)}{(a-\alpha)^3} = 0 \Rightarrow 2\left(-\frac{x^2}{\alpha^3}+\frac{y^2}{(a-\alpha)^3}\right)=0$$

$$\Rightarrow \frac{y^2}{(a-\alpha)^3} = \frac{x^2}{\alpha^3} \Rightarrow \frac{y^2}{x^2} = \frac{(a-\alpha)^3}{\alpha^3}$$

$$\Rightarrow \left(\frac{y}{x}\right) = \left(\frac{a-\alpha}{\alpha}\right)^3 \Rightarrow \left(\frac{y}{x}\right)^{2/3} = \left(\frac{a-\alpha}{\alpha}\right)$$

$$\Rightarrow \frac{y^{2/3}}{x^{2/3}} = \frac{a}{\alpha}-1 \Rightarrow 1+\frac{y^{2/3}}{x^{2/3}} = \frac{a}{\alpha}$$

$$\Rightarrow \frac{x^{2/3}+y^{2/3}}{x^{2/3}} = \frac{a}{\alpha} \Rightarrow \frac{x^{2/3}+y^{2/3}}{a x^{2/3}} = \frac{1}{\alpha}$$

$$\Rightarrow \alpha = \frac{a x^{2/3}}{x^{2/3}+y^{2/3}} \quad \text{(2)}$$

For envelope we eliminate α from (1) & (2)

from (1)

$$\frac{x^2}{\alpha^2}+\frac{y^2}{(a-\alpha)^2}-1=0$$

$$\frac{x^2}{\left(\frac{ax^{2/3}}{x^{2/3}+y^{2/3}}\right)^2}+\frac{y^2}{\left(a-\frac{ax^{2/3}}{x^{2/3}+y^{2/3}}\right)^2}=1$$

$$\frac{x^2\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{a^2\,x^{4/3}}+\frac{y^2\left(x^{2/3}+y^{2/3}\right)^2}{\left(\cancel{ax^{2/3}}+ay^{2/3}-\cancel{ax^{2/3}}\right)^2}=1$$

$$\frac{x^{2-\frac{4}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{a^2}+\frac{y^2\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{a^2\,y^{4/3}}=1$$

$$\frac{x^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{a^2}+\frac{y^{2-\frac{4}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2}{a^2}=1$$

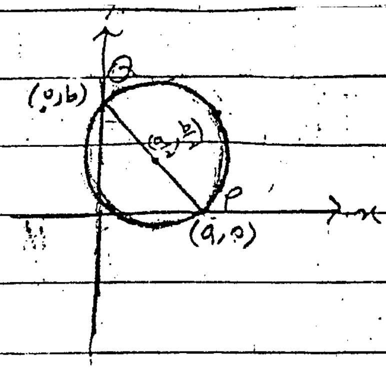
$$\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^2\left[\frac{x^{2/3}}{a^2}+\frac{y^{2/3}}{a^2}\right]=1$$

$$\Rightarrow\left(x^{\frac{2}{3}}+y^{2/3}\right)^2\left(x^{\frac{2}{3}}+y^{2/3}\right)=a^2$$

$$\Rightarrow\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)^3=a^2$$

$$\Rightarrow x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$ Ans.

(7) Let the fixed st lines at right angles be the coord axes. Let $P(a,0)$ & $Q(0,b)$ be the coord of the extremities of the sliding line. Centre of PQ is $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $|PQ| = c$ $\therefore$ radius $= \frac{c}{2}$.



$\sqrt{(0-a)^2+(b-0)^2} = c$ Distance Formula

$a^2+b^2=c^2$ Squaring

Eq of circle with PQ as diameter is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{c}{2}\right)^2$$

$$x^2+\frac{a^2}{4}-\cancel{2}x\frac{a}{\cancel{2}}+y^2+\frac{b^2}{4}-\cancel{2}\frac{by}{\cancel{2}}=\frac{c^2}{4}$$

$$x^2+y^2-ax-by+\left(\frac{a^2+b^2}{4}\right)=\frac{c^2}{4}$$

$$x^2+y^2-ax-by=\cancel{\frac{c^2}{4}}-\cancel{\frac{c^2}{4}}=0 \qquad \because c^2=a^2+b^2$$

$$x^2+y^2-ax-by=0.$$

$\because c^2=a^2+b^2$

$c^2-a^2=b^2$

$$f(x,y,a)=x^2+y^2-ax-\sqrt{c^2-a^2}\,y=0 \quad \text{---(1)}$$

a is ~~parameter~~ one parameter

c is const

$$\frac{\partial f}{\partial a}=-x-\frac{1}{2\sqrt{c^2-a^2}}(-2a)\,y=0$$

$$-x+\frac{ay}{\sqrt{c^2-a^2}}=0 \quad \Rightarrow x=\frac{ay}{\sqrt{c^2-a^2}} \quad \Rightarrow x^2=\frac{a^2y^2}{c^2-a^2}$$

$$\Rightarrow \frac{1}{x^2}=\frac{c^2-a^2}{a^2y^2} \quad \Rightarrow \frac{1}{x^2}=\frac{c^2}{a^2y^2}-\frac{1}{y^2} \quad \Rightarrow \frac{1}{x^2}+\frac{1}{y^2}=\frac{c^2}{a^2y^2}$$

$$\frac{y^2+x^2}{x^2y^2} = \frac{c^2}{a^2y^2} \Rightarrow a^2 = \frac{c^2x^2\cancel{y^2}}{\cancel{y^2}(y^2+x^2)}$$

$$\Rightarrow a^2 = \frac{c^2x^2}{y^2+x^2} \quad \text{——— (2)}$$

Eliminating 'a' for envelope

from (1) $x^2+y^2-ax-\sqrt{c^2-a^2}\,y=0$

$$x^2+y^2-\frac{(cx)x}{\sqrt{y^2+x^2}}-\sqrt{c^2-\frac{c^2x^2}{x^2+y^2}}\,y=0.$$

$$x^2+y^2-\frac{cx^2}{\sqrt{y^2+x^2}}-\frac{\sqrt{\cancel{c^2x^2}+c^2y^2-\cancel{c^2x^2}}}{\sqrt{x^2+y^2}}\,y=0.$$

$$\frac{(x^2+y^2)(\sqrt{x^2+y^2})-cx^2-(cy)y}{\sqrt{x^2+y^2}}=0.$$

$$(x^2+y^2)(\sqrt{x^2+y^2})-c(x^2+y^2)=0.$$

$$(x^2+y^2)\left[\sqrt{x^2+y^2}-c\right]=0.$$

$$\sqrt{x^2+y^2}-c=0.$$

$$\sqrt{x^2+y^2}=c$$

$$x^2+y^2=c^2 \quad \text{Ans.}$$

(8) $\frac{x}{a}+\frac{y}{b}=1$

(i) $a+b=c$ $\qquad b=c-a.$

$\frac{x}{a}+\frac{y}{(c-a)}=1$ $\qquad$ One parameter family of lines

$f(x,y,a)=\frac{x}{a}+\frac{y}{c-a}-1=0$ ——— (1)

$\frac{\partial f}{\partial a}=-\frac{x}{a^2}-\frac{y}{(c-a)^2}(-1)=0$

$-\frac{x}{a^2}+\frac{y}{(c-a)^2}=0 \qquad \Rightarrow \frac{x}{a^2}=\frac{y}{(c-a)^2}$

$\Rightarrow \frac{(c-a)^2}{a^2}=\frac{y}{x} \qquad \Rightarrow \left(\frac{c-a}{a}\right)^2=\frac{y}{x}$

$\Rightarrow \left(\frac{c}{a}-1\right)^2=\frac{y}{x} \qquad \Rightarrow \frac{c}{a}-1=\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

$\frac{c}{a}=1+\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \qquad \Rightarrow \frac{c}{a}=\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$

$\frac{cx^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}=a.$ ——— (2)

Eliminating 'a' from (1) & (2) for Envelope

$\frac{x}{a}+\frac{y}{c-a}-1=0. \qquad \Rightarrow \frac{x}{\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}}+\frac{y}{c-\frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}}-1=0.$

$\Rightarrow \frac{(\sqrt{x}+\sqrt{y})x}{c\sqrt{x}}+\frac{y(\sqrt{x}+\sqrt{y})}{\cancel{c\sqrt{x}}+c\sqrt{y}-\cancel{c\sqrt{x}}}-1=0$

7-9-14

$$x = \frac{y\,a^{n-1}a^{2}}{(c^n-a^n)^{\frac{n+1}{n}}} \Rightarrow \frac{x}{y} = \frac{a^{n+1}}{(c^n-a^n)^{\frac{n+1}{n}}}$$

Taking power $\frac{n}{n+1}$ on both sides

$$\left(\frac{x}{y}\right)^{\frac{n}{n+1}} = \left[\frac{a^{n+1}}{(c^n-a^n)^{\frac{n+1}{n}}}\right]^{\frac{n}{n+1}}$$

$$\frac{x^{\frac{n}{n+1}}}{y^{\frac{n}{n+1}}} = \frac{a^n}{c^n-a^n} \Rightarrow \frac{y^{\frac{n}{n+1}}}{x^{\frac{n}{n+1}}} = \frac{c^n-a^n}{a^n}$$

$$\frac{y^{\frac{n}{n+1}}}{x^{n/n+1}} = \frac{c^n}{a^n} - 1 \Rightarrow 1 + \frac{y^{n/n+1}}{x^{n/n+1}} = \frac{c^n}{a^n}$$

$$\frac{x^{n/n+1} + y^{n/n+1}}{x^{n/n+1}} = \frac{c^n}{a^n} \Rightarrow a^n = \frac{c^n x^{\frac{n}{n+1}}}{x^{n/n+1} + y^{n/n+1}}$$

$$[a^n]^{\frac{1}{n}} = \left[\frac{c^n x^{\frac{n}{n+1}}}{x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}}\right]^{\frac{1}{n}} \Rightarrow a = \frac{c\,x^{\frac{1}{n+1}}}{\left[x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}\right]^{\frac{1}{n}}} \quad (2)$$

Eliminating a from (1) using (2)

$$\frac{x}{a} + \frac{y}{(c^n-a^n)^{\frac{1}{n}}} - 1 = 0$$

$$\frac{x}{\dfrac{c\,x^{\frac{1}{n+1}}}{\left(x^{n/n+1} + y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}} + \frac{y}{\left[c^n - \dfrac{c^n x^{n/n+1}}{\left(x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}\right)}\right]^{\frac{1}{n}}} - 1 = 0$$

7.9-15

$$\frac{x\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c\,x^{\frac{1}{n+1}}}+\frac{y\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{\left[c^{n}x^{\frac{n}{n+1}}+c^{n}y^{\frac{n}{n+1}}-c^{n}x^{\frac{n}{n+1}}\right]^{\frac{1}{n}}}-1=0$$

$$\therefore \frac{x\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c\,x^{\frac{1}{n+1}}}+\frac{y\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{\left[c^{n}y^{\frac{n}{n+1}}\right]^{\frac{1}{n}}}-1=0$$

$$\Rightarrow \frac{x^{1-\frac{1}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c}+\frac{y\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c\,y^{\frac{1}{n+1}}}-1=0$$

$$\frac{x^{\frac{n}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c}+\frac{y^{1-\frac{1}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}}{c}-1=0$$

$$x^{\frac{n}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}+y^{\frac{n}{n+1}}\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}-c=0 \quad \text{By LCM}$$

$$\Rightarrow \left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{\frac{1}{n}}\left[x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right]-c=0$$

$$\left(x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right)^{1+\frac{1}{n}}-c=0 \quad\Rightarrow\quad \left[x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right]^{\frac{n+1}{n}}=c$$

Taking $\frac{n}{n+1}$ power on both sides

$$\left[x^{\frac{n}{n+1}}+y^{\frac{n}{n+1}}\right]=c^{\frac{n}{n+1}} \quad \text{Ans.}$$

7·9-16

~~Inf~~

Q9 $x = a(\cos t + t\sin t)$ $\qquad$ $y = a(\sin t - t\cos t)$

$\frac{dx}{dt} = a(-\cancel{\sin t} + \cancel{\sin t} + t\cos t)$ $\qquad$ $\frac{dy}{dt} = a(\cancel{\cos t} - \cancel{\cos t} - t(-\sin t))$

$\frac{dx}{dt} = a\,t\cos t$ $\qquad$ $\frac{dy}{dt} = a\,t\sin t$

$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{\cancel{a}\cancel{t}\sin t}{\cancel{a}\cancel{t}\cos t} = \frac{\sin t}{\cos t} = m$

Slope of the normal $= -\frac{1}{m} = -\frac{\cos t}{\sin t}$

Eq of Normal $\quad y - y_1 = -\frac{1}{m}(x - x_1)$

$y - a(\sin t - t\cos t) = -\frac{\cos t}{\sin t}\left(x - a(\cos t + t\sin t)\right)$

$y\sin t - a\sin t(\sin t - t\cos t) = -x\cos t + a\cos t(\cos t + t\sin t)$

$x\cos t + y\sin t = a\sin t(\sin t - t\cos t) + a\cos t(\cos t + t\sin t)$

$x\cos t + y\sin t = a\sin^2 t - \cancel{a t\sin t\cos t} + a\cos^2 t + \cancel{a\cos t\, t\sin t}$

$x\cos t + y\sin t = a(\sin^2 t + \cos^2 t)$

$x\cos t + y\sin t = a \quad \text{———} \; (1)$

$f(x,y,t) = x\cos t + y\sin t - a = 0$

$\frac{\partial f}{\partial t} = x(-\sin t) + y\cos t = 0$

7.9-17

$$-x\sin t + y\cos t = 0 \quad \text{--- (2)}$$

Squaring (1) & (2) & adding

$$(x\cos t + y\sin t)^2 + (-x\sin t + y\cos t)^2 = a^2$$

$$x^2\cos^2 t + y^2\sin^2 t + \cancel{2xy\cos t\sin t} + x^2\sin^2 t + y^2\cos^2 t - \cancel{2xy\sin t\cos t} = a^2$$

$$x^2(\cos^2 t + \sin^2 t) + y^2(\sin^2 t + \cos^2 t) = a^2$$

$$x^2 + y^2 = a^2 \quad \text{Required Evolute.}$$

Q10. Parametric eqs of astroid

$$x = a\cos^3\theta \qquad \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$$

$$y = a\sin^3\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta(\cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{-\cancel{3a}\sin^2\theta\cos\theta}{\cancel{3a}\cos^2\theta\sin\theta} = -\tan\theta$$

$$\text{Slope of Normal} = \frac{-1}{dy/dx} = \cot\theta$$

~~Suppose normal makes angle φ with x-axis then slope of Normal is $\tan\varphi$~~

$$\therefore \tan\varphi = \cot\theta$$

$$\tan\varphi = \tan\left(\frac{\pi}{2}-\theta\right) \Rightarrow \varphi = \left(\frac{\pi}{2}-\theta\right)$$

$$\therefore x = a\cos^3\theta = a\cos^3\left(\frac{\pi}{2}-\varphi\right) = a\sin^3\varphi$$

$$y = a\sin^3\theta = a\sin^3\left(\frac{\pi}{2}-\varphi\right) = a\cos^3\varphi$$

Eq of Normal $\quad y - a\cos^3\varphi = \tan\varphi\,(x - a\sin^3\varphi)$

[(1) + (2)]

7·9-18

$$y - a\cos^3\phi = \frac{\sin\phi}{\cos\phi}(x - a\sin^3\phi)$$

$$y\cos\phi - a\cos^4 q = x\sin q - a\sin^4 q$$

$$x\sin q - y\cos q + a\cos^4 q - a\sin^4 q = 0$$

$$x\sin q - y\cos q + a(\cos^4 q - \sin^4 q) = 0$$

$$x\sin q - y\cos q + a(\cos^2 q - \sin^2 q)(\cos^2 q + \sin^2 q) = 0$$

$$x\sin q - y\cos q + a\cos 2q = 0. \quad \text{——— } (1)$$

$$f(x, y, q) = x\sin q - y\cos q + a\cos 2q = 0$$

$$\frac{\partial f}{\partial q} = x\cos q - y(-\sin q) + a(-\sin 2q)(2) = 0.$$

$$x\cos q + y\sin q - 2a\sin 2q = 0 \quad \text{——— } (2)$$

Eliminate q from (1) and (2) for Envelope Normal (which is Evolute of Astroid).

$$x\sin q - y\cos q + a\cos 2q = 0$$

$$x\cos q + y\sin q - 2a\sin 2q = 0$$

$$\frac{x}{2a\sin 2q\cos q - a\cos 2q\sin q} = \frac{-y}{-2a\sin 2q\sin q - \cos q\, a\cos 2q} = \frac{1}{\sin^2 q + \cos^2 q}$$

$$\frac{x}{2a\sin 2q\cos q - a\cos 2q\sin q} = \frac{y}{2a\sin 2q\sin q + \cos q\, a\cos 2q} = 1$$

$$x = 2a\sin 2q\cos q - a\cos 2q\sin q \quad \text{——— } (3)$$

$$y = 2a\sin 2q\sin q + a\cos 2q\cos q \quad \text{——— } (4)$$

7.9-19

from (3) $x = 2a(2\sin q\cos q)\cos q - a(1-2\sin^2 q)\sin q$

$= 4a\sin q\cos^2 q - a\sin q(1-2\sin^2 q)$

$= 4a\sin q(1-\sin^2 q) - a\sin q[1-2\sin^2 q]$

$= 4a\sin q - 4a\sin^3 q - a\sin q + 2a\sin^3 q$

$x = 3a\sin q - 2a\sin^3 q$ ——— (5)

from (4) $y = 2a(2\sin q\cos q)\sin q + a(2\cos^2 q - 1)\cos q$

$= 4a\sin^2 q\cos q + a\cos q(2\cos^2 q - 1)$

$= 4a(1-\cos^2 q)\cos q + 2a\cos^3 q - a\cos q$

$= 4a\cos q - 4a\cos^3 q + 2a\cos^3 q - a\cos q$

$y = 3a\cos q - 2a\cos^3 q$ ——— (6)

from (5) & (6)

Now $x+y = 3a\sin q - 2a\sin^3 q + 3a\cos q - 2a\cos^3 q$

$= 3a\sin q - 2a\sin^3 q - (a\sin^3 q) + (a\sin^3 q) + 3a\cos q - 2a\cos^3 q - (a\cos^3 q) + (a\cos^3 q)$

$= 3a\sin q - 3a\sin^3 q + a\sin^3 q + 3a\cos q - 3a\cos^3 q + a\cos^3 q$

$= 3a\sin q(1-\sin^2 q) + a\sin^3 q + 3a\cos q(1-\cos^2 q) + a\cos^3 q$

$= 3a\sin q(\cos^2 q) + a\sin^3 q + 3a\cos q(\sin^2 q) + a\cos^3 q$

$= a(3\sin q\cos^2 q + 3\cos q\sin^2 q + \sin^3 q + \cos^3 q)$

7.9-20

$= a\left[3\sin q\cos q(\cos q+\sin q)+\cos^3 q+\sin^3 q\right]$ ⑧

$(a+b)^3 = a^3+b^3+3ab(a+b)$

$x+y = a(\cos q+\sin q)^3$ ——— ⑦

from ⑤ & ⑥

$x-y = 3a\sin q - 2a\sin^3 q - 3a\cos q + 2a\cos^3 q$

$= 3a\sin q - 2a\sin^3 q - (a\sin^3 q) + (a\sin^3 q) - 3a\cos q + 2a\cos^3 q + (a\cos^3 q) - (a\cos^3 q)$

$= 3a\sin q - 3a\sin^3 q + a\sin^3 q - 3a\cos q + 3a\cos^3 q - a\cos^3 q$

$= 3a\sin q(1-\sin^2 q) + a\sin^3 q - 3a\cos q(1-\cos^2 q) - a\cos^3 q$

$= 3a\sin q\cos^2 q + a\sin^3 q - 3a\cos q(\sin^2 q) - a\cos^3 q$

$= 3a\sin q\cos q[\cos q - \sin q] + a\sin^3 q - a\cos^3 q$

$= a\left[\sin^3 q - \cos^3 q - 3a\sin q\cos q(\sin q - \cos q)\right]$

$x-y = a(\sin q - \cos q)^3$ ——— 8

Taking power $\frac{2}{3}$ and adding ⑦ & ⑧

$(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = a^{\frac{2}{3}}\left[(\cos q+\sin q)^3\right]^{\frac{2}{3}} + a^{\frac{2}{3}}\left[(\sin q-\cos q)^3\right]^{2/3}$

$= a^{\frac{2}{3}}\left[(\cos q+\sin q)^2 + (\sin q-\cos q)^2\right]$

$= a^{\frac{2}{3}}\left[\cos^2 q+\sin^2 q+2\sin q\cos q+\sin^2 q+\cos^2 q-2\sin q\cos q\right]$

$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$ Ans.