$$\Rightarrow \frac{1}{\sqrt{x} + 2x^{1/3}} \Rightarrow \int \frac{1}{t^3 + 2t^2} 6t^5 dt$$

$$\int \frac{6t^5}{t^3 + 2t^2} dt \Rightarrow 6 \int \frac{t^5}{t^2(t+2)} dt$$

$$= 6 \int \frac{t^3}{t+2} dt$$

$$= 6 \left\{ \int (t^2 - 2t + 4) dt + \int \frac{-8}{t+2} dt \right.$$

$$\begin{array}{r|l} & t^2 - 2t + 4 \\ \hline t+2 & t^3 \\ & \underline{-t^3 \pm 2t^2} \\ & -2t^2 \\ & \underline{\mp 2t^2 \mp 4t} \\ & 4t \\ & \underline{-4t \pm 8} \\ & -8 \end{array}$$

$$6 \left\{ \frac{t^3}{3} - \frac{2t^2}{2} + 4t - 8 \int \frac{1}{t+2} dt \right\}$$

$$6 \left\{ \frac{t^3}{3} - t^2 + 4t - 8 \ln|t+2| \right\} + c$$

$$= 6 \left\{ \frac{x^{3/6}}{3} - x^{2/6} + 4x^{1/6} - 8 \ln|x^{1/6} + 2| \right\} + c$$

$$= \left\{ 6\frac{x^{3/6}}{3} - 6x^{2/6} + 24x^{1/6} - 48 \ln|x^{1/6} + 2| \right\} + c$$

$$= 2x^{3/6} - 6x^{2/6} + 24x^{1/6} - 48 \ln|x^{1/6} + 2| + c \text{ Ans.}$$

Reduction formula.

$$\int \sin^n x \, dx = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

where n an integ

Exercise NO 4.6

(1) $\int \sin^5 x \, dx$

$$= \int \sin^4 x \cdot \sin x \, dx$$

$$\int (\sin^2 x)^2 \sin x \, dx$$

$$\int (1-\cos^2 x)^2 \sin x \, dx$$

put $\cos x = t$

$-\sin x \, dx = dt$

$\sin x \, dx = -dt$

$$\int (1-t^2)^2 (-dt)$$

$$-\int (1+t^4-2t^2) \, dt$$ 3

$$= -t - \frac{t^5}{5} + 2\frac{t^3}{3} + c$$

$$-\cos x - \frac{(\cos x)^5}{5} + \frac{2}{3}(\cos x)^3 + C$$

$$-\left(\cos x + \frac{1}{5}\cos^5 x - \frac{2}{3}\cos^3 x\right) + C \text{ Ans.}$$

2

$$\int \cos^7 x \, dx$$

$$\int \cos^6 x \cdot \cos x \, dx$$

$$\int (\cos^2 x)^3 \cos x \, dx$$

$$= \int (1-\sin^2 x)^3 \cos x \, dx$$

put $\sin x = t$

$\cos x \, dx = dt$

$$\int (1-t^2)^3 dt$$

$$= \int (1 - t^6 - 3(1)(t^2)(1-t^2)) dt$$

$$\int (1 - t^6 - 3t^2 + 3t^4)\, dt$$

$$= t - \frac{t^7}{7} - \frac{3t^3}{3} + \frac{3t^5}{5} + c$$

$$= \sin x - \frac{\sin^7 x}{7} - \sin^3 x + \frac{3}{5}\sin^5 x + C$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C \quad \text{Ans.}$$

3 $$\int \sin^8 x\, dx$$

by using reduction formula

$$= -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

$$= -\frac{\cos x \sin^{8-1} x}{8} + \frac{(8-1)}{8}\int \sin^{8-2} x\, dx$$

$$\int \sin^8 x\, dx = -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\int \sin^6 x\, dx$$

$$= -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\left[-\frac{\cos x \sin^5 x}{6} + \frac{5}{6}\int \sin^4 x\, dx\right]$$

$$= \text{"} \quad \text{"} - \frac{7}{48}\cos x \sin^5 x + \frac{35}{48}\int \sin^4 x\, dx$$

$$= \text{"} \quad \text{"} \quad \text{"} \quad \text{"} + \frac{35}{48}\left[\frac{-\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\, dx\right]$$

$$\text{"} \quad \text{"} \quad \text{"} - \frac{35}{192}\cos x \sin^3 x + \frac{105}{192}\int \sin^2 x\, dx$$

$$\text{"} \quad \text{"} \quad \text{"} \quad \text{"} + \frac{105}{192}\int \left(\frac{1 - \cos 2x}{2}\right) dx$$

$$\text{"} \quad \text{"} \quad \text{"} + \frac{105}{192}\left[\frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos 2x\, dx\right]$$

$$\text{"} \quad \text{"} \quad \text{"} \quad \frac{\cancel{105}^{\,35}}{\cancel{384}_{\,128}} x - \frac{\cancel{105}^{\,35}}{\cancel{384}_{\,128}} \frac{\sin 2x}{2} + c$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}x$$ 5

$$-\frac{35}{128}\,\frac{\cancel{2}\sin x \cos x}{\cancel{2}} + c$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}x$$
$$-\frac{35}{128}\cos x \sin x + c \text{ Ans.}$$

4 $$\int \cos^6 x\, dx$$

by using reduction formula.

$$= \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$$

$$\frac{\sin x \cos^5 x}{6} + \frac{5}{6}\int \cos^4 x\, dx$$

$$\frac{\sin x \cos^5 x}{6} + \frac{5}{6}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\, dx\right]$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{15}{24}\int \cos^2 x\, dx$$ 6

$$\frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{15}{24}\int\left(\frac{1+\cos 2x}{2}\right) dx$$

$$'' \quad '' \quad '' + \frac{15}{24}\left[\frac{1}{2}\int 1\, dx + \frac{1}{2}\int \cos 2x\, dx\right]$$

$$'' \quad '' + \frac{15}{24}\left[\frac{1}{2}x + \frac{1}{2}\,\frac{\sin 2x}{2}\right]$$

$$= + \frac{\overset{5}{\cancel{15}}}{\underset{16}{\cancel{48}}}x + \frac{\overset{5}{\cancel{15}}}{\underset{16}{\cancel{48}}}\,\frac{\cancel{2}\sin x \cos x}{\cancel{2}} + c$$

$$= + \frac{5}{16}\sin x \cos x + \frac{5}{16}x + c$$

$$\int \cos^6 x\, dx = \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}\sin x \cos x$$
$$+ \frac{5}{16}x + c \text{ Ans.}$$

5 $$\int \tan^n x \, dx$$

$$I = \int \tan^{n-2+2} x \, dx$$

$$= \int \tan^{n-2} x \, . \, \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$\int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-2+1} x}{n-2+1} - \int \tan^{n-2} x \, dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$$

6 $$\int \sec^n x \, dx$$

$$I = \int \sec^{n-2+2} x \, dx$$

$$\int \sec^{n-2} x \, . \, \sec^2 x \, dx$$

$$= \sec^{n-2} x \int \sec^2 x \, dx - \int \frac{d}{dx} (\sec^{n-2} x) \int \sec^2 x \, dx$$

$$\sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \, (\sec x \tan x) . \tan$$

$$\text{''} \quad \text{''} \quad - (n-2) \int \sec^{n-2} x \, (\sec^2 x - 1) \, dx$$

$$\text{''} \quad \text{''} \quad - (n-2) \int \sec^n x + (n-2) \int \sec^{n-2} x \, dx$$

$$\sec^{n-2}x \tan x - (n-2)I + (n-2)\int \sec^{n-2}x\,dx$$

$$I + I(n-2) = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\,dx$$

$$I(n-1) = \sec^{n-2}x \tan x + (n-2)\int \sec^{n-2}x\,dx$$

$$= \frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$$

$$\frac{\sec^{n-2}x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx \text{ Ans.}$$

7 $$\int \cot^n x\,dx$$

$$I = \int \cot^n x\,dx$$

$$= \int \cot^{n-2+2} x\,dx$$

$$\int \cot^{n-2}x \cot^2 x\,dx \Rightarrow \int \cot^{n-2}x(\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \cot^{n-2}x \operatorname{cosec}^2 x\,dx - \int \cot^{n-2}x\,dx$$

$$= \frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx$$

$$= -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx \text{ Ans.}$$

(8) $$\int \operatorname{cosec}^n x\,dx$$

$$I = \int \operatorname{cosec}^{n-2+2} x\,dx$$

$$= \int \operatorname{cosec}^{n-2}x \cdot \operatorname{cosec}^2 x\,dx$$

$^{-2}x\,dx$

dx

dx

$$\operatorname{cosec}^{n-2}x\int \operatorname{cosec}^2x\,dx - \int\left[\frac{d}{dx}(\operatorname{cosec}^{n-2}x)\int \operatorname{cosec}^2x\right]d$$

$$\operatorname{cosec}^{n-2}x(-\cot x) - \int (n-2)\operatorname{cosec}^{n-3}x(-\operatorname{cosec}x\cot x)\cot x$$

$$\text{"} \quad \text{"} \quad -(n-2)\int \operatorname{cosec}^{n-2}x\cot^2x\,dx$$

$$\text{"} \quad \text{"} \quad -(n-2)\int \operatorname{cosec}^{n-2}x(\operatorname{cosec}^2x-1)\,dx$$

Ans.

$$-(n-2)\int \operatorname{cosec}^n x\,dx + (n-2)\int \operatorname{cosec}^{n-2}x\,dx$$

$$= \text{"} \quad \text{"} \quad -(n-2)I + (n-2)\int \operatorname{cosec}^{n-2}x\,dx$$

$$I + (n-2)I = -\operatorname{cosec}^{n-2}x\cot x + (n-2)\int \operatorname{cosec}^{n-2}x\,dx$$

$$I(n-1) = -\operatorname{cosec}^{n-2}x\cot x + (n-2)\int \operatorname{cosec}^{n-2}x\,dx$$

$$-\frac{\operatorname{cosec}^{n-2}x\cot x}{n-1} + \frac{(n-2)}{(n-1)}\int \operatorname{cosec}^{n-2}x\,dx$$

$-1)\,dx$

9 $$\int \tan^6 x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$$

$$= \frac{\tan^5 x}{5} - \int \tan^4 x\,dx$$

$$\frac{\tan^5x}{5} - \left[\frac{\tan^3x}{3} + \int \tan^2x\,dx\right]$$

$$\frac{\tan^5x}{5} - \frac{\tan^3x}{3} + \int (\sec^2x - 1)\,dx$$

$$\frac{\tan^5x}{5} - \frac{\tan^3x}{3} + \tan x - x + c$$

(10) $$\int \cot^5 x\, dx$$

$$\int \cot^n x\, dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\, dx$$

$$\int \cot^5 x\, dx = -\frac{\cot^4 x}{4} - \int \cot^3 x\, dx$$

$$= -\frac{\cot^4 x}{4} - \left[-\frac{\cot^2 x}{2} - \int \cot x\, dx\right]$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \int \cot x\, dx$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \int \frac{\cos x}{\sin x}\, dx$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln|\sin x| + c \text{ Ans.}$$

11 $$\int \sec^6 x\, dx$$

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$$

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\int \sec^4 x\, dx$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\, dx\right]$$

$$= \text{"} + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + c$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + c \text{ Ans.}$$

12 $$\int \operatorname{cosec}^5 x\, dx$$

$$\operatorname{cosec}^n x\, dx = -$$

$$\int \operatorname{Cosec}^5 x\,dx = -\frac{\operatorname{Cosec}^3 x \operatorname{Cot} x}{4} + \frac{3}{4}\int \operatorname{Cosec}^3 x\,dx$$

$$= -\frac{\operatorname{Cosec}^3 x \operatorname{Cot} x}{4} + \frac{3}{4}\left[-\frac{\operatorname{Cosec} x \cot x}{2} + \frac{1}{2}\int \operatorname{Cosec} x\,dx\right]$$

$$= -\frac{\operatorname{Cosec}^3 x \operatorname{Cot} x}{4} - \frac{3}{8}\operatorname{Cosec} x \cot x + \frac{3}{8}\ln|\operatorname{Cosec} x - \cot x|$$

$$= \;\; '' \;\; '' \;\; '' \;\; '' + \frac{3}{8}\ln|\tan(x/2)| + c$$

$$-\frac{\operatorname{Cosec}^3 x \operatorname{Cot} x}{4} - \frac{3}{8}\operatorname{Cosec} x \operatorname{Cot} x + \frac{3}{8}\ln|\tan x/2| + c \text{ Ans.}$$

Useful Subsitions (only Trignometric functions)

$$dz = \frac{2dz}{1+z^2} \qquad \operatorname{Cos} x = \frac{1-z^2}{1+z^2}$$

$$\sin x = \frac{2z}{1+z^2} \qquad \tan x = \frac{2z}{1-z^2}$$

suppose:-

$$z = \tan(x/2)$$

$$dz = \sec^2(x/2)\cdot\frac{1}{2}\,dx$$

$$dx = \frac{2dz}{\sec^2(x/2)}$$

$$dx = \frac{2dz}{1+\tan^2(x/2)}$$

$$dx = \frac{2dz}{1+z^2}$$

$$\sin x = 2\sin x/2 \cos x/2$$

$$'' \; '' \; = 2\sin x/2 \cos x/2$$

Dividing by $\cos^2 x/2$

$$\sin x = \frac{2\sin x/2 / \cos x/2}{1 + \tan^2 x/2}$$

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$$

$$\sin x = \frac{2z}{1+z^2} \qquad \because \tan x/2 = z$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$$

$$\cos x = \frac{\cos^2 x/2 - \sin^2 x/2}{\cos^2 x/2 + \sin^2 x/2}$$

Dividing by $\cos^2 x/2$ in R.H.S.

$$= \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

Now

$$\tan x = \frac{2\tan x/2}{1 - \tan^2 x/2}$$

$$\tan x = \frac{2z}{1-z^2}$$

13 $$\int \frac{dx}{a + b\sin x}$$

$$z = \tan x/2$$

$$dx = \frac{2dz}{1+z^2}$$

$$\sin x = \frac{2z}{1+z^2}$$

$$I = \int \frac{2dz/1+z^2}{a + b(2z/1+z^2)}$$

$$I = \int \frac{2dz/\cancel{1+z^2}}{a(1+z^2)+2bz/\cancel{1+z^2}}$$

$$= \int \frac{2dz}{a + az^2 + 2bz}$$

$= z$

$$\frac{2}{a}\int \frac{dz}{z^2 + \frac{2b}{a}z + (b/a)^2 - (b/a)^2 + 1}$$

$$\frac{2}{a}\int \frac{dz}{(z+b/a)^2 + \frac{a^2-b^2}{a^2}}$$

Case I :- if $a > b$

$\because a^2 - b^2$ is true.

$$\frac{2}{a}\int \frac{dz}{(z+b/a)^2 + \left(\frac{\sqrt{a^2-b^2}}{a}\right)^2}$$

$$\frac{2}{a}\,\frac{1}{\frac{\sqrt{a^2-b^2}}{a}}\tan^{-1}\left(\frac{(z+b/a)}{\frac{\sqrt{a^2-b^2}}{a}}\right) + c$$

$$\frac{2}{a} \times \frac{a}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{az+b/\cancel{a}}{\sqrt{a^2-b^2}/\cancel{a}}\right)$$

$$\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{az+b}{\sqrt{a^2-b^2}}\right) + c$$

$$\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\frac{a\tan x/2 + b}{\sqrt{a^2-b^2}}\right) + c$$

Case II if $a < b$

$\because b^2 - a^2$ is true

$$\frac{2}{a}\int \frac{dz}{(z+b/a)^2 - \left(\frac{b^2-a^2}{a^2}\right)}$$

$$\frac{2}{a}\int \frac{dz}{(z+b/a)^2 - (\sqrt{b^2-a^2}/a)^2}$$

$$\because \int \frac{1}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$$

$$\frac{2}{a} \cdot \frac{1}{\frac{2\sqrt{b^2-a^2}}{a}} \ln \left| \frac{z + b/a - \sqrt{b^2-a^2}/a}{z + b/a + \sqrt{b^2-a^2}/a} \right| + c$$

$$\frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{az + b - \sqrt{b^2-a^2}}{az + b + \sqrt{b^2-a^2}} \right| + c$$

$$\frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{a\tan x/2 + b - \sqrt{b^2-a^2}}{a\tan x/2 + b + \sqrt{b^2-a^2}} \right| + c$$

Useful Substitions:-
(In case of hyperbolic function)

$$z = \tanh(x/2)$$

$$dz = \operatorname{Sech}^2(x/2) \cdot \tfrac{1}{2}\, dx$$

$$dx = \frac{2\,dz}{\operatorname{Sech}^2(x/2)}$$

$$dx = \frac{2\,dz}{1 - \tanh^2(x/2)}$$

$$dz = \frac{2\,dz}{1 - z^2}$$

Now

$$\operatorname{Sinh} x = \frac{2\sinh x/2 \cosh x/2}{\cosh^2 x/2 - \sinh^2 x/2}$$

Dividing by $\cosh^2 x/2$

$$= \frac{2\tanh x/2}{1 - \tanh^2 x/2}$$

$$\operatorname{Sinh} x = \frac{2z}{1 - z^2}$$

Similarly

$$\operatorname{Cosh} x = \frac{1 + z^2}{1 - z^2}$$

$$\tanh x = \frac{2z}{1+z^2}$$

14 $$\int \frac{1}{a + b\cosh x}\,dx$$

put $x = \tanh x/2$

$$dx = \frac{2dz}{1-z^2} \quad ; \quad \cosh x = \frac{1+z^2}{1-z^2}$$

$$I = \int \frac{2dz/1-z^2}{a + b\left(\frac{1+z^2}{1-z^2}\right)}$$

$$= \int \frac{2dz}{a(1-z^2) + b(1+z^2)}$$

$$\int \frac{2dz}{a - az^2 + b + bz^2}$$

$$= \int \frac{2dz}{(b-a)z^2 + (b+a)}$$

Case I if $a < b$

$\because$ $b-a$ is true

$$\frac{2}{b-a} \int \frac{dz}{z^2 + \left(\sqrt{b+a/b-a}\right)^2}$$

$$= \frac{2}{b-a} \cdot \frac{1}{\sqrt{\frac{b+a}{b-a}}} \tan^{-1}\left(\frac{z}{\sqrt{\frac{b+a}{b-a}}}\right) + c$$

$$\frac{2}{\sqrt{b^2-a^2}} \tan^{-1}\left(\frac{\sqrt{b-a}\,z}{\sqrt{b+a}}\right) + c$$

$$\frac{2}{\sqrt{b^2-a^2}} \tan^{-1}\left(\frac{\sqrt{b-a}\tanh(x/2)}{\sqrt{b+a}}\right) + c$$

Case II :- if $a > b$

$\because$ $a-b$ is true.

$$\int \frac{2dz}{-(a-b)z^2+b+a} \Rightarrow \int \frac{2dz}{(b+a)-(a-b)z^2}$$

$$\frac{1}{a-b}\int \frac{2dz}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2-(z)^2}$$

$$\because \int \frac{1}{a^2-z^2}dz = \frac{1}{2a}\ln\left|\frac{z+a}{z-a}\right|$$

$$= \frac{2}{a-b}\,\frac{1}{2\sqrt{\frac{a+b}{a-b}}}\ln\left|\frac{z+\sqrt{a+b/a-b}}{z-\sqrt{a+b/a-b}}\right| + c$$

$$\frac{1}{a-b}\,\frac{1}{\sqrt{a+b/a-b}}\ln\left|\frac{\sqrt{a-b}\,z+\sqrt{a+b}}{\sqrt{a-b}\,z-\sqrt{a+b}}\right|$$

$$= \frac{1}{\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{a+b}+\sqrt{a-b}\tan x/2}{\sqrt{a+b}-\sqrt{a-b}\tan x/2}\right| + c$$

15 $$\int \frac{\cot x}{1+\sin x}dx$$

$$\int \frac{\cos x}{\sin x(1+\sin x)}dx$$

put $\sin x = t$

$\cos x\,dx = dt$

$$\int \frac{dt}{t(1+t)}$$ 16

Take

$$\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t}$$

$$1 = A(1+t) + B(t)$$

putting $t = 0$

$$1 = A(1+0)$$

$$A = 1$$

putting $1+t = 0 \Rightarrow t = -1$

$$1 = B(-1)$$

$$B = -1$$

$$\int \frac{dt}{t(1+t)} = \int \frac{1}{t}\,dt - \int \frac{1}{1+t}\,dt$$

$$= \ln|t| - \ln|1+t|$$

$$= \ln\left|\frac{t}{1+t}\right|$$

$$= \ln\left|\frac{\sin x}{1+\sin x}\right| \Rightarrow \ln\left|\frac{2\sin x/2 \cos x/2}{1 + 2\sin x/2 \cos x/2}\right|$$

$$\ln\left|\frac{2\sin x/2 \cos x/2}{\sin^2 x/2 + \cos^2 x/2 + 2\sin x/2 \cos x/2}\right|$$

$$\ln \frac{2\sin x/2 \cos x/2}{(\cos x/2 + \sin x/2)^2}$$

Dividing $\cos^2 x/2$

$$\ln \frac{2\tan x/2}{(1+\tan x/2)^2}$$

$$= \ln|2\tan x/2| - 2\ln|1+\tan x/2| + c$$

16 $$\int \frac{2-\cos x}{2+\cos x}\,dx$$

$$= \int \left(-1 + \frac{4}{2+\cos x}\right)$$

$$\begin{array}{r|l} & -1 \\ \cos x + 2 & -\cos x + 2 \\ & \underline{\mp\cos x \mp 2} \\ & 4 \end{array}$$

Now by integration

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$$-\int 1\,dx + \int \frac{4}{2+\cos x}\,dx$$

$$= -x + \int \frac{4}{2+\cos x}\,dx \quad \text{---} \text{(i)}$$

Now let

$$I_1 = \int \frac{4}{2+\cos x}\,dx$$

put $z = \tan x/2$

$$dx = \frac{2\,dz}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$= \int \frac{4\left(2dz/1+z^2\right)}{2+\left(1-z^2/1+z^2\right)}$$

$$8\int \frac{dz}{2+2z^2+1-z^2}$$

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$$= 8\int \frac{dz}{z^2+3}$$

$$8\int \frac{dz}{(z)^2+(\sqrt{3})^2}$$

$$8\,\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)$$

$$\int \frac{4}{2+\cos x}\,dx = \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right)$$

putting this value in equation (i)

$$= -x + \frac{8}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

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$$z = \tan(x/2)$$

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$= \int \frac{2dz/1+z^2}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} \Rightarrow \int \frac{2dz/1+z^2}{\frac{1+z^2+2z+1-z^2}{1+z^2}}$$

$$= \int \frac{2dz}{2z+2} \Rightarrow \int \frac{dz}{z+1}$$

$$= \ln|z+1| + c$$

$$\ln|\tan x/2 + 1| + c$$

18

$$\int \frac{\cos x}{2-\cos x}\,dx$$

$$-\int 1\,dx + \int \frac{2}{2-\cos x}\,dx \qquad -\cos x + 2 \overline{\big)\ \cos x\ }^{-1}$$

$$\pm\cancel{\cos x} \mp 2$$

$$-x + \int \frac{2}{2-\cos x}\,dx \quad \text{—— (i)} \qquad 2$$

Take

$$\int \frac{2}{2-\cos x}\,dx$$

put $z = \tan\frac{x}{2}$

$$dx = \frac{2dz}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$= \int \frac{2}{2} \frac{2dz/1+z^2}{-(1-z^2/1+z^2)}$$

$$= \int \frac{4dz/1+z^2}{\frac{2+2z^2-1+z^2}{1+z^2}} \Rightarrow \int \frac{4dz}{3z^2+1}$$

$$\int \frac{4dz}{3z^2+1} \Rightarrow \frac{4}{3}\int \frac{dz}{z^2+1/3}$$

$$\frac{4}{3}\int \frac{dz}{z^2+(1/\sqrt{3})^2} \Rightarrow \frac{4}{3} \cdot \frac{1}{1/\sqrt{3}} \tan^{-1}\left(\frac{z}{1/\sqrt{3}}\right)$$

$$\frac{4}{3}\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}\tan x/2}{1}\right)$$

Now use this value in (i)

$$-x + \frac{4}{3}\sqrt{3}\tan^{-1}(\sqrt{3}\tan x/2) + c$$

So

$$\int \frac{\cos x}{2-\cos x}dx = -x + \frac{4}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x/2) + c$$

9
$$\int \frac{1}{4\sin x - 3\cos x}dx$$

$$dx = \frac{2dz}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$

$$= \int \frac{2dz/1+z^2}{4(2z/1+z^2) - 3(1-z^2/1+z^2)}$$

$$\int \frac{2dz/1+z^2}{8z-3+3z^2/1+z^2} \Rightarrow \int \frac{2dz}{3z^2+8z-3}$$

$$\frac{2}{3}\int \frac{dz}{z^2+(8/3)z-1} \Rightarrow \frac{2}{3}\int \frac{dz}{z^2+(8/3)z+(4/3)^2-(4/3)^2-1}$$

$$\frac{2}{3}\int \frac{dz}{(z+4/3)^2-\frac{16}{9}-1} \Rightarrow \frac{2}{3}\int \frac{dz}{(z+4/3)^2-25/9}$$

$$\frac{2}{3}\int \frac{dz}{(z+4/3)^2-(5/3)^2} \Rightarrow \frac{2}{3}\cdot\frac{3}{2\times5}\ln\left|\frac{(z+4/3)-(5/3)}{(z+4/3)+(5/3)}\right|$$

$$\frac{1}{5}\ln\left|\frac{\tan x/2 - 1/3}{\tan x/2 + 9/3}\right| \Rightarrow \frac{1}{5}\ln\left|\frac{3\tan x/2 - 1}{3\tan x/2 + 9}\right|$$

So $\frac{1}{5}\ln\left|\frac{3\tan x/2 - 1}{3\tan x/2 + 9}\right|$ Ans.

20 $$\int \frac{dx}{\tan x - \sin x}$$

$z = \tan x/2$, $\sin x = \frac{2z}{1+z^2}$

$\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2dz}{1+z^2}$

$\tan x = \frac{2z}{1-z^2}$

$$= \int \frac{2dz/1+z^2}{2z/1-z^2 - 2z/1+z^2} \Rightarrow \int \frac{2dz/1+z^2}{2z\left[\frac{1}{1-z^2}-\frac{1}{1+z^2}\right]}$$

$$= \int \frac{2\,dz/1+z^2}{2z\left[\frac{1+z^2-1+z^2}{(1-z^2)(1+z^2)}\right]} \Rightarrow \int \frac{2dz/1+z^2}{2z\left(\frac{2z^2}{(1-z^2)(1+z^2)}\right)}$$

$$\int \frac{2dz/1+z^2}{\frac{4z^2}{(1+z^2)(1-z^2)}} \Rightarrow \int \frac{dz}{2z^3/1-z^2}$$

$$\frac{1}{2}\int\left(\frac{1-z^2}{z^3}\right)dz \Rightarrow \frac{1}{2}\int\left(\frac{1}{z^3}-\frac{z^2}{z^3}\right)dz$$

$$= \frac{1}{2}\int z^{-3}dz - \frac{1}{2}\int \frac{1}{z}dz$$

$$= \frac{1}{2}\frac{z^{-3+1}}{-3+1} - \frac{1}{2}\ln|z| + c.$$

$$= \frac{1}{2}\frac{z^{-2}}{-2} - \frac{1}{2}\ln|z| + c$$

$$\frac{-1}{4}z^{-2} - \frac{1}{2}\ln|z| + c$$

$$= \frac{-1}{4}\left[\tan(x/2)\right]^{-2} - \frac{1}{2}\ln\left[\tan(x/2)\right] + c$$

So

$$\int \frac{dx}{\tan x - \sin x} = \frac{-1}{4}(\tan x/2)^{-2} - \frac{1}{2}\ln\left(\tan\frac{x}{2}\right) + c$$

21 $$\int \frac{dx}{2\cosh x + \sinh x}$$

$$\int \frac{dx}{2(\cosh^2 x/2 + \sinh^2 x/2) + 2\sinh x/2 \cosh x/2}$$

Multiplying and dividing by $\cosh^2 x/2$

$$\frac{1}{2}\int \frac{1/\cosh^2 x/2\, dx}{(1+\tanh^2 x/2) + \tan x/2}$$

$$\frac{1}{2}\int \frac{dx}{\cosh^2 x/2\,(1+\tanh^2 x/2)+\tanh x/2}$$

$$\frac{1}{2}\int \frac{\operatorname{sech}^2 x/2}{(1+\tanh^2 x/2)+\tanh x/2}$$

$$\int \frac{1/2\,\operatorname{sech}^2(x/2)}{(1+\tanh^2 x/2)+\tanh x/2}$$

$$\tanh\frac{x}{2} = u$$

$$\operatorname{sech}^2\frac{x}{2}\cdot\frac{1}{2}\,dx = du$$

$$\int \frac{du}{u+(1+u^2)} \Rightarrow \int \frac{du}{u^2+u+1}$$

$$\int \frac{du}{u^2+2(u)(1/2)+(1/2)^2-(1/2)^2+1}$$

$$\int \frac{du}{(u+1/2)^2+(\sqrt{3}/2)^2}$$

$$\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{u+1/2}{\sqrt{3}/2}\right) \Rightarrow \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2u+1/2}{\sqrt{3}/2}\right]$$

$$\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2u+1}{\sqrt{3}}\right]$$

So

$$\int \frac{dx}{2\cosh x+\sinh x} = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2\tanh(x/2)+1}{\sqrt{3}}\right]+c$$

22 $$\int \frac{\sin x+\cos x}{\tan x}\,dx$$

Sol:- $\int\left(\frac{\sin x}{\tan x}+\frac{\cos x}{\tan x}\right)dx$

$$\int\left(\frac{\sin x}{\sin x/\cos x}+\frac{\cos x}{\sin x/\cos x}\right)dx$$

$$\int\left(\cos x+\frac{\cos^2 x}{\sin x}\right)dx$$

$$\int\cos x\,dx+\int\frac{(1-\sin^2 x)\,dx}{\sin x}$$

$$\sin x+\int\left(\frac{1}{\sin x}-\frac{\sin^{\cancel{2}} x}{\cancel{\sin x}}\right)dx$$

$$\sin x+\int\operatorname{cosec} x\,dx-\int\sin x\,dx$$

$$\sin x+\ln|\operatorname{cosec} x-\cot x|+\cos x+c$$

$$\sin x+\cos x+\ln\left|\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right|+c$$

$$\sin x+\cos x+\ln\left|\frac{1-\cos x}{\sin x}\right|+c$$

$$\sin x+\cos x+\ln\left|\frac{\sin^2 x/2+\cos^2 x/2-(\cos^2 x/2-\sin^2 x/2)}{2\sin x/2\cos x/2}\right|$$

$$\sin x+\cos x+\ln\left|\frac{\sin^2 x/2+\cancel{\cos^2 x/2}-\cancel{\cos^2 x/2}+\sin^2 x/2}{2\sin x/2\cos x/2}\right|$$

$$\sin x+\cos x+\ln\left|\frac{\sin x/2}{\cos x/2}\right|+c$$

$$\sin x+\cos x+\ln|\tan x/2|+c$$

so

$$\int\frac{\sin x+\cos x}{\tan x}dx=\sin x+\cos x+\ln|\tan x/2|+c$$ 24

23 $$\int\cos x\sqrt{1-\cos x}\,dx$$

$$=\int(1-2\sin^2 x/2)\sqrt{2\sin^2 x/2}\,dx$$

$$\int \sqrt{2}\sin x/2\,(1-2\sin^2 x/2)\,dx$$

$$\sqrt{2}\int \sin x/2\,dx - 2\sqrt{2}\int \sin^2 x/2\,\sin x/2\,dx$$

$$\sqrt{2}\left(\frac{-\cos x/2}{1/2}\right) - 2\sqrt{2}\int (1-\cos^2 x/2)\sin x/2\,dx$$

$$-2\sqrt{2}\cos x/2 - 2\sqrt{2}\int \sin x/2\,dx + 2\sqrt{2}\int \cos^2 x/2\,\sin x/2\,dx$$

$$-2\sqrt{2}\cos x/2 + 4\sqrt{2}\cos x/2 - 4\sqrt{2}\int \cos^2 x/2\,(-1/2\sin x/2)\,dx$$

$$-2\sqrt{2}\cos x/2 + 4\sqrt{2}\cos x/2 - 4\sqrt{2}\,\frac{\cos^3 x/2}{3}$$

$$2\sqrt{2}\cos x/2 - \frac{4\sqrt{2}}{3}\cos^3 x/2 + c$$

$$2\sqrt{2}\cos x/2\,[1 - 2/3\cos^2 x/2]$$

$$\frac{2\sqrt{2}}{3}\cos x/2\,(3-2\cos^2 x/2) \Rightarrow \frac{2\sqrt{2}}{3}\sqrt{\frac{1+\cos x}{2}}\left[3-(1+\cos x)\right]$$

$$\frac{2\cancel{\sqrt{2}}}{3}\,\frac{\sqrt{1+\cos x}}{\cancel{\sqrt{2}}}\,[3-1-\cos x] \Rightarrow \frac{2}{3}\sqrt{1+\cos x}\,(2-\cos x)+c$$

so

$$\int \cos x\sqrt{1-\cos x}\,dx = \frac{2}{3}(2-\cos x)\sqrt{1+\cos x}+c$$

$$\int \sqrt{a+\sec^2 x}\,dx$$

put $\sqrt{a+\sec^2 x} = z$

by squaring

$a+\sec^2 x = z^2$

$\Rightarrow \sec^2 x = z^2 - a$

$2\sec x\cdot\sec x\tan x\,dx = 2z\,dz$

$\therefore dx = \dfrac{2\,dz}{\sec^2 x\tan x}$

$dx = \dfrac{z\,dz}{\sec^2 x\sqrt{\sec^2 x - 1}}$

$= \dfrac{z\,dz}{(z^2-a)\sqrt{z^2-a-1}}$

$$= \int \frac{z\,dz \cdot z}{(z^2-a)\sqrt{z^2-(a+1)}} \qquad \because dx = \frac{z\,dz}{(z^2-a)\sqrt{z^2-(a+1)}}$$

$$\int \frac{z^2 dx}{(z^2-a)\sqrt{z^2-(a+1)}} \Rightarrow \int \frac{z^2}{z^2-a} \cdot \frac{1}{\sqrt{z^2-(a+1)}}$$

$$\int\left(1+\frac{a}{z^2-a}\right)\frac{dz}{\sqrt{z^2-(a+1)}}$$

$$= \int \frac{dz}{\sqrt{z^2-(a+1)}} + a\int \frac{dz}{(z^2-a)\sqrt{z^2-(a+1)}}$$

$$I = I_1 + aI_2$$

Take

$$I_1 = \int \frac{dz}{\sqrt{z^2-(\sqrt{a+1})^2}}$$

$$I_1 = \ln\left|\frac{z+\sqrt{z^2-(a+1)}}{\sqrt{a+1}}\right| \Rightarrow \ln\left|\frac{\sqrt{a+\sec^2x}+\tan x}{\sqrt{a+1}}\right|$$

Now

$$I_2 = \int \frac{dz}{(z^2-a)\sqrt{z^2-(a+1)}}$$

$$z = \frac{1}{t} \Rightarrow dz = \frac{-1}{t^2}\,dt$$

$$z^2 - a = \frac{1}{t^2} - a \Rightarrow \frac{1-at^2}{t^2}$$

$$z^2-(a+1) = \frac{1}{t^2}-(a+1) \Rightarrow \frac{1-(a+1)t^2}{t^2}$$

$$I_2 = \frac{-1/t^2\,dt}{\frac{1-at^2}{t^2}\cdot\frac{\sqrt{1-(a+1)t^2}}{t}}$$

$$\frac{-t\,dt}{(1-at^2)\sqrt{1-(a+1)t^2}}$$

$$\Rightarrow \sqrt{1-(a+1)t^2} = u$$

$$1 - \cos\theta = 2\sin^2\theta/2$$

$$\cos\alpha - \cos\theta = (2\cos^2\alpha/2 - 1) - (2\cos^2\theta/2 - 1)$$
$$= 2\cos^2\alpha/2 - 1 - 2\cos^2\theta/2 + 1$$
$$= 2\cos^2\alpha/2 - 2\cos^2\theta/2$$

$$I = \int \frac{\cancel{2}\sin^2\theta/2}{\sqrt{2\cos^2\alpha/2 - \cancel{2}\cos^2\theta/2}}\, d\theta$$

$$I = \frac{\sin\theta/2}{\sqrt{\cos^2\alpha/2 - \cos^2\theta/2}}\, d\theta$$

Taking common from Denominator $\sqrt{\cos^2\alpha/2 \cos^2\theta/2}$

$$= \int \frac{\sin\theta/2}{\cos\alpha/2 \cos\theta/2 \sqrt{1/\cos^2\theta/2 - 1/\cos^2\alpha/2}}$$

$$\frac{1}{\cos\alpha/2} \int \frac{\tan\theta/2}{\sqrt{\sec^2\theta/2 - \sec^2\alpha/2}}$$

$$\sec\alpha/2 \int \frac{\tan\theta/2}{\sqrt{\sec^2\theta/2 - \sec^2\alpha/2}}$$

$$\sqrt{\sec^2\theta/2 - \sec^2\alpha/2} = u$$
$$\sec^2\theta/2 - \sec^2\alpha/2 = u^2$$
$$\sec^2\theta/2 = u^2 + \sec^2\alpha/2$$
$$2\sec\theta/2 \cdot \sec\theta/2 \tan\theta/2 \cdot 1/2\, d\theta = 2u\,du + 0$$
$$\tan\theta/2\, d\theta = 2u\,du / \sec^2\theta/2$$
$$\tan\theta/2\, d\theta = 2u\,du / u^2 + \sec^2\alpha/2$$

$$= \sec\frac{\alpha}{2} \int \frac{2u\,du}{(u^2 + \sec^2\alpha/2)u}$$

$$2\sec\frac{\alpha}{2} \int \frac{1}{u^2 + \sec^2\alpha/2}\, du$$

26 (i)

(ii)

149 4.6

$1-(a+1)t^2 = u^2 \Rightarrow -2(a+1)t\,dt = 2u\,du$

$-t\,dt = \cancel{2}u\,du/\cancel{2}(a+1) \Rightarrow -t\,dt = \frac{u\,du}{a+1}$

$1-at^2 = 1-a\left[\frac{1-u^2}{a+1}\right] \Rightarrow \frac{a+1-a+au^2}{a+1}$

$1-at^2 = \frac{1+au^2}{a+1}$

$\Rightarrow 1-(a+1)t^2 = 1-\cancel{(a+1)}\left(\frac{1-u^2}{\cancel{a+1}}\right) \Rightarrow 1-1+u^2$

$1-(a+1)t^2 = u^2$

$I_2 = \int \frac{u\,du/a+1}{\frac{1+au^2}{a+1}\cdot\sqrt{u^2}} \Rightarrow \int \frac{du}{1+au^2}$

$\frac{1}{a}\int \frac{du}{1/a+u^2} \Rightarrow \frac{1}{a}\int \frac{du}{u^2+(1/\sqrt{a})^2}$

$\frac{1}{a}\tan^{-1}\left(\frac{u}{1/\sqrt{a}}\right) \Rightarrow \frac{1}{a}\tan^{-1}(\sqrt{a}\,u)$

$\frac{1}{a}\tan^{-1}(\sqrt{a}\sqrt{1-(a+1)t^2}$ $\quad \because u^2 = 1-(a+1)t^2$

$\frac{1}{a}\tan^{-1}(\sqrt{a}\sqrt{1-(a+1)\cdot 1/z^2})$ $\quad \because t = 1/z \quad \because z^2 = a+\sec^2 x$

$\frac{1}{a}\tan^{-1}\left(\frac{\sqrt{a}\sqrt{z^2-(a+1)}}{z}\right) \Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{\sqrt{a}\sqrt{\sec^2 x+a-a-1}}{\sqrt{a+\sec^2 x}}\right)$

$I_2 = \frac{1}{a}\tan^{-1}\left(\frac{\sqrt{a}\tan x}{\sqrt{a+\sec^2 x}}\right)$

put I_1 & I_2 in (i)

$I = \ln\left|\frac{\sqrt{\sec^2 x+a}+\tan x}{\sqrt{a+1}}\right| + a\cdot\frac{1}{a}\tan^{-1}\left(\frac{\sqrt{a}\tan x}{\sqrt{a+\sec^2 x}}\right)+c$

$I = \ln\left|\frac{\sqrt{\sec^2 x+a}+\tan x}{\sqrt{a+1}}\right| + \tan^{-1}\left(\frac{\sqrt{a}\tan x}{\sqrt{a+\sec^2 x}}\right)+c$ Ans.

25 $\quad I = \int\sqrt{\frac{1-\cos\theta}{\cos\alpha-\cos\theta}}\,d\theta$

$$= 2\sec\alpha/2 \tan^{-1}\left(\frac{u}{\sec\alpha/2}\right)$$

$$= 2\sec\alpha/2 \tan^{-1}\left(\frac{\sqrt{\sec^2\theta/2 - \sec^2\alpha/2}}{\sec\alpha/2}\right)$$

$$= 2\sec\alpha/2 \tan^{-1}\left[\frac{\sec\theta/2 \sec\alpha/2 \sqrt{1/\sec^2\alpha/2 - 1/\sec^2\theta/2}}{\sec\alpha/2}\right]$$

$$= 2\sec\alpha/2 \tan^{-1}\left[\frac{\sqrt{\cos^2\alpha/2 - \cos^2\theta/2}}{\cos\theta/2}\right] + c$$

26 (i)

$$z = \tan(x/2) \quad , \quad dx = 2dz/1+z^2$$

$$\cos x = 1-z^2/1+z^2 \Rightarrow \sec x = 1+z^2/1-z^2$$

$$\int \sec x\, dx = \int \frac{(1+z^2)}{(1-z^2)} + \frac{2dz}{(1+z^2)}$$

$$= 2\int \frac{dz}{1-z^2} \Rightarrow 2\cdot\frac{1}{2(1)} \ln\left|\frac{1+z}{1-z}\right|$$

$$\ln\left|\frac{1+\tan x/2}{1-\tan x/2}\right| \Rightarrow \ln\left|\frac{1+\sin x/2/\cos x/2}{1-\sin x/2/\cos x/2}\right|$$

$$= \ln\left|\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{\cos x/2 + \sin x/2}{\cos x/2 - \sin x/2}\right|^2$$

$$\frac{1}{2}\ln\left|\frac{\cos^2 x/2 + \sin^2 x/2 + 2\sin x/2\cos x/2}{\cos^2 x/2 + \sin^2 x/2 - 2\sin x/2 \cos x/2}\right|$$

$$\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$$

$$\int \sec x\, dx = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| \quad \text{Ans.}$$

(ii)

$$\int \operatorname{cosec} x\, dx$$

$$z = \tan x/2 \quad , \quad dx = 2dz/1+z^2$$

$$\cos x = \frac{1-z^2}{1+z^2} \quad , \quad \sin x = \frac{2z}{1+z^2}$$

$\text{Cosec}x\,dx = 1+z^2/2z$

$\text{Cosec}x\,dx = \int \frac{1+z^2}{2z} \times \frac{2dz}{1+z^2}$

$= \int \frac{1}{z}dz \Rightarrow \ln|z| + c$

$\ln|\tan x/2| \Rightarrow \ln|\sin x/2 / \cos x/2|$

$\frac{1}{2} \cdot 2\ln\left|\frac{\sin x/2}{\cos x/2}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{\sin x/2}{\cos x/2}\right|^2$

$\frac{1}{2}\ln\left|\frac{\sin^2 x/2}{\cos^2 x/2}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{1-\cos x/\cancel{2}}{1+\cos x/\cancel{2}}\right|$

$\int \text{cosec}x\,dx = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|$ Ans.

$\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$

$\int (\sqrt{\tan x} + 1/\sqrt{\tan x})\,dx$

$\int \left(\frac{\tan x + 1}{\sqrt{\tan x}}\right) dx$

$\Rightarrow \sqrt{\tan x} = t \Rightarrow \tan x = t^2$

$\sec^2 x\,dx = 2t\,dt \Rightarrow dx = \frac{2t\,dt}{\sec^2 x}$

$dx = \frac{2t\,dt}{1+\tan^2 x}$

$dx = \frac{2t}{1+(t^2)^2}dt \Rightarrow dx = \frac{2t}{1+t^4}dt$

$\int \frac{t^2+1}{t} \cdot \frac{2t}{(1+t^4)}dt \Rightarrow 2\int \frac{t^2+1}{1+t^4}dt$

$2\int \frac{t^2(1+1/t^2)}{t^2(t^2+1/t^2)}dt$

$2\int \frac{(1+1/t^2)}{t^2+1/t^2}dt$

$t - \frac{1}{t} = u \Rightarrow (1+\frac{1}{t^2})dt = du$

by suquaring

$$t^2+\frac{1}{t^2}-2=u^2 \Rightarrow t^2+\frac{1}{t^2}=u^2+2$$

$$=2\int\frac{du}{u^2+2} \Rightarrow 2\int\frac{1}{(u)^2+(\sqrt{2})^2}du$$

$$2\cdot\frac{1}{\sqrt{2}}\tan^{-1}(u/\sqrt{2}) \Rightarrow \sqrt{2}\tan^{-1}\left(\frac{t-1/t}{\sqrt{2}}\right)$$

$$\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\sqrt{\tan x}}\right) \Rightarrow \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\sqrt{\tan x}}\right)$$

so

$$=\sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)$$

Now

(Method of Sir Farooq)

$$\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$$

$$\int\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}\right)dx$$

$$\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx$$

let $\sin x-\cos x = t$

$$(\cos x+\sin x)\,dx = dt$$

$$\sin^2x+\cos^2x-2\sin x\cos x = t^2$$

$$1-2\sin x\cos x = t^2$$

$$\frac{1-t^2}{2} \quad \sin x\cos x$$

$$\sqrt{\frac{1-t^2}{2}} = \sqrt{\sin x\cos x}$$

$$= \int \frac{dt}{\frac{\sqrt{1-t^2}}{\sqrt{2}}}$$

$$\sqrt{2} \int \frac{1}{\sqrt{1-t^2}}$$

$$\sqrt{2} \sin^{-1}(t)$$

$$\sqrt{2} \sin^{-1}(\sin x - \cos x) + c \text{ Ans.}$$