Exercise Set 5.5

In each of **Problems 1 – 12**, use the trapezoidal rule to approximate the given integral:

1. $\int_1^4 \frac{dx}{x} = \ln 4$ with $n = 3$
2. $\int_0^{\pi/3} \cos x \, dx = \sqrt{3}/2$ with $n = 4$
3. $\int_0^2 e^{-x^2} \, dx$ with $n = 4$
4. $\int_0^4 x^2 \, dx$ with $n = 8$
5. $\int_0^{\pi} \sin x \, dx$ with $n = 6$
6. $\int_0^2 \frac{dx}{1+x^3}$ with $n = 4$
7. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ with $n = 4$
8. $\int_{-2}^2 (2x^2+1)\, dx$ with $n = 4$
9. $\int_1^5 \frac{dx}{x^2}$ with $n = 4$
10. $\int_0^1 e^{-x} dx$ with $n = 6$
11. $\int_1^2 \ln x \, dx$ with $n = 4$
12. $\int_0^2 \frac{1}{\sqrt{1+x^2}} \, dx$ with $n = 4$
13. Use Simpson's rule to approximate the integrals of Problems 3, 4, 10, 11 and 12.

Find a bound on the error in approximating the given integral using (i) the trapezoidal rule (ii) Simpson's rule. (**Problems 14 – 16**):

14. $\int_{-1}^2 x^5 \, dx$ with $n = 10$
15. $\int_1^3 \frac{dx}{x}$ with $n = 10$
16. $\int_0^2 \frac{dx}{\sqrt{1+x}}$ with $n = 8$

17. With $n = 8$, find the area under the semicircle $y = \sqrt{4 - x^2}$ and above the x-axis by (i) the trapezoidal rule (ii) Simpson's rule.

18. A reading of the velocity of a ship was made every quarter hour as shown below:

Time t (hours) =	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Velocity $v(t)$ (mph) =	19.5	24.3	34.2	40.5	38.4
$t =$	5/4	3/2	7/4	2	
$v(t) =$	26.2	18	16	8	

Estimate the distance travelled by the ship during the 2-hour period.

Numerical Integration:-

Trapezoidal rule:-

Let a function f be continuous on $[a,b]$ and let $[a,b]$ be partioned into n equal subintervals $[x_0, x_1], [x_1, x_2] \ldots [x_{n-1}, x_n]$ each of length $\frac{b-a}{n}$, Then,

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{n}\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + \frac{1}{2}f(x_n)\right]$$

Error:- $\approx \frac{(b-a)^3 M}{12n^2}$

$M = \max|f''(x)|$ on $[a,b]$

Simpson's Rule:-

$\Rightarrow$ Lemma:-

$$\text{Area} = \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

Then:-

If f is continuous on $[a,b]$ and $[a,b]$ is Partitioned into n even number of equal Subintervals by the Points,

$$a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b$$

So,

$$\int_a^b f(x)\,dx \simeq \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Wher Coefficient Pattern is that:-

1, 4, 2, 4, 2, 4, ... 2, 4, 2, 4, 1

Error:-

$$\text{Error} \leq \frac{M(b-a)^5}{180n^4} \Rightarrow M = \text{Max}|f^4(x)|$$

Ex # 5.5

Q (1 → 12) use trapezoidal rule.

Q No: 1:-

$$\int_1^4 \frac{dx}{x} = \ln 4 \quad \therefore n=3$$

Length of each sub Interval $\Delta x = \frac{4-1}{3} = 1$

Partionted $= [1, 2, 3, 4,] \;; [x_0, x_1, x_2, x_3]$

where $f(x_0) = 1 \qquad \because f(x) = \frac{1}{x}$

$f(x_1) = \frac{1}{2}, f(x_2) = \frac{1}{3} \;; f(x_3) = \frac{1}{4}$

By trapezoidal rule:-

$$\int_1^4 \frac{dx}{x} \approx \frac{4-1}{3}\left[\frac{1}{2}(1) + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{4}\right)\right]$$

$$\approx \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{35}{24}$$

$$\approx 1.4583 \approx \ln 4 \approx 1.3863 \quad \text{Ans}$$

Q No. 2:-

$$\int_0^{\pi/3} \cos x\, dx \quad ; n=4$$

Length of each sub-Interval $= \Delta x = \frac{\pi/3 - 0}{4} = \frac{\pi}{12}$

Partition $= \left\{0, \frac{\pi}{12}, \frac{2\pi}{12}, \frac{3\pi}{12}, \frac{\pi}{3}\right\}$

$f(x_0) = \cos 0 = 1 \qquad \Rightarrow f(x) = \cos x$

$f(x_1) = \cos\frac{\pi}{12} = 0.9659 \;, f(x_2) = \cos 2\frac{\pi}{12} = 0.8660$

$f(x_3) = \cos\frac{3\pi}{12} = 0.7071 \;, f(x_4) = \cos\frac{\pi}{3} = 0.5$

Using trapezoidal rule:-

$$\int_0^{\pi/3} \cos x\, dx \approx \frac{\pi/3 - 0}{4}\left[\frac{1}{2}(1) + 0.9659 + 0.8660 + 0.7071 + \frac{1}{2}(0.5)\right]$$

$$\approx \frac{\pi}{12}\left[3.039 + 0.25\right] = \frac{\pi}{12}(3.289)$$

$$\approx 0.8611 \quad \text{Ans}$$

Q No. 3 → Q. No. 12

Same method.

Q No. 13:- Use Simpson's rule to

i) Q No. 3:-
$$\int_0^2 e^{-x^2} dx \quad ; n=4$$

Length of each interval $= \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

Partition will be $= \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

$f(x_0) = 1$, $f(x_1) = \frac{1}{e^{1/4}} = 0.7788$, $f(x_2) = \frac{1}{e} = 0.3679$

$f(x_3) = \frac{1}{e^{9/4}} = 0.1054$, $f(x_4) = \frac{1}{e^4} = 0.01834$.

Using Simpson rule:-

$$\int_0^2 e^{-x^2} dx \approx \frac{2-0}{3\cdot 4}\left[1 + 4(0.7788) + 2(0.3679) + 4(0.1055) + 0.1059\right]$$

$$\approx \frac{2}{12}(5.2913) = \frac{1}{6}(5.2913)$$

$$\approx 0.8819 \text{ Ans}$$

ii) Q No. 11:- $\int_1^2 \ln x \, dx \quad ; n=4$

Length of each interval $= \Delta x = \frac{b-a}{n} = \frac{1}{4}$

Partition $= \{1, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4} = 2\}$

$f(x_0) = \ln 1 = 0$, $f(x_1) = 0.2231$

$f(x_2) = 0.4055$, $f(x_3) = 0.5596$, $f(x_4) = 0.6931$

By Simpson rule:-

$$\int_1^2 \ln x \, dx \approx \frac{1-0}{3\times 4}\left[0 + 4(0.2231) + 2(0.4055) + 4(0.5596) + 0.6931\right]$$

$$\approx \frac{1}{12}\left[0.8924 + 0.8110 + 2.2384 + 0.6931\right]$$

$$\approx \frac{1}{12}\left[4.6349\right]$$

$$\approx 0.3862 \text{ Ans}$$

Remaining Question is solved by same method of Previous Questions.

Q No. 14:-

Find error:-

$$\int_{-1}^{2} x^5 dx \quad ; n=10$$

$f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$

$f'''(x) = 60x^2$, $f^4(x) = 120x$

Now,

i):- $\max |f''(x)| = 20x^3$ on $[-1,2]$ at $x=2$

$$\text{So, error} \approx \frac{(b-a)^3 M}{12n^2} = \frac{(2-(-1))^3 \cdot 160}{12 \cdot 100}$$

$$\approx \frac{27 \times 160}{12 \times 100} = \frac{36}{10}$$

$$\approx 3.6 \text{ Ans}$$

ii):- For simpson's rule:-

$$M = \max |f^4(x)| \text{ on } [-1,2]$$

$$= \max |120x| \text{ at } 2$$

$$= 240$$

$$\text{error} \approx \frac{M(b-a)^5}{180n^4} = \frac{240 \times 3^5}{180 \times 10^4} = \frac{240 \times 243}{180 \times 10000}$$

$$\approx \frac{324}{10000}$$

$$\approx 0.0324 \text{ Ans}$$

Next Q No. 15 and Q No. 16 are the same pattern:-

Q No. 17:- Find area with $n=8$

$$y = \sqrt{4-x^2}$$

Required Area:-

$$A = \int_{-2}^{2} \sqrt{4-x^2}\, dx \quad , n=8$$

$$\text{Length of each subinterval} = \frac{2-(-2)}{8} = \frac{1}{2}$$

$\therefore \quad y = \sqrt{4-x^2}$

$$y^2 = 4-x^2 \Rightarrow x^2+y^2 = (2)^2 \quad \therefore a = 2$$

$$\text{Then, } \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

Partition will be $= \left\{-2, -\frac{3}{2}, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$

$f(x_0) = f(-2) = 0$, $f(x_1) = 1.32287$, $f(x_2) = 1.73210$

$f(x_3) = 1.9365$, $f(x_4) = 2$, $f(x_5) = 1.9563$

$f(x_6) = 1.7321$, $f(x_7) = 1.3229$, $f(x_8) = 0$

i):- By Trapezoidal rule:

$$A \approx \frac{2-(-2)}{8}\left[\frac{1}{2}(0)+1.3229+1.7321+1.9365+2+1.9365 + 1.7321+1.3229+\frac{1}{2}(0)\right]$$

$$\approx \frac{1}{2}[11.9830] = 5.9915$$

$$\approx 5.9915 \text{ Ans}$$

ii):- By Simpson rule:-

$$A \approx \frac{2-(-2)}{3\cdot 8}[0+4(1.3229)+2(1.7321)+4(1.9365)+ 2(2)+4(1.9365)+2(1.7321)+4(1.3229)+0]$$

$$\approx \frac{1}{6}[5.2916+3.4642+7.7460+3.4642+5.2916]$$

$$\approx \frac{37.0036}{6}$$

$$\approx 6.1673 \text{ Ans}$$

Q No. 18:-

Sol:-

The total distance travelled by the ship during 2-hour period is $\int_0^2 v(t)\,dt$; $n=8$

⇒ By trapezoidal rule:-

Length of each sub-interval $= \frac{2-0}{8} = \frac{1}{4}$

So,

$$\int_0^2 v(t)\,dt \approx \frac{2-0}{8}\left[\frac{1}{2}(19.5)+24.3+34.2+26.2 + 40.5+38.4+18+16+\frac{1}{2}(8)\right]$$

$$\approx \frac{1}{4}[9.75+24.3+34.2+40.5+38.4+26.2+18 +16+4]$$

$$\approx \frac{1}{4}(211.35)$$

$$\approx 52.8375 \approx 52.84 \text{ Ans}$$

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