Exercise 6.7

Written by
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Parametric representation of curves

Let $y = f(x)$ ——— I

be any curve in rectangular coordinates.

Then $x = x(t)$ and $y = y(t)$

represent parametric form of (I)

We know that Eq. of the circle is

$$x^2+y^2 = a^2$$

Parametric form of this circle is

$$x = a\cos t \ , \ y = a\sin t.$$

Similarly parametric form the parabola

$$y^2 = 4ax$$

is

$$x = at^2, \ y = 2at.$$

Similarly parametric form of the Ellipse

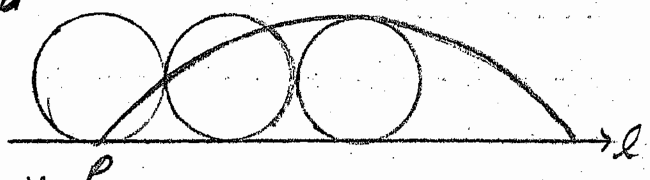
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$x = a\cos\theta \ , \ y = b\sin\theta .$$

The Cycloid.

Def: The cycloid is a curve described by a point marked on the circumference of a circle as it rolls without slipping or sliding along a fixed right line.

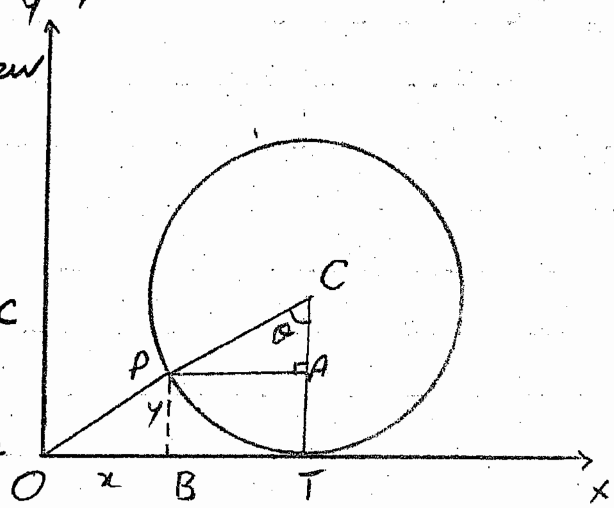


Let the line l be taken as x-axis and let the marked point P be at O. Let the new position be as shown.

Draw PA and $PB \perp$ to CT and x-axis.

Let $P\hat{C}A = \theta$ and $CT = a = PC$ the radius of the circle. Let the coordinates of P are (x, y) distance

We note that the ~~diameter~~ travelled along x-axis $= OT$



and Distance along the arc of the circle $= \widehat{PT}$

$OT = \widehat{PT} = a\theta$ $\qquad | \; l = r\theta$

$OT = a\theta$

$x = OB = OT - BT$

$x = OT - PA$ $\qquad \because PT = BT$

$x = a\theta - a\sin\theta$ $\qquad \dfrac{PA}{PC} = \sin\theta$

Now $y = PB = AT$ $\qquad PA = \sin\theta \, PC$

$y = CT - CA$

$y = a - a\cos\theta$

So $\quad x = a(\theta - \sin\theta)$

$\quad y = a(1 - \cos\theta)$

are the parametric equations of the cycloid.

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Exercise 6.7

Find parametric eqs. of the given curves (1-3)

Q#1 $r = a\sin 2\theta \quad , \quad 0 \le \theta \le 2\pi$

Soln. For parametric eqs.

Put $\quad x = r\cos\theta$

$\Rightarrow \quad x = a\sin 2\theta\cos\theta$

and $\quad y = r\sin\theta$

$\Rightarrow \quad y = a\sin 2\theta\sin\theta$

Hence $\quad x = a\sin 2\theta\cos\theta$

$y = a\sin 2\theta\sin\theta$ are the required parametric eqs.

Q#2. $r = \theta$

Soln:

$x = r\cos\theta$

$x = \theta\cos\theta$

and $y = r\sin\theta$

$y = \theta\sin\theta$

So parametric eqs. of $r = \theta$ are

$x = \theta\cos\theta$

$y = \theta\sin\theta$.

Q#3. $r = 2 + 3\sin\theta$.

Soln. $\because$ For parametric eqs.

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$\Rightarrow \quad x = (2+3\sin\theta)\cos\theta, \quad y = (2+3\sin\theta)\sin\theta$

are the required parametric eqs.

Q#4. Show that the equations

$x = a + r\cos\theta, \quad y = b + r\sin\theta$

are parametric eqs. for a circle with centre (a,b) and radius $|r|$.

Soln. Here $\quad x = a + r\cos\theta$ ——— (1)

$y = b + r\sin\theta$ ——— (2)

1) $\Rightarrow \quad x - a = r\cos\theta$ ——— (3)

2) $\Rightarrow \quad y - b = r\sin\theta$ ——— (4)

Now $(3)^2 + (4)^2 \Rightarrow$

$$(x-a)^2 + (y-b)^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$(x-a)^2 + (y-b)^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$(x-a)^2 + (y-b)^2 = r^2$$

which represent a circle with centre at (a,b) and radius $|r|$.

Q#5 Show that the curve whose parametric eqs. are

$$\left.\begin{array}{l} x = a\cos\theta + h \\ y = b\sin\theta + k \end{array}\right\}, \quad 0 \le \theta \le 2\pi$$

is an ellipse with centre (h,k)

Soln: Here $x = a\cos\theta + h$ ——— (1)

$y = b\sin\theta + k$ ——— (2)

1) $\Rightarrow$ $x - h = a\cos\theta$

$\frac{x-h}{a} = \cos\theta$ ——— (3)

2) $\Rightarrow$ $y - k = b\sin\theta$

$\frac{y-k}{b} = \sin\theta$ ——— (4)

Now $(3)^2 + (4)^2 \Rightarrow$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \cos^2\theta + \sin^2\theta$$

$$\Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

which represent an ellipse with centre (h,k).

Q#6. A wheel of radius a rolls on a straight line without slipping or sliding. Let P be a fixed point on the wheel, at a distance b from the centre of the wheel. Find parametric eqs. of the curve described by the point P. The curve is called a trochoid. Hence deduce parametric equations for a cycloid.

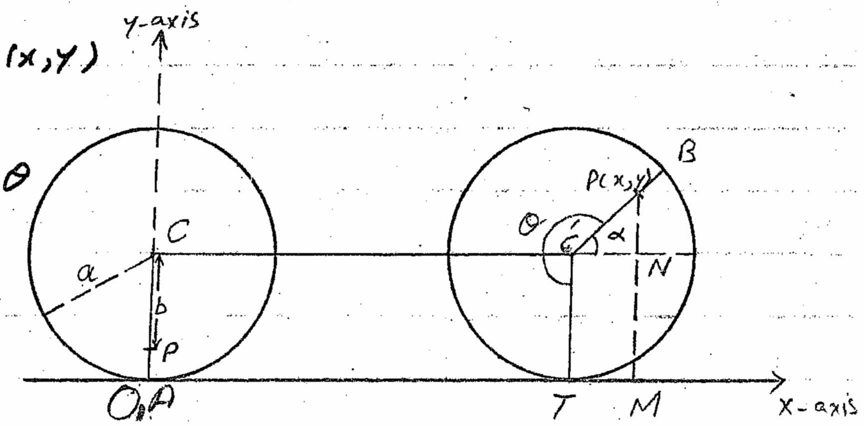
Soln: Let the wheel be initially in contact with the line at O. where the line is x-axis, $CP = b$. Let the wheel rotate through an angle θ as reflected in the 2nd fig.

let $P\hat{C}N = \alpha$ and $P = (x, y)$

Then $B\hat{C'}T = \theta$

and $OT = TB = a\theta$ $\quad \because l = r\theta$

[because the radial distance of P is equal to the distance along x-axis]



Now
$$x = OT + TM$$
$$x = OT + C'N$$
$$x = a\theta + b\cos\alpha \quad \text{———} \quad (1)$$

and
$$y = PN + NM \qquad \because NM = C'T = a$$
$$y = b\sin\alpha + a \quad \text{———} \quad (2)$$

$$\because \quad \alpha + \theta = \frac{3\pi}{2}$$
$$\Rightarrow \quad \alpha = \frac{3\pi}{2} - \theta$$

Putting $\alpha = \frac{3\pi}{2} - \theta$ into (1) and (2), we have

$$x = a\theta + b\cos\left(\frac{3\pi}{2} - \theta\right) \Rightarrow x = a\theta - b\sin\theta$$
$$y = b\sin\left(\frac{3\pi}{2} - \theta\right) + a \Rightarrow y = -b\cos\theta + a$$

Hence the parametric eqs. of trochoid are
$$x = a(\theta) - b\sin\theta \quad \text{———} \quad (3)$$
$$y = a - b\cos\theta \quad \text{———} \quad (4)$$

Put $b = a$ in (3) and (4), we get
$$x = a\theta - a\sin\theta \Rightarrow x = a(\theta - \sin\theta)$$
$$y = a - a\cos\theta \Rightarrow y = a(1 - \cos\theta)$$

So
$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

are eqs. of a cycloid.

Q#7. Find the points at which $r = 1+\cos\theta$ has horizontal and vertical tangents.

Here $r = 1+\cos\theta$ ——— (1)

Diff. w.r.t. 'θ' $\dfrac{dr}{d\theta} = -\sin\theta$

$$\Rightarrow \frac{d\theta}{dr} = -\frac{1}{\sin\theta}$$

So
$$\tan\psi = r\frac{d\theta}{dr}$$
$$= 1+\cos\theta \cdot -\frac{1}{\sin\theta}$$
$$= -\frac{2\cos^2\theta/2}{2\sin\theta/2\,\cos\theta/2}$$
$$= -\cot\theta/2$$
$$= \tan\left(\frac{\pi}{2}+\theta/2\right)$$
$$\Rightarrow \psi = \pi/2 + \theta/2$$

But $\alpha = \theta + \psi$
$$\Rightarrow \alpha = \theta + \pi/2 + \theta/2$$
$$\alpha = 3\theta/2 + \pi/2 \quad \text{——— (2)}$$

__For horizontal tangents.__

For these tangents we may have the following two cases.

__Case 1.__ If $\alpha = 0$ put in (2)

$$(2) \Rightarrow 0 = 3\theta/2 + \pi/2$$
$$0 = 3\theta + \pi$$
$$\theta = -\pi/3 = 2\pi - \pi/3 = 5\pi/3$$

$\Rightarrow$ at $\theta = -\pi/3$ the tangent is $\parallel$ to x-axis.

Put $5\pi/3$ in (1)
$\Rightarrow r = \frac{3}{2}$
$\Rightarrow$ The tangent is horizontal at $\left(\frac{3}{2}, 5\pi/3\right)$

__Case 2.__ If $\alpha = \pi$

Put in (2) $\Rightarrow \pi = \dfrac{3\theta}{2} + \dfrac{\pi}{2}$

$$\frac{3\theta}{2} = \pi - \pi/2 = \pi/2$$
$$\Rightarrow \theta = \frac{\pi}{3}$$

$\Rightarrow$ at $\theta = \pi/3$ the tangent is horizontal.

Put $\pi/3$ in (1)
$\Rightarrow r = 3/2$
$\Rightarrow$ The tangent is horizontal at $(3/2, \pi/3)$

For Vertical Tangents.

Two cases arise here

Case 1. if $\alpha = \pi/2$, put in (2)

$$\Rightarrow \quad \pi/2 = \frac{3\theta}{2} + \pi/2$$

$$\Rightarrow \quad \frac{3\theta}{2} = 0$$

$$\Rightarrow \quad \theta = 0$$

$\Rightarrow$ at $\theta = 0$ the tangent is vertical

Put $\theta = 0$ in (1)
$\Rightarrow r = 2$
$\Rightarrow$ The tangent is vertical at $(2, 0)$

Case 2. if $\alpha = \frac{3\pi}{2}$, put in (2)

$$\Rightarrow \quad \frac{3\pi}{2} = \frac{3\theta}{2} + \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{3\theta}{2} = \frac{3\pi}{2} - \frac{\pi}{2}$$

$$\Rightarrow \quad 3\theta = 3\pi - \pi$$

$$\Rightarrow \quad 3\theta = 2\pi$$

$$\Rightarrow \quad \theta = \frac{2\pi}{3}$$

$\Rightarrow$ at $\theta = \frac{2\pi}{3}$ the tangent is vertical.

Put $\theta = \frac{2\pi}{3}$ in (1)
$\Rightarrow r = \frac{1}{2}$
$\Rightarrow$ The tangent is vertical at $(\frac{1}{2}, 2\pi/3)$

Q#8. Find the points on the curve

$$x(t) = t^2 + 4 , \quad y(t) = 3t^2 - 6t + 2$$

where tangents are horizontal and vertical.

Soln: Here $x = t^2 + 4$ ——— (I)

and $y = 3t^2 - 6t + 2$ ——— (II)

Now Diff. (I) w.r.t. t

$$\frac{dx}{dt} = 2t \quad \text{——— (III)}$$

Also Diff. (II) w.r.t. 't'

$$\frac{dy}{dt} = 6t - 6 \quad \text{——— (IV)}$$

Now
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-6}{2t} = \frac{6(t-1)}{2t}$$

$$\frac{dy}{dx} = \frac{3(t-1)}{t} \quad \text{——— (V)}$$

Now for horizontal tangent:

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad dy/dt = 0 \quad \text{and} \quad \frac{dx}{dt} \neq 0$$

So for horizontal tangent

$$\frac{dy}{dt} = 0$$

$$6t - 6 = 0$$

$$6t = 6$$

$$\Rightarrow t = 1$$

Put $t = 1$ in (1) and (2)

$\Rightarrow \quad x = 1 + 4 = 5$

and $\quad y = 3 - 6 + 2 = -1$

$\Rightarrow$ at $(5, -1)$ the tangent is $\parallel$ to x-axis.

<u>For Vertical Tangent</u>:

$$\frac{dy}{dx} = \infty$$

$$\Rightarrow \quad \frac{dx}{dt} = 0$$

$$2t = 0$$

$$t = 0$$

Put in (1) and (2)

$\Rightarrow \quad x = 0 + 4 = 4$

$y = 0 - 0 + 2 = 2$

$\Rightarrow$ at $(4, 2)$ the tangent is vertical.

Find equations of the tangent and Normal to each of the given curves at indicated point. (9-11)

Q#9 $\quad x = 2a\cos\theta - a\cos 2\theta, \; y = 2a\sin\theta - a\sin 2\theta$ at $\theta = \frac{\pi}{2}$

Soln. Here $x = 2a\cos\theta - a\cos 2\theta$ ——— (I)

$y = 2a\sin\theta - a\sin 2\theta$ ——— (II)

$$\text{I)} \Rightarrow \frac{dx}{d\theta} = -2a\sin\theta + 2a\sin 2\theta$$

$$= 2a(\sin 2\theta - \sin\theta)$$

$$= 2a \cdot 2\cos\frac{2\theta + \theta}{2} \cdot \sin\frac{2\theta - \theta}{2}$$

$$= 4a\cos\frac{3\theta}{2}\sin\theta/2 \quad \text{——— III}$$

$$\text{II)} \Rightarrow \frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta$$

$$= 2a(\cos\theta - \cos 2\theta)$$

$$= 2a\left(-2\sin\frac{\theta + 2\theta}{2}\sin\frac{\theta - 2\theta}{2}\right)$$

$$\frac{dy}{d\theta} = -4a \sin\frac{3\theta}{2} \sin\left(-\frac{\theta}{2}\right)$$
$$= 4a \sin\frac{3\theta}{2} \sin\frac{\theta}{2}$$

Now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{4a \sin 3\theta/2 \sin \theta/2}{4a \cos 3\theta/2 \sin \theta/2}$$

$$\frac{dy}{dx} = \frac{\sin 3\theta/2}{\cos 3\theta/2}$$

$$\frac{dy}{dx} = \tan 3\theta/2$$

$$\left.\frac{dy}{dx}\right|_{\pi/2} = \tan \frac{3}{2} \cdot \frac{\pi}{2}$$

$$\left.\frac{dy}{dx}\right|_{\pi/2} = \tan \frac{3\pi}{4}$$
$$= -1$$

Also $\quad x_1 = 2a\cos\frac{\pi}{2} - a\cos 2\cdot\frac{\pi}{2} = 0 - a(-1) = a$

$\quad\quad y_1 = 2a\sin\frac{\pi}{2} - a\sin 2\cdot\frac{\pi}{2} = 2a - 0 = 2a$

Thus the eq. of the tangent is

$$y - y_1 = \left.\frac{dy}{dx}\right|_{P} (x - x_1)$$

$$y - 2a = -1(x - a)$$
$$y - 2a = -x + a$$
$$x + y - 2a - a = 0$$
$$x + y - 3a = 0$$

IS the eq. of the tangent at $\pi/2$

Now eq. of the Normal

$$y - y_1 = -\frac{1}{dy/dx} \cdot (x - x_1)$$

$$y - 2a = -\frac{1}{-1}(x - a)$$
$$y - 2a = x - a$$
$$x - y - a + 2a = 0$$
$$x - y + a = 0$$

Is the eq. of the tangent at $\pi/2$

(i) #10. $x = \frac{2at^2}{1+t^2}$, $y\ \frac{2at^3}{1+t^2}$ at $t = \frac{1}{2}$

Soln:

$$x = \frac{2at^2}{1+t^2} \quad \text{———} \quad (1)$$

$$y = \frac{2at^3}{1+t^2} \quad \text{———} \quad (2)$$

1) ⇒ $$\frac{dx}{dt} = \frac{(1+t^2)\,4at - 2at^2 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at + 4at^3 - 4at^3}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{4at}{(1+t^2)^2} \quad \text{———} \quad (3)$$

2) ⇒ $$\frac{dy}{dt} = \frac{(1+t^2)\,6at^2 - 2at^3 \cdot 2t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at^2 + 6at^4 - 4at^4}{(1+t^2)^2}.$$

$$\frac{dy}{dt} = \frac{6at^2 + 2at^4}{(1+t^2)^2}$$

$$= \frac{2a(3t^2 + t^4)}{(1+t^2)^2}$$

Now

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2a(3t^2+t^4)}{(1+t^2)^2} \cdot \frac{(1+t^2)^2}{4at}$$

$$\frac{dy}{dx} = \frac{2at(3t + t^3)}{4at}$$

$$\frac{dy}{dx} = \frac{3t + t^3}{2}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \frac{3(1/2) + (1/2)^3}{2}$$

$$= \frac{3/2 + 1/8}{2} = \frac{\frac{12+1}{8}}{2}$$

$$= \frac{13}{16}$$

Now

$$x_1 = \frac{2a(1/2)^2}{1+(1/2)^2}, \qquad y_1 = \frac{2a(1/2)^3}{1+(1/2)^2}$$

$$x_1 = \frac{2a \cdot 1/4}{1+1/4}, \qquad y_1 = \frac{2a(1/8)}{1+1/4}$$

$$x_1 = \frac{a/2}{5/4}, \qquad y_1 = \frac{a/4}{5/4}$$

$$x_1 = \frac{2a}{5}, \qquad y_1 = \frac{a}{5}$$

$$\Rightarrow \quad (x_1, y_1) = \left(\frac{2a}{5}, \frac{a}{5}\right)$$

So Eq. of the tangent at $t = \frac{1}{2}$ is

$$y - \frac{a}{5} = \frac{13}{16}\left(x - \frac{2a}{5}\right)$$

$$16y - \frac{16a}{5} = 13\left(x - \frac{2a}{5}\right)$$

$$80y - 16a = 65x - 26a$$

$$65x - 80y - 10a = 0$$

$$13x - 16y - 2a = 0$$

which is the eq. of tangent.

Now Eq. of the Normal is

$$y - \frac{a}{5} = -\frac{16}{13}\left(x - \frac{2a}{5}\right)$$

$$65y - 13a = -80x + 32a$$

$$80x + 65y - 45a = 0$$

$$16x + 13y - 9a = 0$$

which is the eq. of the Normal at $t = \frac{1}{2}$

Q# 11. $x = (t-1)^{3/2}, \quad y = 3t \quad$ at $t = 5$

Here

$$x = (t-1)^{3/2} \quad \text{———— (1)}$$

$$y = 3t \quad \text{———— II}$$

Now $1) \Rightarrow$ $\frac{dx}{dt} = \frac{3}{2}(t-1)^{1/2}$

$2) \Rightarrow$ $\frac{dy}{dt} = 3$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{3}{\frac{3}{2}\sqrt{t-1}} = \dfrac{2}{\sqrt{t-1}}$

$\left.\dfrac{dy}{dx}\right|_{t=5} = \dfrac{2}{\sqrt{5-1}} = \dfrac{2}{\sqrt{4}} = \dfrac{2}{2} = 1$

The point $P(x_1, y_1)$ at $t = 5$

$x_1 = (5-1)^{3/2}$

$= (4)^{3/2} = 8$

and $y_1 = 3(5)$

$y_1 = 15$

Hence the point $P(x_1, y_1)$ at $t = 5$ is $(8, 15)$

So the Eq. of the tangent is given by

$y - 15 = 1\,(x - 8)$

$x - y + 7 = 0$

Eq. of the Normal is

$y - 15 = -1\;(x - 8)$

$x + y - 23 = 0$

Q.12 Show that the Normal at any point of the curve $x = a\cos\theta + a\theta\sin\theta$

$y = a\sin\theta - a\theta\cos\theta$

is at a constant distance from the origin.

Soln: Here $x = a\cos\theta + a\theta\sin\theta$ ——— (I)

$y = a\sin\theta - a\theta\cos\theta$ ——— (II)

1) => $\dfrac{dx}{d\theta} = -a\sin\theta + a\theta\cos\theta + a\sin\theta$

$= a\theta\cos\theta$

2) => $\dfrac{dy}{d\theta} = a\cos\theta + a\theta\sin\theta - a\cos\theta$

$= a\theta\sin\theta$

But $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \cdot \dfrac{d\theta}{dx} = \dfrac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$

Now the Eq. of the Normal at $(a\cos\theta + a\theta\sin\theta, a\sin\theta - a\theta\cos\theta)$

$y - y_1 = -\dfrac{1}{dy/dx}\,(x - x_1)$

$y - (a\sin\theta - a\theta\cos\theta) = -\dfrac{1}{\tan\theta}\,(x - (a\cos\theta + a\theta\sin\theta))$

$$y - a\sin\theta + a\theta\cos\theta = -\frac{\cos\theta}{\sin\theta}\left[x - a\cos\theta - a\theta\sin\theta\right]$$

X by $\sin\theta$

$$y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

$$\Rightarrow \quad x\cos\theta + y\sin\theta - a\sin^2\theta - a\cos^2\theta = 0$$

$$x\cos\theta + y\sin\theta - a(\sin^2\theta + \cos^2\theta) = 0$$

$$x\cos\theta + y\sin\theta - a = 0 \quad \text{———} \quad \text{III}$$

which Is the Eq. of Normal

Now If d is the distance of this normal form $O(0,0)$ then

$$d = \frac{|0 + 0 - a|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$d = \frac{a}{1} = a$$

$= \text{Constant}.$

Q#13. Prove that an equation of the Normal to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ can be written in the form

$$x\sin t - y\cos t + a\cos 2t = 0, \quad t \text{ being}$$

parameter.

Soln: The parametric eq. of the astroid is

$$x = a\sin^3 t \quad \text{———} (1) \quad \text{and} \quad y = a\cos^3 t \quad \text{———} (2)$$

$$1) \Rightarrow \frac{dx}{dt} = 3a\sin^2 t\cos t$$

$$2) \Rightarrow \frac{dy}{dt} = -3a\cos^2 t\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{3a\cos^2 t\sin t}{3a\sin^2 t\cos t}$$

$$= -\frac{\cos t}{\sin t}$$

So eq. of the Normal at $(a\sin^3 t, a\cos^3 t)$ is

$$y - a\cos^3 t = \frac{\sin t}{\cos t}(x - a\sin^3 t)$$

$$\Rightarrow \quad y\cos t - a\cos^4 t = x\sin t - a\sin^4 t$$

$$\Rightarrow \quad x \sin t - y \cos t + a\cos^4 t - a\sin^4 t = 0$$

$$x \sin t - y \cos t + a(\cos^4 t - \sin^4 t) = 0$$

$$x \sin t - y \cos t + a\left[(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)\right] = 0$$

$$x \sin t - y \cos t + a(\cos 2t) = 0$$

$$x \sin t - y \cos t + a\cos 2t = 0$$

which is the required eq. of the Normal.

Q#14. Show that the pedal eq. of the astroid

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

is $\quad r^2 = a^2 - 3p^2$

Soln.

Here $x = a\cos^3\theta$ ——— (1)

$y = a\sin^3\theta$ ——— (2)

$$1) \Rightarrow \quad \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$2) \Rightarrow \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

But
$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

Now Eq. of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is given by

$$y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

$$y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\cos^3\theta\sin\theta$$

$$\sin\theta\, x + \cos\theta\, y - a\left[\sin^3\theta\cos\theta + \cos^3\theta\sin\theta\right] = 0$$

$$\sin\theta\, x + \cos\theta\, y - a\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta) = 0$$

$$\sin\theta\, x + \cos\theta\, y - a\sin\theta\cos\theta = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 - a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta + \cos^2\theta}}$$

$$p = a\sin\theta\cos\theta$$

$$\frac{p}{a} = \sin\theta\cos\theta \quad \text{———— (3)}$$

$\because$ $$r^2 = x^2+y^2$$

$\therefore$ $$r^2 = a^2\cos^6\theta + a^2\sin^6\theta$$

$$r^2 = a^2(\cos^6\theta+\sin^6\theta)$$

$$r^2 = a^2\left[(\cos^2\theta)^3+(\sin^2\theta)^3\right]$$

$$r^2 = a^2\left[(\cos^2\theta+\sin^2\theta)^3 - 3(\cos^2\theta\sin^2\theta)(\cos^2\theta+\sin^2\theta)\right]$$

$$r^2 = a^2\left[1-3\cos^2\theta\sin^2\theta\right] \quad \text{———— (4)}$$

$a^3+b^3 = (a+b)^3 - 3ab(a+b)$

Put (3) in (4)

$$r^2 = a^2\left[1 - 3\frac{p^2}{a^2}\right]$$

$$\Rightarrow r^2 = a^2 - 3p^2$$

Is the required pedal Equation.

Q#15. Prove that the pedal Eq. of the curve $x = 2a\cos\theta - a\cos2\theta$, $y = 2a\sin\theta - a\sin2\theta$ is $9(r^2-a^2) = 8p^2$

Soln: Here

$$x = 2a\cos\theta - a\cos2\theta \quad \text{———— I}$$

$$y = 2a\sin\theta - a\sin2\theta \quad \text{———— II}$$

I) $\Rightarrow$ $$\frac{dx}{d\theta} = -2a\sin\theta + 2a\sin2\theta$$

$$= 2a\left[\sin2\theta - \sin\theta\right]$$

II) $\Rightarrow$ $$\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos2\theta$$

$$= 2a\left[\cos\theta - \cos2\theta\right]$$

But $$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= \frac{2a\left[\cos\theta - \cos2\theta\right]}{2a\left[\sin2\theta - \sin\theta\right]}$$

$$= \frac{-2\sin\frac{\theta+2\theta}{2}\sin\frac{\theta-2\theta}{2}}{2\cos\frac{2\theta+\theta}{2}\sin\frac{2\theta-\theta}{2}}$$

$$= -\frac{\sin 3\theta/2 \cdot \sin(-\theta/2)}{\cos 3\theta/2 \ \sin\theta/2}$$

$$\frac{dy}{dx} = \frac{\sin 3\theta/2}{\cos 3\theta/2}$$

Eq. of the tangent at $(2a\cos\theta - a\cos 2\theta,\ 2a\sin\theta - a\sin 2\theta)$ is

$$y - 2a\sin\theta + a\sin 2\theta = \frac{\sin 3\theta/2}{\cos 3\theta/2}(x - 2a\cos\theta + a\cos 2\theta)$$

$$\Rightarrow \quad \cos\tfrac{3\theta}{2}\, y - 2a\sin\theta\cos\frac{3\theta}{2} + a\sin 2\theta\cos\frac{3\theta}{2}$$

$$= x\sin\frac{3\theta}{2} - 2a\cos\theta\sin\frac{3\theta}{2} + a\cos 2\theta\sin\frac{3\theta}{2}$$

$$\Rightarrow \quad x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\left[\sin\frac{3\theta}{2}\cos\theta - \sin\theta\cos\frac{3\theta}{2}\right]$$

$$+ a\left[\sin\frac{3\theta}{2}\cos 2\theta - \sin 2\theta\cos\frac{3\theta}{2}\right] = 0$$

$$\Rightarrow \quad x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\left[\sin\left(\frac{3\theta}{2} - \theta\right)\right] + a\left[\sin\left(\frac{3\theta}{2} - 2\theta\right)\right] = 0$$

$$\Rightarrow \quad x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\sin\frac{\theta}{2} - a\sin\frac{\theta}{2} = 0$$

$$\Rightarrow \quad x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 3a\sin\frac{\theta}{2} = 0$$

which is the equation of tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{\left|0 - 0 - 3a\sin\frac{\theta}{2}\right|}{\sqrt{\sin^2\frac{3\theta}{2} + \cos^2\frac{3\theta}{2}}}$$

$$p = 3a\sin\frac{\theta}{2} \quad \text{———} \quad III$$

Now

$$r^2 = x^2 + y^2$$

$$r^2 = (2a\cos\theta - a\cos 2\theta)^2 + (2a\sin\theta - a\sin 2\theta)^2$$

$$r^2 = 4a^2\cos^2\theta + a^2\cos^2 2\theta - 4a^2\cos\theta\cos 2\theta + 4a^2\sin^2\theta + a^2\sin^2 2\theta - 4a^2\sin\theta\sin 2\theta.$$

$$r^2 = 4a^2(\cos^2\theta + \sin^2\theta) + a^2(\cos^2 2\theta + \sin^2 2\theta). - 4a^2[\cos 2\theta\cos\theta + \sin 2\theta\sin\theta]$$

$$r^2 = 4a^2 + a^2 - 4a^2\cos(2\theta - \theta)$$
$$r^2 = 5a^2 - 4a^2\cos\theta$$
$$r^2 = 5a^2 - 4a^2[1 - 2\sin^2\theta/2]$$
$$r^2 = 5a^2 - 4a^2 + 8a^2\sin^2\frac{\theta}{2}$$
$$r^2 = a^2 + 8a^2\sin^2\theta/2$$
$$r^2 - a^2 = 8a^2\sin^2\theta/2 \qquad \text{——— IV}$$

$1 - \cos\theta = 2\sin^2\theta/2$

$\Rightarrow 1 - 2\sin^2\theta/2 = \cos\theta$

Put (3) in (4)

$$\Rightarrow \quad r^2 - a^2 = 8a^2\frac{p^2}{9a^2}$$

$$9(r^2 - a^2) = 8p^2$$

which is the required pedal Equation.

Q#16. Show that the pedal Equation of the curve $x = ae^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$ is $r = \sqrt{2}\,p$

Soln: Here
$$x = ae^{\theta}(\sin\theta - \cos\theta) \qquad \text{——— (1)}$$
$$y = ae^{\theta}(\sin\theta + \cos\theta) \qquad \text{——— (2)}$$

$$1) \Rightarrow \quad \frac{dx}{d\theta} = ae^{\theta}(\sin\theta - \cos\theta) + ae^{\theta}(\cos\theta + \sin\theta)$$
$$= ae^{\theta}\sin\theta - ae^{\theta}\cos\theta + ae^{\theta}\cos\theta + ae^{\theta}\sin\theta$$
$$= 2ae^{\theta}\sin\theta$$

$$2) \Rightarrow \quad \frac{dy}{d\theta} = ae^{\theta}(\sin\theta + \cos\theta) + ae^{\theta}(\cos\theta - \sin\theta)$$
$$= ae^{\theta}\sin\theta + ae^{\theta}\cos\theta + ae^{\theta}\cos\theta - ae^{\theta}\sin\theta$$
$$= 2ae^{\theta}\cos\theta$$

$$\text{But } \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$
$$= \frac{\cos\theta}{\sin\theta}.$$

Eq. of the tangent at $((ae^{\theta}(\sin\theta - \cos\theta)), ae^{\theta}(\sin\theta + \cos\theta))$ is

$$y - ae^{\theta}\sin\theta - ae^{\theta}\cos\theta = \frac{\cos\theta}{\sin\theta}[x - ae^{\theta}\sin\theta + ae^{\theta}\cos\theta]$$

$$y\sin\theta - ae^{\theta}\sin^2\theta - ae^{\theta}\sin\theta\cos\theta = x\cos\theta - ae^{\theta}\sin\theta\cos\theta + ae^{\theta}\cos^2\theta$$
$$x\cos\theta - y\sin\theta + ae^{\theta}(\cos^2\theta + \sin^2\theta) = 0$$

Written by
Shahid Javed

$$x\cos\theta - y\sin\theta - ae^{\theta} = 0$$

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 - 0 - ae^{\theta}|}{\sqrt{\cos^2\theta + \sin^2\theta}}$$

$$p = ae^{\theta} \quad \text{——— III}$$

$\because$

$$r^2 = x^2 + y^2$$

$$r^2 = a^2e^{2\theta}\left[(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2\right]$$

$$r^2 = a^2e^{2\theta}\left[\sin^2\theta + \cos^2\theta - \cancel{2\sin\theta\cos\theta} + \sin^2\theta + \cos^2\theta + \cancel{2\sin\theta\cos\theta}\right]$$

$$r^2 = 2a^2e^{2\theta} \quad \text{——— IV}$$

Put III in IV

$$r^2 = 2p^2$$

$$r = \sqrt{2}\,p$$

which is the required pedal Equation.

Q#17. Prove that the pedal Equation of the Curve

$$x = a(3\cos\theta - \cos^3\theta), \quad y = a(3\sin\theta - \sin^3\theta)$$

is $\quad 3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$.

Soln. Here

$$x = a(3\cos\theta - \cos^3\theta) \quad \text{——— (1)}$$

$$y = a(3\sin\theta - \sin^3\theta) \quad \text{——— (2)}$$

$$1) \Rightarrow \quad \frac{dx}{d\theta} = -3a\sin\theta + 3a\cos^2\theta\sin\theta$$

$$= 3a\sin\theta\left[\cos^2\theta - 1\right]$$

$$= 3a\sin\theta\left(-\sin^2\theta\right)$$

$$= -3a\sin^3\theta$$

$$2) \Rightarrow \quad \frac{dy}{d\theta} = 3a\cos\theta - 3a\sin^2\theta\cos\theta.$$

$$= 3a\cos\theta\left[1 - \sin^2\theta\right]$$

$$= 3a\cos\theta\left[\cos^2\theta\right]$$

$$= 3a\cos^3\theta.$$

But $\quad \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \cdot \dfrac{d\theta}{dx}$

$$\frac{dy}{dx} = \frac{3a\cos^3\theta}{-3a\sin^3\theta}$$

$$\frac{dy}{dx} = -\frac{\cos^3\theta}{\sin^3\theta}$$

Eq. of the tangent at $(a(3\cos\theta - \cos^3\theta),\ a(3\sin\theta - \sin^3\theta))$

$$y - 3a\sin\theta + a\sin^3\theta = -\frac{\cos^3\theta}{\sin^3\theta}\left(x - 3a\cos\theta + a\cos^3\theta\right)$$

$$y\sin^3\theta - 3a\sin^4\theta + a\sin^6\theta = -x\cos^3\theta + 3a\cos^4\theta - a\cos^6\theta$$

$$x\cos^3\theta + y\sin^3\theta + a[\sin^6\theta + \cos^6\theta] = 3a[\sin^4\theta + \cos^4\theta]$$

$$x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta)^3 + (\cos^2\theta)^3] = 3a[(\sin^2\theta)^2 + (\cos^2\theta)^2]$$

$$x\cos^3\theta + y\sin^3\theta + a[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$$

$$= 3a[(\sin^2\theta + \cos^2\theta)^2 - 2\cos^2\theta\sin^2\theta]$$

$$x\cos^3\theta + y\sin^3\theta + a[1 - 3\sin^2\theta\cos^2\theta] = 3a[1 - 2\cos^2\theta\sin^2\theta]$$

$$x\cos^3\theta + y\sin^3\theta + a - 3a\sin^2\theta\cos^2\theta = 3a - 6a\sin^2\theta\cos^2\theta$$

$$x\cos^3\theta + y\sin^3\theta + a - 3a - 3a\sin^2\theta\cos^2\theta + 6a\sin^2\theta\cos^2\theta = 0$$

$$x\cos^3\theta + y\sin^3\theta - 2a + 3a\sin^2\theta\cos^2\theta = 0$$

$$x\cos^3\theta + y\sin^3\theta + 3a\sin^2\theta\cos^2\theta - 2a = 0$$

which is the Eq. of the tangent.

If p is the length of the perpendicular from $O(0,0)$ to the tangent then

$$p = \frac{|0 + 0 + 3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{\cos^6\theta + \sin^6\theta}}$$

$$p = \frac{|3a\sin^2\theta\cos^2\theta - 2a|}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}} \qquad \text{——— (3)}$$

Now $\quad r^2 = x^2 + y^2$

$$r^2 = a^2(3\cos\theta - \cos^3\theta)^2 + a^2(3\sin\theta - \sin^3\theta)^2$$

$$r^2 = a^2(9\cos^2\theta + \cos^6\theta - 6\cos^4\theta) + a^2(9\sin^2\theta + \sin^6\theta - 6\sin^4\theta)$$

$$r^2 = 9a^2(\cos^2\theta + \sin^2\theta) + a^2 6(-\cos^4\theta - \sin^4\theta) + a^2(\sin^6\theta + \cos^6\theta)$$

$$r^2 = a^2[9 - 6(\cos^4\theta + \sin^4\theta) + (\sin^6\theta + \cos^6\theta)]$$

$$r^2 = a^2[9 - 6\{(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta\} + (\sin^2\theta + \cos^2\theta)^3$$

$$- 3\cos^2\theta\sin^2\theta(\sin^2\theta + \cos^2\theta)]$$

$a^2 + b^2 = (a+b)^2 - 2ab$

$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$$r^2 = a^2\left[9 - 6\{1 - 2\sin^2\theta\cos^2\theta\} + 1 - 3\sin^2\theta\cos^2\theta\right]$$
$$r^2 = 9a^2 - 6a^2 + 12a^2\sin^2\theta\cos^2\theta + a^2 - 3a^2\sin^2\theta\cos^2\theta$$
$$r^2 = 4a^2 + 9a^2\cos^2\theta\sin^2\theta$$
$$r^2 = a^2\left[4 + 9\cos^2\theta\sin^2\theta\right]$$
$$r^2 - 4a^2 = 9a^2\cos^2\theta\sin^2\theta$$
$$\Rightarrow \quad \cos^2\theta\sin^2\theta = \frac{r^2 - 4a^2}{9a^2} \quad \text{——— (4)}$$

put (4) in (3)

$$p = \frac{\left|3a \cdot \frac{r^2-4a^2}{9a^2} - 2a\right|}{\sqrt{1 - 3\left(\frac{r^2-4a^2}{9a^2}\right)}}$$

$$p = \frac{\left|\frac{r^2-4a^2}{3a} - 2a\right|}{\sqrt{1 - \left(\frac{r^2-4a^2}{3a^2}\right)}}$$

$$p = \frac{r^2 - 4a^2 - 6a^2}{3a} \cdot \frac{\sqrt{3}\,a}{\sqrt{3a^2 - r^2 + 4a^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3}} \times \frac{1}{\sqrt{7a^2 - r^2}}$$

$$p = \frac{r^2 - 10a^2}{\sqrt{3}\sqrt{7a^2 - r^2}}$$

$$\Rightarrow \quad p^2 = \frac{(r^2 - 10a^2)^2}{3(7a^2 - r^2)}$$

$$3p^2(7a^2 - r^2) = (r^2 - 10a^2)^2$$

is the required pedal Equation.

Q #18 If $x = a\cos g(t)$, $y = b\sin g(t)$, prove that

$$xy^3\frac{d^2y}{dx^2} = b^2\frac{dy}{dx}$$

Soln:

Here $x = a\cos g(t)$ ——— I

$y = b\sin g(t)$ ——— II

$$1) \Rightarrow \quad \frac{dx}{dt} = -a\sin g(t)\cdot g'(t)$$

$$2) \Rightarrow \quad \frac{dy}{dt} = b\cos g(t)\cdot g'(t)$$

$$\frac{dy}{dx} = \frac{b\cos g(t)\cdot g'(t)}{-a\sin g(t)\cdot g'(t)}$$

$$= -\frac{b}{a}\cot g(t) \qquad \text{——— } III$$

Again Diff. w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{b}{a}\operatorname{cosec}^2 g(t)\ g'(t)\ \frac{dt}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2 g(t)\ g'(t)\ \frac{1}{-a\sin g(t)\cdot g'(t)}$$

$$= -\frac{b}{a^2\sin^3 g(t)}$$

Consider $\quad xy^2\dfrac{d^2y}{dx^2} = a\cos g(t)\cdot b^2\sin^2 g(t)\cdot -\dfrac{b}{a^2\sin^3 g(t)}$

$$= -\frac{b^3\cos g(t)}{a\sin g(t)}$$

$$= b^2\left(-\frac{b\cos g(t)}{a\sin g(t)}\right)$$

$$= b^2\frac{dy}{dx} \qquad \text{from } III$$

Hence $\quad xy^2\dfrac{d^2y}{dx^2} = b^2\dfrac{dy}{dx}$

is as required.