Exercise Set 5.3 MathCity.org

Determine whether the following improper integrals converge. Evaluate the integrals that converge (Problems 1 – 33):

Notes of Chapter 05
Calculus with Analytic Geometry
by Ilmi Kitab Khana, Lahore.

1. $\displaystyle\int_0^{\infty} e^{-x}\, dx$

2. $\displaystyle\int_0^{\infty} e^{-x}\sin x\, dx$

3. $\displaystyle\int_{-\infty}^{0} \frac{dx}{1+x^2}$

4. $\displaystyle\int_0^{\infty} e^{-2x}\cos 2x\, dx$

5. $\displaystyle\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

6. $\displaystyle\int_{-\infty}^{2} e^{2x}\, dx$

7. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

8. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}}$

9. $\displaystyle\int_{-\infty}^{\infty} x^3\, dx$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^4+1)}\, dx$

11. $\displaystyle\int_0^{1} \frac{dx}{x}$

12. $\displaystyle\int_0^{a} \frac{dx}{x\sqrt{a^2-x^2}}$

13. $\displaystyle\int_0^{1} \frac{dx}{(x-1)^2}$

14. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

15. $\displaystyle\int_0^{e} x^2\ln x\, dx$

16. $\displaystyle\int_{-1}^{8} \frac{dx}{x^{1/3}}$

17. $\displaystyle\int_{-2}^{2} \frac{dx}{x}$

18. $\displaystyle\int_0^{3} \frac{dx}{x^2+2x-3}$

19. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$

20. $\displaystyle\int_0^{2} \frac{x}{x^2-5x+6}\, dx$

21. $\displaystyle\int_0^{\infty} \frac{\ln(1+x^2)}{1+x^2}\, dx$

22. $\displaystyle\int_0^{\infty} \frac{x\, dx}{(1+x)(1+x^2)}$

23. $\displaystyle\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx$

24. $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

25. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx$

26. $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

27. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12}$

28. $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$

29. $\displaystyle\int_{2}^{\infty} \frac{dx}{x(\ln x)^3}$

30. $\displaystyle\int_{0}^{\infty} xe^{-x}\,dx$

31. $\displaystyle\int_{1}^{\infty} \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$

32. $\displaystyle\int_{0}^{1} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

33. $\displaystyle\int_{0}^{\infty} \frac{x^3}{x^3+1}\,dx$

34. Let $I_n = \displaystyle\int_{0}^{\infty} x^n e^{-x}\,dx$, where n is a positive integer. Prove that $I_n = nI_{n-1}$. Hence show that $I_n = n\,!$.

35. Evaluate $\displaystyle\int_{1}^{5} [\,x\,]\,dx$, where $[\,x\,]$ denotes the greatest integer less than or equal to x.

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Improper Integrals :-

i) :- Integrals which have infinite Intervals of Integration.

ii) :- Integrals in which the integrands become Infinite within the intervals of integration.

These are called Improper Integrals.

Definition :-

Let f be continuous on $[a,\infty]$. The improper Integral $\int_a^\infty f(x)dx$ is defined as the limit

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

So, it is finite.

In this case, the Integral $\int_a^\infty f(x)dx$ is said to Converge. If the limit does not exist (Infinite), the Integral is said to diverge.

The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx \quad \text{--- (i)}$$

The Integral $\int_{-\infty}^{\infty} f(x)\,dx$ is said to converge only when both the integrals on the right side of eq (i) Converge, otherwise it is said to diverge.

For Example :- Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}\,dx$

Sol :- $$\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}\,dx = \int_{-\infty}^{0} \frac{x}{(x^2+3)^2}\,dx + \int_{0}^{\infty} \frac{x\,dx}{(x^2+3)^2}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{x}{(x^2+3)^2}dx = \lim_{t \to -\infty} \int_t^0 \frac{x}{(x^2+3)^2}dx$$

$$= \lim_{t \to -\infty} 1/2 \int_t^0 \frac{2x}{(x^2+3)^2}dx$$

$$= \lim_{t \to -\infty} \frac{1}{2}\left[\frac{(x^2+3)^{-2+1}}{-2+1}\right]_t^0$$

$$= \lim_{t \to -\infty} \frac{-1}{2(x^2+3)}\Big|_t^0$$

$$= \lim_{t \to -\infty} \frac{-1}{2(0+3)} - \left(\frac{-1}{2(t^2+3)}\right)$$

$$= \lim_{t \to -\infty} \frac{-1}{6} + \frac{1}{2(t^2+3)}$$

$$= \frac{-1}{6} + 0 = \frac{-1}{6}$$

$$\Rightarrow \int_0^{\infty} \frac{x}{(x^2+3)^2}dx = \lim_{t \to \infty} \int_0^t \frac{x}{(x^2+3)^2}dx$$

$$= \lim_{t \to \infty} \frac{-1}{2(x^2+3)}\Big|_0^t$$

$$= \lim_{t \to \infty} \frac{-1}{2(t^2+3)} + \frac{1}{6}$$

$$= \frac{1}{6}$$

So, $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+3)^2}dx = -\frac{1}{6} + \frac{1}{6}$

$= 0$ Ans

$\Rightarrow$ If f is continuous on $[a,b[$ and $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to b^-$, then we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

$\Rightarrow$ If f is continuous on $]a,b]$ and $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^+$, then Improper integral is defined as

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

The improper integrals converge if the limits are finite, otherwise they are said to diverge

$\Rightarrow$ let f be continuous on $[a,b]$ except some points $c \in]a,b[$ and let $f(x)$ become infinite as $x \to c^-$ or $x \to c^+$. The improper Integral is that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex # 5.3

Determine whether the following improper integral converge Evaluate (Q 1 $\to$ Q 33):-

Q No. 1:-

$$\int_0^\infty e^{-x}\,dx$$

Sol:- $\int_0^t e^{-x}\,dx = -e^{-x}\Big|_0^t = -e^{-t} - (-e^{-0}) = -e^{-t}+1$

$\Rightarrow \int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx$

$= \lim_{t\to\infty} (-e^{-t}+1) = \frac{-1}{e^\infty}+1$

$= 1$ Ans

Converges

QNo.2:- $\int_0^{\infty} e^{-x} \sin x \, dx$

$$\Rightarrow \int e^{-x} \sin x \, dx = e^{-x} \int \sin x \, dx - \int \left(\frac{d}{dx}(e^{-x}) \int \sin x \, dx \right) dx$$

$$= e^{-x}(-\cos x) - \int -\cos x \, (-e^{-x}) \, dx$$

$$= -e^{-x}\cos x - \left[e^{-x}(\sin x) - \int \sin x \, (-e^{-x}) dx \right]$$

$$= -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x} \sin x \, dx$$

$$2\int e^{-x} \sin x \, dx = -e^{-x}\cos x - e^{-x}\sin x$$

$$\int e^{-x} \sin x \, dx = -\frac{e^{-x}}{2}(\sin x + \cos x)$$

$$\Rightarrow \int_0^{\infty} e^{-x} \sin x \, dx = \lim_{t \to \infty} \int_0^t e^{-x} \sin x \, dx$$

$$= \lim_{t \to \infty} \left[-\frac{e^{-x}}{2}(\sin x + \cos x) \right]_0^t$$

$$= \lim_{t \to \infty} \left[-\frac{e^{-t}}{2}(\sin t + \cos t) + \frac{1}{2} \right]$$

$$= 0 + \frac{1}{2}$$

$$= \frac{1}{2} \text{ Ans}$$

Given Integral Converges

QNO.3:- $\int_{-\infty}^{0} \frac{dx}{1+x^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x$$

$$\Rightarrow \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{t \to -\infty} \int_t^0 \frac{dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \left[-\tan^{-1} t \right]$$

$$= -\left(\lim_{t \to -\infty} \tan^{-1} t \right)$$

$$= -\left(-\frac{\pi}{2} \right) = \frac{\pi}{2} \text{ Ans}$$

Given Integral Converges.

Q NO:4:- $\int_0^\infty e^{-2x}\cos 2x\,dx$

$$\int e^{-2x}\cos 2x\,dx = \cos 2x \int e^{-2x}dx \quad \int\left(\frac{d}{dx}(\cos 2x)\int e^{-2x}dx\right)dx$$

$$\int e^{-2x}\cos 2x\,dx = \cos 2x \cdot \frac{-e^{-2x}}{2} - \int -2\sin 2x \cdot \frac{-e^{-2x}}{2}\,dx$$

$$\int e^{-2x}\cos 2x\,dx = \frac{-e^{-2x}}{2}\cos 2x - \int \sin 2x\, e^{-2x}\,dx \quad \text{—(i)}$$

So,

$$\int \sin 2x\, e^{-2x}\,dx = \sin 2x \cdot \frac{-e^{-2x}}{2} - \int 2\cos 2x \cdot \frac{-e^{-2x}}{2}\,dx$$

$$= \sin 2x \cdot \frac{-e^{-2x}}{2} + \int e^{-2x}\cos 2x\,dx$$

By (i):-

$$\Rightarrow \int e^{-2x}\cos 2x\,dx = -\frac{e^{-2x}}{2}\cos 2x - \left[\frac{-e^{-2x}\sin 2x}{2} + \int e^{-2x}\cos 2x\,dx\right]$$

$$= \frac{-e^{-2x}\cos 2x}{2} + \frac{e^{-2x}\sin 2x}{2} - \int e^{-2x}\cos 2x\,dx$$

$$2\int e^{-2x}\cos 2x\,dx = \frac{-e^{-2x}\cos 2x}{2} + \frac{e^{-2x}\sin 2x}{2}$$

$$\int e^{-2x}\cos 2x\,dx = \frac{e^{-2x}}{4}\left[-\cos 2x + \sin 2x\right]$$

Hence,

$$\int_0^\infty e^{-2x}\cos 2x\,dx = \lim_{t\to\infty}\int_0^t e^{-2x}\cos 2x\,dx$$

$$= \lim_{t\to\infty}\left[\frac{e^{-2x}}{4}(-\cos 2x + \sin 2x)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\frac{e^{-2t}}{4}(-\cos 2t + \sin 2t) + \frac{1}{4}\right]$$

$$= 0 + \frac{1}{4} = \frac{1}{4}$$

Thus Given Integral Converges

QNo.5:- $\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$

$$\int_{-\infty}^{0} \frac{dx}{(2x-1)^3} = \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{(2x-1)^3}$$

$$= \lim_{t \to -\infty} \frac{1}{2} \int_{t}^{0} \frac{2}{(2x-1)^3} dx$$

$$= \frac{1}{2} \lim_{t \to -\infty} \left. \frac{(2x-1)^{-3+1}}{-3+1} \right|_{t}^{0}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \lim_{t \to -\infty} \left. \frac{-1}{(2x-1)^2} \right|_{t}^{0}$$

$$= \frac{1}{4} \lim_{t \to -\infty} \left. \frac{-1}{(2x-1)^2} \right|_{t}^{0}$$

$$= \frac{1}{4} \lim_{t \to -\infty} \left[-1 + \frac{1}{(2t-1)^2} \right]$$

$$= \frac{-1}{4} + \lim_{t \to -\infty} \frac{1}{(2t-1)^2}$$

$$= \frac{-1}{4} + 0 = \frac{-1}{4} \text{ Ans}$$

Thus Given Integral Converges

QNo.6:- $\int_{-\infty}^{2} e^{2x} dx$

$$\int_{-\infty}^{2} e^{2x} dx = \lim_{t \to -\infty} \int_{t}^{2} e^{2x} dx$$

$$= \lim_{t \to -\infty} \left. \frac{e^{2x}}{2} \right|_{t}^{2}$$

$$= \lim_{t \to -\infty} \left(\frac{e^4}{2} - \frac{e^{2t}}{2} \right)$$

$$= e^4/2 \text{ Ans}$$

Converge

Q No. 7:- $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{t \to -\infty} \int_{t}^{0} \frac{dx}{1+x^2} + \lim_{t \to \infty} \int_{0}^{t} \frac{dx}{1+x^2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{t \to \infty} \tan^{-1} x \Big|_{t}^{0} + \lim_{t \to \infty} \tan^{-1} x \Big|_{0}^{t}$$

$$= \lim_{t \to -\infty} 0 - \tan^{-1} t + \lim_{t \to \infty} \tan^{-1} t - 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{2} = \pi \text{ Ans.}$$

Thus Integral Converges.

Q No. 8:-

$$\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}}$$

$$\int_{-\infty}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}} = \int_{-\infty}^{0} \frac{x\,dx}{\sqrt{x^2+2}} + \int_{0}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}}$$

Now, $\int_{-\infty}^{0} \frac{x\,dx}{\sqrt{x^2+2}} = \lim_{t \to -\infty} \int_{t}^{0} \frac{x\,dx}{\sqrt{x^2+2}}$

$$= \lim_{t \to -\infty} \frac{1}{2} \int_{t}^{0} \frac{2x}{\sqrt{x^2+2}}\,dx$$

$$= \lim_{t \to -\infty} \frac{1}{2} \frac{\sqrt{x^2+2}}{1/2} \Big|_{t}^{0}$$

$$= \lim_{t \to -\infty} \left(\sqrt{2} - \sqrt{t^2+2}\right)$$

$$= -\infty = \int_{0}^{\infty} \frac{x\,dx}{\sqrt{x^2+2}}$$

Hence,

Given Integral diverges.

QNo.9:- $\int_{-\infty}^{\infty} x^3 \, dx$

$$\int_{-\infty}^{\infty} x^3 dx = \int_{-\infty}^{0} x^3 dx + \int_{0}^{\infty} x^3 dx$$

$$\int_{-\infty}^{\infty} x^3 dx = \lim_{t\to -\infty} \int_{t}^{0} x^3 dx + \lim_{t\to\infty} \int_{0}^{t} x^3 dx$$

$$= \lim_{t\to -\infty} \frac{x^4}{4}\Big|_{t}^{0} + \lim_{t\to\infty} \frac{x^4}{4}\Big|_{0}^{t}$$

$$\int_{-\infty}^{\infty} x^3 dx = -\lim_{t\to -\infty} \frac{t^4}{4} + \lim_{t\to\infty} \frac{t^4}{4}$$

$\Rightarrow$ which Gives $-\infty$ and ∞

So, Given Integral diverges.

QNo.10:- $\int_{-\infty}^{\infty} \frac{x}{(x^4+1)} \, dx$

$\int \frac{x}{1+x^4} dx \Rightarrow$ Put $x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{1}{2} dz$

$$\int \frac{x}{x^4+1} dx = \int \frac{1/2 \, dz}{1+z^2}$$

$$= \frac{1}{2}\int \frac{dz}{1+z^2} = \frac{1}{2}\tan^{-1} z$$

$$= \frac{1}{2}\tan^{-1} x^2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x}{(x^4+1)} dx = \int_{-\infty}^{0} \frac{x}{(x^4+1)} dx + \int_{0}^{\infty} \frac{x}{(x^4+1)} dx$$

$$= \lim_{t\to -\infty} \left(\frac{1}{2}\tan^{-1}x^2\right)\Big|_{t}^{0} + \lim_{t\to\infty} \left(\frac{1}{2}\tan^{-1}x^2\right)\Big|_{0}^{t}$$

$$= \lim_{t\to -\infty} \left[0 - \frac{1}{2}\tan^{-1}t^2\right] + \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}t^2\right]$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) = 0 \text{ Ans}$$

Thus Given Integral Converges.

Q No. 11: $\int_0^1 \frac{dx}{x}$

$$\int_0^1 \frac{dx}{x} = \lim_{t \to 0^+} \int_t^1 \frac{dx}{x}$$

$$= \lim_{t \to 0^+} [\ln x]_t^1$$

$$\int_0^1 \frac{dx}{x} = \lim_{t \to 0^+} [\ln 1 - \ln t]$$

$$= \lim_{t \to 0^+} (0 - \ln t) = \lim_{t \to 0^+} (-\ln t)$$

$$\int_0^1 dx/x = \infty$$

Thus, Given Integral diverges

Q No. 12: $\int_0^a \frac{dx}{x\sqrt{a^2-x^2}}$

$\int \frac{dx}{x\sqrt{a^2-x^2}}$ $\Rightarrow$ Put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{a\sin\theta\sqrt{a^2-(a\sin\theta)^2}}$$

$$= \int \frac{\cos\theta\, d\theta}{a\sin\theta \cdot \sqrt{\cos^2\theta}} = \int \frac{1}{a\sin\theta}\, d\theta$$

$$= \frac{1}{a}\int \csc\theta\, d\theta = \frac{1}{a}\ln|\csc\theta - \cot\theta|$$

So, $\sin\theta = \frac{x}{a} \Rightarrow \csc\theta = \frac{a}{x}$

$\cos\theta = \sqrt{1-\sin^2\theta} = \frac{\sqrt{a^2-x^2}}{a}$

Then, $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{a^2-x^2}/\cancel{a}}{x/\cancel{a}}$

$$= \frac{\sqrt{a^2-x^2}}{x}$$

So, $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} - \ln\left|\frac{\sqrt{a^2-x^2}}{x} - \frac{a}{x}\right|$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a} \ln\left| \frac{a-\sqrt{a^2-x^2}}{x} \right|$$

$$\Rightarrow \int_0^a \frac{dx}{x\sqrt{a^2-x^2}} = \lim_{t \to 0^+} \int_t^a \frac{dx}{x\sqrt{a^2-x^2}}$$

$$= \lim_{t \to 0^+} \frac{1}{a} \ln\left| \frac{a-\sqrt{a^2-x^2}}{x} \right|_t^a$$

$$= \lim_{t \to 0^+} \frac{1}{a} \left\{ 0 - \ln\left(\frac{a-\sqrt{a^2-t^2}}{t} \right) \right\}$$

$$= \frac{-1}{a} \lim_{t \to 0^+} \left(\ln\left(\frac{a-\sqrt{a^2-t^2}}{t} \right) \right)$$

$$= \frac{-1}{a} \ln\left(\frac{a-\sqrt{a^2-0^2}}{0} \right)$$

$$= \frac{-1}{a} \ln 0 = \infty \text{ Ans}$$

Thus Given integral diverges

QNo. 13:- $\int_0^1 \frac{dx}{(x-1)^2}$

$$\int_0^1 \frac{dx}{(x-1)^2} = \lim_{t \to 1^-} \int_0^t \frac{dx}{(x-1)^2}$$

$$= \lim_{t \to 1} \left[\frac{(x-1)^{-2+1}}{-2+1} \right]_0^t$$

$$= \lim_{t \to 1^-} \left[\frac{-1}{(x-1)} \right]_0^t$$

$$= \lim_{t \to 1} \left[\left(\frac{-1}{t-1} \right) - 1 \right]$$

$$= \infty$$

Hence Given Integral diverges.

Written by Abrar Mustafa

Q No. 14:- $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}}$$

$$= \lim_{t\to 1^-} \left[\sin^{-1}x\right]_0^t$$

$$= \lim_{t\to 1^-} \sin^{-1}t - \sin^{-1}0$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}1 - 0 = \frac{\pi}{2} \text{ Ans}$$

Thus, Given integral Converges

Q No. 15:- $\int_0^e x^2 \ln x\,dx$

$$\int (\ln x)x^2 dx = \ln x \cdot \frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$$

$$= \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3}$$

$$\Rightarrow \int_0^e x^2 \ln x\,dx = \lim_{t\to 0^+}\int_t^e x^2 \ln x\,dx$$

$$= \lim_{t\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_t^e$$

$$= \lim_{t\to 0^+}\left[\left(\frac{e^3}{3}\ln e - \frac{e^3}{9}\right) - \left(\frac{t^3}{3}\ln t - \frac{t^3}{9}\right)\right]$$

$$\int_0^e x^2 \ln x\,dx = \lim_{t\to 0^+}\left[\frac{e^3}{3}\ln e - \frac{e^3}{9} - \frac{t^3}{3}\ln t + \frac{t^3}{9}\right]$$

$$= 2e^3/9 - \lim_{t\to 0^+}\left(\frac{t^3}{3}\ln t\right) + 0$$

Since, $\lim_{t \to 0^+} \frac{\ln t}{1/t^3}$

By L. Hospital rule

$$= \lim_{t \to 0^+} \frac{1/t}{-3 \cdot \frac{1}{t^4}} = \lim_{t \to 0^+} \frac{t^3}{-3} = 0$$

Thus, $\int_0^e x^2 \ln x \, dx = \frac{2e^3}{9} - 0$ to

$$= \frac{2e^3}{9} \text{ Ans}$$

Given Integral Converges.

QNO. 16:- $\int_{-1}^{8} \frac{dx}{x^{1/3}}$

$$\int_{-1}^{8} \frac{dx}{x^{1/3}} = \int_{-1}^{0} \frac{dx}{x^{1/3}} + \int_{0}^{8} \frac{dx}{x^{1/3}}$$

$$= \lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^{1/3}} + \lim_{t \to 0^+} \int_{t}^{8} \frac{dx}{x^{1/3}}$$

$$= \lim_{t \to 0^-} \frac{x^{-1/3+1}}{-\frac{1}{3}+1} \Big|_{-1}^{t} + \lim_{t \to 0^+} \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \Big|_{t}^{8}$$

$$= \lim_{t \to 0^-} \frac{3}{2} x^{2/3} \Big|_{-1}^{t} + \lim_{t \to 0^+} \frac{3}{2} x^{2/3} \Big|_{t}^{8}$$

$$= \lim_{t \to 0^-} \left[\frac{3}{2} t^{2/3} - \frac{3}{2} (-1)^{2/3}\right] + \lim_{t \to 0^+} \left[\frac{3}{2} (8)^{2/3} - \frac{3}{2} t^{2/3}\right]$$

$$= 0 - \frac{3}{2} + \frac{3}{2} \cdot \frac{1}{2} \cdot 8 = -\frac{3}{2} + 6 = \frac{9}{2} \text{ Ans}$$

QNo. 17:- $\int_{-2}^{2} \frac{dx}{x}$

$$\int_{-2}^{2} \frac{dx}{x} = \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{2} \frac{dx}{x}$$

$$= \lim_{t \to 0^-} [\ln|x|]_{-2}^{t} + \lim_{t \to 0^+} [\ln|x|]_{t}^{2}$$

QNO. 17:- $\int_{-2}^{2} \frac{dx}{x}$

Sol:-
$$\int_{-2}^{2} \frac{dx}{x} = \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{2} \frac{dx}{x}$$

$$= \lim_{t \to 0^-} \int_{-2}^{t} \frac{dx}{x} + \lim_{t \to 0^+} \int_{t}^{2} \frac{dx}{x}$$

$$= \lim_{t \to 0^-} [\ln|x|]_{-2}^{t} + \lim_{t \to 0^+} [\ln|x|]_{t}^{2}$$

$$= \lim_{t \to 0^-} [\ln|t| - \ln 2] + \lim_{t \to 0^+} [\ln 2 - \ln|t|]$$

$$\int_{-2}^{2} \frac{dx}{x} = \lim_{t \to 0^-} \ln|t| - \lim_{t \to 0^+} \ln|t| + (\ln 2 - \ln 2)$$

$$= \lim_{t \to 0^-} \ln|t| - \lim_{t \to 0^+} \ln|t|$$

$$\int_{-2}^{2} \frac{dx}{x} = \ln 0 - \ln 0 = \infty$$

Thus, Given Integral diverges

QNO. 18:- $\int_{0}^{3} \frac{dx}{x^2+2x-3}$

$$\int_{0}^{3} \frac{dx}{x^2+2x-3} = \int_{0}^{3} \frac{dx}{x^2+3x-x-3} = \int_{0}^{3} \frac{dx}{x(x+3)-1(x+3)}$$

$$= \int_{0}^{3} \frac{dx}{(x+3)(x-1)} = \int_{0}^{1} \frac{dx}{(x+3)(x-1)} + \int_{1}^{3} \frac{dx}{(x+3)(x-1)}$$

Now, $\int_{0}^{1} \frac{dx}{(x+3)(x-1)}$

$$\int \frac{dx}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} \quad \text{(i)}$$

$$1 = A(x-1) + B(x+3) \quad \text{(ii)}$$

Put $x=-3$ in (ii)

$1 = A(-3-1) + 0$

$\Rightarrow A = -\frac{1}{4}$

Put $x=1$ in (ii)

$1 = 0 + B(1+3)$

$\Rightarrow B = \frac{1}{4}$

Hence,

$$\int_0^1 \frac{dx}{(x+3)(x-1)} = \lim_{t\to 1}\int_0^t \left(\frac{-1}{4(x+3)} + \frac{1}{4(x-1)}\right)dx$$

$$= \lim_{t\to 1}\int_0^t \frac{dx}{4(x-1)} - \lim_{t\to 1}\int_0^t \frac{dx}{4(x+3)}$$

$$= \lim_{t\to 1}\left(\frac{1}{4}\ln|x-1|\right)_0^t - \lim_{t\to 1}\left(\frac{1}{4}\ln|x+3|\right)_0^t$$

$$= \lim_{t\to 1}\left(\frac{1}{4}\ln|1-t|\right) - \lim_{t\to 1}\left(\frac{1}{4}\ln|t+3| - \frac{1}{4}\ln 3\right)$$

$$= \lim_{t\to 1}\left(\frac{1}{4}\ln|1-t|\right) - \frac{1}{4}(\ln 4 - \ln 3)$$

$$= \lim_{t\to 1}\left(\frac{1}{4}\ln|1-t|\right) - \frac{1}{4}\ln\frac{4}{3}$$

$$= \frac{1}{4}\ln 0 - \frac{1}{4}\ln\frac{4}{3}$$

$= \infty$ which is not finite

Hence, $\int_0^3 \frac{dx}{x^2+2x-3}$ diverges

QNo. 19 :- $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx = \lim_{t\to(\pi/2)^-} -\int_0^t (1-\sin x)^{-\frac{1}{2}}(-\cos x)\,dx$$

$$= \lim_{t\to(\frac{\pi}{2})^-} \left[-\frac{(1-\sin x)^{1/2}}{1/2}\right]_0^t$$

$$= \lim_{t\to(\frac{\pi}{2})^-} \left[-2\sqrt{1-\sin x}\right]_0^t$$

$$= \lim_{t\to(\frac{\pi}{2})^-} \left[-2\sqrt{1-\sin t}+2\right]$$

$$\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1-\sin x}} = 0+2 = 2 \text{ Ans}$$

Given Integral Converges

QNo. 20:- $\int_0^2 \frac{x}{x^2-5x+6}\,dx$

$$\int_0^2 \frac{x\,dx}{x^2-5x+6} \Rightarrow \int_0^2 \frac{x\,dx}{x^2-3x-2x+6}$$

$$= \int_0^2 \frac{x\,dx}{x(x-3)-2(x-3)} = \int_0^2 \frac{x\,dx}{(x-3)(x-2)}$$

$$\Rightarrow \int \frac{x\,dx}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2} \quad \text{(i)}$$

$$x = A(x-2)+B(x-3) \quad \text{(ii)}$$

Put $x=3$ in (ii)

$$3 = A(3-2)+0$$

$$\Rightarrow A = 3$$

Put $x=2$ in (ii)

$$2 = 0+B(2-3)$$

$$\Rightarrow B = -2$$

So, $\int_0^2 \frac{x}{x^2-5x+6}\,dx = \int_0^2 \frac{3}{x-3}\,dx - \int_0^2 \frac{2}{x-2}\,dx$

$$\text{Now, } 2\int_0^2 \frac{dx}{x-2} = 2\lim_{t\to 2^-} \int_0^t \frac{dx}{x-2}$$

$$2\int_0^2 \frac{dx}{x-2} = 2\lim_{t\to 2^-} \left[\ln|x-2|\right]_0^t$$

$$= 2\lim_{t\to 2} \left(\ln|(t-2)| - \ln 2\right)$$

$$2\int_0^2 \frac{dx}{x-2} = 2\lim_{t\to 2^-} \ln\left(\frac{2-t}{2}\right)$$

$$= 2\lim_{t\to 2^-} \ln\left(1-\frac{t}{2}\right)$$

$$2\int_0^2 \frac{dx}{x-2} = 2\ln\left(1-\frac{2}{2}\right) = 2\ln 0 = \infty$$

which is infinite,

Thus,

$$\int_0^2 \frac{dx}{x^2-5x+6}\, dx \quad \text{diverges.}$$

QNo.21:-

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\, dx$$

$$I = \int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\, dx = \lim_{t\to\infty} \int_0^t \frac{\ln(1+x^2)}{(1+x^2)}\, dx$$

Put $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

when $x \to 0$, $\theta \to 0$

$x \to t$, $\theta \to \tan^{-1} t$

$$\text{Now, } \int_0^t \frac{\ln(1+x^2)\,dx}{(1+x^2)} = \int_0^{\tan^{-1}t} \frac{\ln\sec^2\theta}{\sec^2\theta}\, \sec^2\theta\, d\theta$$

$$I = \lim_{t\to\infty}\int_0^t \frac{\ln(1+x^2)}{(1+x^2)} = \lim_{t\to\infty}\int_0^{\tan^{-1}t} \frac{1}{\ln\cos^2\theta}\, d\theta = (-2)\lim_{t\to\infty}\int_0^{\pi/2} \ln(\cos\theta)\, d\theta$$

$$= -2\int_0^{\pi/2} \ln(\cos\theta)\, d\theta = -2\left(-\frac{\pi}{2}\ln 2\right)$$

$$I = \pi\ln 2 \quad \text{Ans} \quad \text{It's Converges}$$

Q No. 22:- $\int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)}$

$$I = \int_0^\infty \frac{x\,dx}{(1+x)(1+x^2)} = \lim_{t\to\infty}\int_0^t \frac{x\,dx}{(1+x)(1+x^2)}$$

Put $x = \operatorname{Tan}\theta$, $dx = \sec^2\theta\,d\theta$

when $x \to 0$, $\theta \to 0$

$x \to t$, $\theta \to \operatorname{Tan}^{-1} t$

$$\text{Now, } \int_0^t \frac{x\,dx}{(1+x)(1+x^2)} = \int_0^{\operatorname{Tan}^{-1}t} \frac{\operatorname{Tan}\theta}{(1+\operatorname{Tan}\theta)\sec^2\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\operatorname{Tan}^{-1}t} \frac{\operatorname{Tan}\theta}{1+\operatorname{Tan}\theta}\,d\theta = \int_0^{\operatorname{Tan}^{-1}t} \frac{\operatorname{Sin}\theta/\operatorname{Cos}\theta}{\frac{\operatorname{Cos}\theta+\operatorname{Sin}\theta}{\operatorname{Cos}\theta}}\,d\theta$$

$$= \int_0^{\operatorname{Tan}^{-1}t} \frac{\operatorname{Sin}\theta}{\operatorname{Sin}\theta+\operatorname{Cos}\theta}\,d\theta$$

$$I = \lim_{t\to\infty}\int_0^{\operatorname{Tan}^{-1}t} \frac{\operatorname{Sin}\theta}{\operatorname{Sin}\theta+\operatorname{Cos}\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\operatorname{Sin}\theta}{\operatorname{Sin}\theta+\operatorname{Cos}\theta}\,d\theta = \int_0^{\pi/2} \frac{\operatorname{Sin}(\pi/2-\theta)}{\operatorname{Sin}(\pi/2-\theta)+\operatorname{Cos}(\frac{\pi}{2}-\theta)}\,d\theta$$

$$I = \int_0^{\pi/2} \frac{\operatorname{Cos}\theta}{\operatorname{Sin}\theta+\operatorname{Cos}\theta}\,d\theta$$

$$I + I = \int_0^{\pi/2} \frac{\operatorname{Sin}\theta+\operatorname{Cos}\theta}{\operatorname{Sin}\theta+\operatorname{Cos}\theta}\,d\theta$$

$$2I = \int_0^{\pi/2} 1\,d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2}$$

$I = \pi/4$ Ans.

Thus Given Integral converges.

Q No. 23:- $\int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx$

$$I = \int_{-\infty}^{0} \frac{e^x}{1+e^x}\,dx = \lim_{t \to -\infty} \int_{t}^{0} \frac{e^x}{1+e^x}\,dx$$

Put $e^x = z$, $e^x dx = dz$

when, $x \to 0$, $z \to 1$

$x \to t$, $z \to e^t$

Now $\int_{t}^{0} \frac{e^x}{1+e^x}\,dx = \int_{e^t}^{1} \frac{dz}{1+z} = \left[\ln|1+z|\right]_{e^t}^{1}$

$$= \ln 2 - \ln|1+e^t|$$

$$\lim_{t \to -\infty} \int_{t}^{0} \frac{e^x}{1+e^x}\,dx = \lim_{t \to -\infty} \left[\ln 2 - \ln|1+e^t|\right]$$

$$= \ln 2 - \ln(1+e^{-\infty})$$

$$= \ln 2 - 0$$

$$\lim_{t \to -\infty} \int_{t}^{0} \frac{e^x}{1+e^x}\,dx = \ln 2 \text{ Ans}$$

Q No. 24:- $\int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

$$I = \int_{0}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

$$I = \frac{1}{a^2-b^2}\left[\int_{0}^{\infty} \frac{a^2-b^2}{(x^2+a^2)(x^2+b^2)}\,dx\right]$$

$$I = \frac{1}{a^2-b^2}\left[\int_{0}^{\infty} \frac{x^2+a^2-x^2-b^2\,dx}{(x^2+a^2)(x^2+b^2)}\right]$$

$$= \frac{1}{a^2-b^2}\left[\int_{0}^{\infty} \frac{(x^2+a^2)\,dx}{(x^2+a^2)(x^2+b^2)} - \int_{0}^{\infty} \frac{(x^2+b^2)\,dx}{(x^2+a^2)(x^2+b^2)}\right]$$

$$= \frac{1}{a^2-b^2}\left[\int_{0}^{\infty} \frac{dx}{x^2+b^2} - \int_{0}^{\infty} \frac{dx}{x^2+a^2}\right]$$

Let $I_1 = \int_0^\infty \frac{dx}{x^2+b^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+b^2}$

Put $x = b\,\mathrm{Tan}\,\theta$, $dx = b\sec^2\theta\, d\theta$

When $x \to 0$, $\theta \to 0$

$x \to t$, $\theta \to \mathrm{Tan}^{-1}\frac{t}{b}$

Now $\int_0^t \frac{dx}{x^2+b^2} = \int_0^{\mathrm{Tan}^{-1}\frac{t}{b}} \frac{b\sec^2\theta\, d\theta}{b^2(1+\mathrm{Tan}^2\theta)}$

$= \int_0^{\mathrm{Tan}^{-1}\frac{t}{b}} \frac{b\sec^2\theta}{b^2\sec^2\theta}\, d\theta = \int_0^{\mathrm{Tan}^{-1}\frac{t}{b}} \frac{1}{b}\, d\theta$

$= \frac{1}{b}[\theta]_0^{\mathrm{Tan}^{-1}\frac{t}{b}} = \frac{1}{b}\left(\mathrm{Tan}^{-1}\frac{t}{b} - 0\right)$

$\lim_{t\to\infty} \int_0^t \frac{dx}{x^2+b^2} = \frac{1}{b} \lim_{t\to\infty} \left(\mathrm{Tan}^{-1}\frac{t}{b}\right)$

$= \frac{1}{b} \cdot \mathrm{Tan}^{-1}\frac{\infty}{b} = \frac{1}{b}\cdot\frac{\pi}{2} = \frac{\pi}{2b}$

$I_1 = \frac{\pi}{2b}$

Similarly $I_2 = \frac{\pi}{2a}$

So,

$I = \frac{1}{a^2-b^2}\left[\frac{\pi}{2b} - \frac{\pi}{2a}\right] = \frac{1}{a^2-b^2}\cdot\frac{\pi}{2}\left(\frac{a-b}{ab}\right)$

$I = \frac{\pi}{2ab(a+b)}$ Ans

Thus, Given Integral Converges

QNo.25:- $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\, dx$

Sol:- $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\, dx$

$= \lim_{t\to\frac{\pi}{2}} \left[- \int_{\pi/4}^{t} (\cos x)^{-\frac{1}{2}} (-\sin x)\, dx\right]$

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx = \lim_{t \to \frac{\pi}{2}} -\int_{t}^{\pi/4} (\cos x)^{-\frac{1}{2}} (-\sin x)\,dx$$

$$= \lim_{t \to \frac{\pi}{2}} \left[\frac{(\cos x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_t^{\pi/4}$$

$$= \lim_{t \to \frac{\pi}{2}} \left[2\sqrt{\cos x} \right]_t^{\pi/4}$$

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx = \lim_{t \to \frac{\pi}{2}} \left[2\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} - 2\sqrt{\cos t} \right]$$

$$= 2\frac{1}{(2)^{\frac{1}{4}}} - 0 = 2^{1-\frac{1}{4}}$$

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}}\,dx = 2^{3/4} \quad \text{Ans}$$

Thus, Given Integral Converges

QNO. 27:- $\int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12}$

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12} = \int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12} = \int_{-\infty}^{\infty} \frac{dx}{x^2+2(3)(x)+(3)^2+12-9}$$

$$= \int_{-\infty}^{\infty} \frac{dx}{(x+3)^2+(\sqrt{3})^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+6x+12} = \int_{-\infty}^{0} \frac{dx}{(x+3)^2+(\sqrt{3})^2} + \int_{0}^{\infty} \frac{dx}{(x+3)^2+(\sqrt{3})^2}$$

Now, $\int \frac{dx}{(x+3)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right)$

$$I = \frac{1}{\sqrt{3}} \left[\lim_{t \to -\infty} \left(\tan^{-1} \frac{x+3}{\sqrt{3}} \right)_t^0 + \lim_{t \to \infty} \left(\tan^{-1} \frac{x+3}{\sqrt{3}} \right)_0^t \right]$$

$$= \frac{1}{\sqrt{3}} \left[\lim_{t \to -\infty} \left(\tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{t+3}{\sqrt{3}}\right) \right) + \lim_{t \to \infty} \left(\tan^{-1}\left(\frac{t+3}{\sqrt{3}}\right) - \tan^{-1}\sqrt{3} \right) \right]$$

$$I = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\pi}{3}\right]$$

$$I = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{3}\right]$$

$$I = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{2}\right] = \frac{1}{\sqrt{3}}(\pi)$$

$$I = \frac{\pi}{\sqrt{3}} \text{ Ans}$$

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Thus, Given Integral Converges.

QNo. 28:– $\int_{-1}^{1} \frac{dx}{x^2}$

$$I = \int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$$

$$= \lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^2} + \lim_{t \to 0^+} \int_{t}^{1} \frac{dx}{x^2}$$

$$I = \lim_{t \to 0^-} \left[\frac{-1}{x}\right]_{-1}^{t} + \lim_{t \to 0^+} \left[-\frac{1}{x}\right]_{t}^{1}$$

$$= \lim_{t \to 0^-} \left[-\frac{1}{t} - 1\right] + \lim_{t \to 0^+} \left[-1 + \frac{1}{t}\right]$$

$$I = \infty$$

Thus, Given Integral diverges

QNo. 29:– $\int_{2}^{\infty} \frac{dx}{x(\ln x)^3}$

$$I = \int_{2}^{\infty} \frac{dx}{x(\ln x)^3} = \lim_{t \to \infty} \int_{2}^{t} \frac{dx}{x(\ln x)^3}$$

$$= \lim_{t \to \infty} \left[\frac{-1}{2(\ln x)^2}\right]_{2}^{t}$$

$$I = \lim_{t \to \infty} \left[\frac{-2}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2} \right]$$

$$I = 0 + \frac{1}{2(\ln 2)^2} = \frac{1}{2(\ln 2)^2} \text{ Ans}$$

Thus,

Given Integral Converges

Q No. 30:- $\int_0^\infty x e^{-x} dx$

$$I = \int_0^\infty x e^{-x} dx = \lim_{t \to \infty} \int_0^t x e^{-x} dx$$

$$I = \lim_{t \to \infty} \left[\left(x \cdot \frac{e^{-x}}{-1} \right)_0^t - \int_0^t 1 \cdot \frac{e^{-x}}{-1} dx \right]$$

$$I = \lim_{t \to \infty} \left[\left(-x e^{-x} \right)_0^t + \int_0^t e^{-x} dx \right]$$

$$I = \lim_{t \to \infty} \left[-x e^{-x} - e^{-x} \right]_0^t$$

$$I = \lim_{t \to \infty} \left[-t e^{-t} - e^{-t} - (0 - 1) \right]$$

$$I = \lim_{t \to \infty} \left[-(t+1) e^{-t} + 1 \right]$$

$$I = \lim_{t \to \infty} \left[-\frac{(t+1)}{e^t} + 1 \right]$$

$$I = \lim_{t \to \infty} \left[-\frac{(t+1)}{e^t} + 1 \right]$$

$$I = 0 + 1 = 0 \text{ Ans}$$

Thus,

Given Integral Converges

Q.No. 31:- $\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

Sol:-

$$\Rightarrow \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

Put $\sqrt{x} = z$

$dx = z\,2\,dz$

$$\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \int \frac{2z\,dz}{z(z+1)}$$

$$= 2\int \frac{dz}{z+1} = 2\ln|1+z|$$

$$= 2\ln|\sqrt{x}+1|$$

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{t\to\infty} \left[2\ln|\sqrt{x}+1|\right]_1^t$$

$$= \lim_{t\to\infty} \left(2\ln(\sqrt{t}+1) - 2\ln 2\right)$$

$$= \lim_{t\to\infty} \left[2\ln\left(\frac{\sqrt{t}+1}{2}\right)\right]$$

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \infty \text{ Ans}$$

Thus, Given Integral diverge.

Q.No. 32:- $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$$I = \int_0^1 e^{\sqrt{x}} \cdot \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} 2\int_t^1 e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$$

$$I = \lim_{t\to 0^+} \left[2e^{\sqrt{x}}\right]_t^1 = \lim_{t\to 0^+} \left(2e - 2e^{\sqrt{t}}\right)$$

$$I = 2e - 2 = 2(e-1) \text{ Ans}$$

Thus,

Given Integral Converges

Q No. 33 :- $\int_0 \frac{x^3}{x^3+1}\,dx$

$$I = \int_0^\infty \frac{x^3}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^3}{x^3+1}\,dx \qquad x^3+1\overline{)\,x^3\,}\ 1,\quad \cancel{x^3}+1,\quad -1$$

Now,

$$\int \frac{x^3}{x^3+1}\,dx = \int\left(1-\frac{1}{x^3+1}\right)dx$$

$$= \int\left(1-\frac{1}{(x+1)(x^2-x+1)}\right)dx \quad \text{—— (i)}$$

$$\Rightarrow \frac{1}{(x+1)(x^2+x+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1} \quad \text{—— (ii)}$$

$$1 = A(x^2-x+1) + Bx+C\,(x+1) \quad \text{—— (iii)}$$

Put $x=-1$ in (iii)

$$1 = A\left((-1)^2-(-1)+1\right)+0$$

$$\Rightarrow A = \frac{1}{3}$$

By (iii) :-

$$1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$\Rightarrow A+B = 0$$

$$B = -A = -\frac{1}{3}$$

$$\Rightarrow -A+B+C = 0$$

$$-\frac{1}{3}-\frac{1}{3}+C = 0$$

$$C = \frac{2}{3}$$

By (ii)

$$\frac{1}{(x+1)(x^2+x+1)} = \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}$$

$$\frac{1}{(x+1)(x^2+x+1)} = \frac{1}{3(x+1)} - \frac{x-2}{3(x^2-x+1)}$$

By (i) We get :-

$$\int\frac{x^3}{x^3+1}\,dx = \int\left(1-\frac{1}{3(x+1)}+\frac{x-2}{3(x^2-x+1)}\right)dx$$

$$\int \frac{x^3}{x^3+1}\,dx = x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}\,dx$$

So, $\Rightarrow \int \frac{2x-1-3}{x^2-x+1}\,dx = \int \frac{2x-1}{x^2-x+1}\,dx - \int \frac{3\,dx}{x^2-x+1+\frac{1}{4}-\frac{1}{4}}$

$$= \ln(x^2-x+1) - \int \frac{3\,dx}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \ln(x^2-x+1) - \frac{3\times 2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

Hence,

$$\int \frac{x^3}{x^3+1}\,dx = x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

$$I = \lim_{t\to\infty}\left[x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]_0^t$$

So that limit does not exist.

Hence, the given integral diverges

QNo. 34:-

$$I_n = \int_0^\infty x^n e^{-x}\,dx \quad : n \pm ve$$

Prove that, $I_n = n I_{n-1}$ Show, $I_n = n!$

Sol:- $I_n = \int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^n e^{-x}\,dx$

Now,

$$\int x^n e^{-x}\,dx = x^n \cdot -e^{-x} - \int (-e^{-x})\cdot n x^{n-1}\,dx$$

$$= -x^n e^{-x} + n\int e^{-x} x^{n-1}\,dx$$

Now, $I_n = \lim_{t\to\infty}\left[-\frac{x^n}{e^x} + n\int_0^t x^{n-1} e^{-x}\,dx\right]$

$$= \lim_{t\to\infty}\left[-\frac{x^n}{e^x}\right]_0^t + n\lim_{t\to\infty}\int_0^t e^{-x} x^{n-1}\,dx$$

$$I_n = \lim_{t\to\infty}\left[\frac{-t^n}{e^t}\right] + n\int_0^\infty x^{n-1} e^{-x}\,dx$$

BY L'Hospital rule:-

$$\Rightarrow \lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] = -\lim_{t\to\infty}\left(\frac{n(n-1)(n-2)\cdots 1}{e^t}\right) = 0$$

Thus, $I_n = 0 + nI_{n-1} = nI_{n-1}$

Hence Proved:-

$$I_n = nI_{n-1}$$

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So, $I_{n-1} = (n-1)I_{n-2}$

$I_{n-2} = (n-2)I_{n-3}$

$I_{n-3} = (n-3)I_{n-4}$

$\vdots \quad \vdots \quad \vdots$

$I_3 = 3\,I_2$

$I_2 = 2\,I_1$

$I_1 = 1\,I_0$

$\Rightarrow I_0 = \int_0^\infty x^0 e^{-x}\,dx = \int_0^\infty e^{-x}\,dx$

$= \lim_{t\to\infty}[-e^{-x}]_0^t$

$I_0 = \lim_{t\to\infty}\left[\frac{-1}{e^t}+1\right] = 0+1$

$I_0 = 1$

Thus,

$I_n = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$

$I_n = n!$ Ans

Q No. 35:- Evaluate $\int_1^5 [x]\,dx$

By definition of the bracket function:-

$$[x] = \begin{cases} 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ 3 & \text{if } 3 \le x < 4 \\ 4 & \text{if } 4 \le x < 5 \\ 5 & \text{if } 5 \le x = 5 \end{cases}$$

Therefore,

$$I = \int_1^2 1 \cdot dx + \int_2^3 2\,dx + \int_3^4 3\,dx + \int_4^5 4\,dx$$

$$= [x]_1^2 + [2x]_2^3 + [3x]_3^4 + [4x]_4^5$$

$$I = (2-1) + (6-4) + (12-9) + (20-16)$$

$$= 1 + 2 + 3 + 4$$

$$I = 3+3+4 = 6+4$$

$$I = 10 \text{ Ans}$$

Reduction formula:-

As we proved in 4th chepter:-

$$\int \sin^n x\,dx = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} \int \tan^{n-2} x\,dx$$

$$\int \cot^n x\,dx = \frac{-\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$$

$$\int \csc^n x\,dx = \frac{-\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$\Rightarrow$ Reductin formula for $\int \sin^p x \cos^q x\,dx$

As we know that, there are two methods. I Connect method

II Integration by parts

We use II method that easily Soluable-

$$\Rightarrow \int \sin^p x \cos^q x \, dx = \int \sin^{p-1} x \cos^q x \sin x \, dx$$

$$I = \int \sin^p x \cos^q x \, dx = -\int \sin^{p-1} x \cos^q x (-\sin x) \, dx$$

we use $u = \sin^{p-1} x$; $v = \cos^q x (-\sin x)$

$$\text{So, } I = -\left\{ \sin^{p-1} x \frac{\cos^{q+1} x}{q+1} - \int (p-1) \sin^{p-2} x \cdot \cos x \cdot \frac{\cos^{q+1} x}{q+1} dx \right\}$$

$$I = -\left\{ \frac{\sin^{p-1} x \cos^{q+1} x}{q+1} - \int \frac{p-1}{q+1} \sin^{p-2} x \cos^{q+2} x \, dx \right\}$$

$$I = \frac{-\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^q x (1 - \sin^2 x) dx$$

$$I = \frac{-\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int (\sin^{p-2} x \cos^q x - \sin^p x \cos^q x) dx$$

$$I = \frac{-\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^q x \, dx - \frac{p-1}{q+1} I$$

$$I + \frac{p-1}{q+1} I = \frac{-\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1} \int \sin^{p-2} x \cos^q x \, dx$$

$$(q + 1 + p - 1) I = -\sin^{p-1} x \cos^{q+1} x + p - 1 \int \sin^{p-2} x \cos^q x \, dx$$

$$I = \frac{-\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx$$

Ans

Evaluate:-

$$I_{p,q} = \int_0^{\pi/2} \sin^p x \cos^q x \, dx$$

Sol:-

By above equation, We have,

$$I_{p,q} = \left[\frac{-\sin^{p-1} x \cos^{q+1} x}{p+q} \right]_0^{\pi/2} + \frac{p-1}{p+q} \int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx$$

$$I_{p,q} = 0 + \frac{p-1}{p+q} \int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx$$

$$I_{p,q} = \frac{p-1}{p+q} I_{p-2,q}$$

$$\text{Now } I_{p-2,q} = \frac{p-3}{p+q-2} I_{p-4,q}$$

$$I_{p-4,q} = \frac{p-5}{p+q-4} I_{p-6,q}$$

$$\vdots$$

$$I_{3,q} = \frac{2}{3+q} I_{1,q} \quad \text{If } p \text{ is odd}, \geq 3$$

$$I_{2,q} = \frac{1}{2+q} I_{0,q} \quad \text{If } p \text{ is even}, \geq 2$$

Hence,

$$I_{p,q} = \begin{cases} \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \cdots \cdot \frac{2}{q+3} I_{1,q} & \text{If } p \text{ is odd and } \geq 3 \\ \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \cdots \cdot \frac{1}{q+2} I_{0,q} & \text{If } p \text{ is even and } \geq 2 \end{cases}$$

So,

$$I_{1,q} = \int_0^{\pi/2} \sin x \cos^q x \, dx$$

$$I_{1,q} = -\left[\frac{\cos^{q+1} x}{q+1}\right]_0^{\pi/2} + 0 = \frac{1}{q+1}$$

And,

$$I_{0,q} = \int_0^{\pi/2} \cos^q x \, dx$$

We shall show in wallis cosine formula.

i) when p is odd, q may be even or odd

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \cdots \cdot \frac{2}{q+3} \cdot \frac{1}{q+1}$$

ii) when p is even, q is odd

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \cdots \cdot \frac{1}{q+2} \cdot \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{2}{3}$$

iii) when p is even and q is even, Then,

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{p-1}{p+q} \cdot \frac{p-3}{p+q-2} \cdot \cdots \cdot \frac{1}{q+2} \cdot \frac{q-1}{q} \cdot \frac{q-3}{q-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

We get by all three results,

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-3)\cdots\cdots(q-1)(q-3)}{(p+q)(p+q-2)\cdots\cdots} \cdot \frac{\pi}{2}$$

⇒ Wallis Cosine Formula And Wallis Sine formula

i):- $\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

$$I_n = \int_0^{\pi/2} \cos^n x\,dx$$

We know that,

$$I_n = \int_0^{\pi/2} \cos^n x\,dx$$

$$\Rightarrow I_n = \left[\frac{\sin x \cos^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

$$I_n = 0 + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\,dx$$

we have,

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3 = \frac{2}{3} I_1 \quad \text{If } n \text{ is odd}$$

$$I_2 = \frac{1}{2} I_0 \quad \text{If } n \text{ is even}$$

Now,

$$I_1 = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1$$

$$I_0 = \int_0^{\pi/2} \cos^0 x\,dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Thus,

$$\int_0^{\pi/2} \cos^n x = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3} & ; n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2} & ; n \text{ is even} \end{cases}$$

This is known as Wallis Cosine formula.

ii):- Similarly:-

$$\int \sin^n x\,dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int_0^{\pi/2} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{2}{3}; & n \text{ is odd} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2}; & n \text{ is even.} \end{cases}$$

which is known as Wallis Sine formula.

$\Rightarrow$ Let n be +ve Integer. Then, by Wallis' Sine formula:-

$\because 2n$ is even:-

$$\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\cdots\frac{1}{2}\cdot\frac{\pi}{2} \quad \text{—(i)}$$

$\because 2n+1$ is odd:-

$$\int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\cdots\frac{2}{3} \quad \text{—(ii)}$$

(i) becomes:-

$$\frac{\pi}{2} = \frac{(2n)(2n-2)(2n-4)\cdots\cdots 2}{(2n-1)(2n-3)(2n-5)\cdots\cdots 1}\int_0^{\pi/2} \sin^{2n} x\,dx \quad \text{—(iii)}$$

(ii) becomes:-

$$1 = \frac{(2n+1)(2n-1)(2n-3)\cdots\cdots 3}{(2n)(2n-2)(2n-4)\cdots\cdots 2}\int_0^{\pi/2} \sin^{2n+1} x\,dx \quad \text{—(iv)}$$

By dividing (iv) to (iii)

$$\frac{\pi}{2} = \frac{(2n)(2n-2)\cdots 2}{(2n-1)(2n-3)\cdots 1}\cdot\frac{(2n)(2n-2)(2n-4)\cdots 2}{(2n+1)(2n-1)(2n-3)\cdots 3}\frac{\int_0^{\pi/2}\sin^{2n} x\,dx}{\int_0^{\pi/2}\sin^{2n+1} x\,dx}$$

$$\frac{\pi}{2} = \frac{(2n)(2n)(2n-2)(2n-2)\cdots 6\cdot 6\cdot 4\cdot 4\cdot 2\cdot 2}{(2n+1)(2n-1)(2n-1)\cdots\cdots 5\cdot 5\cdot 3\cdot 3\cdot 1\cdot 1}\frac{\int_0^{\pi/2}\sin^{2n} x\,dx}{\int_0^{\pi/2}\sin^{2n+1} x\,dx} \quad \text{—☆}$$

So, $0 \le x \le \pi/2$

$\sin 0 \le \sin x \le \sin \pi/2$

$0 \le \sin x \le 1$

$$0 \le \int_0^{\pi/2} \sin^{2n+1} x\,dx \le \int_0^{\pi/2} \sin^{2n} x\,dx \le \int_0^{\pi/2} \sin^{2n-1} x\,dx \le 1 \quad \text{—(V)}$$

we have,

$$\int_0^{\pi/2} \sin^n x\,dx = \left[\frac{-\cos x \sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2n+1} x\,dx = \frac{2n+1-1}{2n+1}\int_0^{\pi/2} \sin^{2n+1-2} x\,dx$$

$$= \frac{2n}{2n+1}\int_0^{\pi/2} \sin^{2n-1} x\,dx$$

We get by (v)

$$1 \le \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} \le 1 + \frac{1}{2n}$$

Therefore:-

By the Sandwich theorem,

$$\lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx} = 1$$

Taking limit of both sides of (A) as $n \to \infty$

$$\lim_{n\to\infty} \frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6 \cdots\cdots (2n)(2n)}{1\cdot3\cdot3\cdot5\cdot5\cdot7 \cdots\cdots (2n-1)(2n+1)} \times \lim_{n\to\infty} \frac{\int_0^{\pi/2} \sin^{2n} x\,dx}{\int_0^{\pi/2} \sin^{2n+1} x\,dx}$$

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7} \cdots \frac{2n}{(2n-1)}\cdot\frac{2n}{(2n+1)} \times 1$$

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7} \cdots\cdots \frac{2n}{2n-1}\cdot\frac{2n}{2n+1}$$

This is known as Wallis' Product formula for $\pi/2$

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