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Exercise No 4.5

1 $$\int x^2\sqrt{25-x^2}\,dx$$

put $x = 5\sin\theta$

$\theta = \sin^{-1}\left(\frac{x}{5}\right)$

$dx = 5\cos\theta\,d\theta$

$\cos\theta = \sqrt{1-\sin^2\theta}$

(i) $\cos\theta = \sqrt{1 - x^2/25}$

$\cos\theta = \frac{\sqrt{25-x^2}}{5}$

dx $$= \int (5\sin\theta)^2\sqrt{(5)^2 - (5\sin\theta)^2}\;5\cos\theta\,d\theta$$

$$= \int 25\sin^2\theta.\sqrt{25-25\sin^2\theta}\;.\;5\cos\theta\,d\theta$$

$$= \int 25\sin^2\theta\;\sqrt{25(1-\sin^2\theta)}\;.\;5\cos\theta\,d\theta$$

$$= \int 25\sin^2\theta\,.(5\cos\theta)(5\cos\theta)\,d\theta$$

$$= 625\int \sin^2\theta\cos^2\theta\,d\theta$$

$\because \sin 2\theta = 2\sin\theta\cos$

$(\sin 2\theta)^2 = 4\sin^2\theta\cos^2\theta$

$\frac{(\sin 2\theta)^2}{4} = \sin^2\theta\cos^2$

$$= 625\int \frac{(\sin 2\theta)^2}{4}\,d\theta$$

$$= \frac{625}{4}\int\left(\frac{1-\cos 4\theta}{2}\right)d\theta$$

$$\frac{625}{8}\int (1)\,d\theta - \int \cos 4\theta\,d\theta$$

$$\frac{625}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]$$

$$\frac{625}{8}\left[\theta - \frac{2\sin 2\theta\cos 2\theta}{4}\right]$$

$$\frac{625}{8}\left[\theta - \frac{2\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)}{2}\right]$$

$$\frac{625}{8}\left[\sin^{-1}\left(\frac{x}{5}\right) - \frac{x}{5}\frac{\sqrt{25-x^2}}{5}\left(1 - 2\left(x^2/25\right)\right)\right]$$

$$\frac{625}{8}\left[\sin^{-1}\left(\frac{x}{5}\right) - \frac{x\sqrt{25-x^2}\,(25-2x^2)}{625}\right]$$

$$\frac{625}{8}\sin^{-1}\left(\frac{x}{5}\right) - \frac{1}{8}\left[x\sqrt{25-x^2}\,(25-x^2) - x^2\right]$$

$$\frac{625}{8}\sin^{-1}\left(\frac{x}{5}\right) - \frac{1}{8}(x)(25-x^2)^{3/2} + \frac{x^3}{8}\sqrt{25-x^2} + c$$

$$\int x(x+4)^{1/3}\,dx$$

let

$$(x+4)^{1/3} = t$$

$$x+4 = t^3$$

$$x = t^3 - 4$$

$$dx = 3t^2\,dt$$

$$\int (t^3-4)\,t\,3t^2\,dt$$

$$= 3\int (t^6 - 4t^3)\,dt$$

$$3\left[\frac{t^7}{7} - \frac{4t^4}{4} + c\right]$$

$$\frac{3}{7}(x+4)^{1/3} - 3(x+4)^{4/3} + c \text{ Ans.}$$

$$\int e^x\sqrt{1-e^{2x}}\,dx$$

let $e^x = t$

$$e^x dx = dt$$

$$= \int \sqrt{1-(e^x)^2}\, e^x dx$$

$$\int \sqrt{1-t^2}\, dt$$

$$t = 1\sin\theta \Rightarrow \theta = \sin^{-1}(t)$$

$$dt = \cos\theta\, d\theta \Rightarrow \theta = \sin^{-1}(e^x)$$

$$= \int \sqrt{1-t^2}\, dt \Rightarrow \int \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= \int \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta \Rightarrow \int \frac{1+\cos 2\theta\, d\theta}{2}$$

$$\frac{1}{2}\int 1\, d\theta + \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2} + c$$

$$\frac{1}{2}\theta + \frac{1}{4} 2\sin\theta\cos\theta + c$$

$$\frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2}(t.\sqrt{1-\sin^2\theta}) + c$$

$$\frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2} e^x\sqrt{1-e^{2x}} + c$$

$$\frac{1}{2}\sin^{-1}(e^x) + \frac{1}{2} e^x\sqrt{1-e^{2x}} + c \text{ Ans.}$$

4 $$\int \frac{x}{(1-x^2)^{3/2}} dx$$

$$= \frac{-1}{2}\int \frac{-2x}{(1-x^2)^{3/2}} dx$$

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$$= \frac{-1}{2}\int (1-x^2)^{-3/2}(-2x)\,dx$$

$$= \frac{-1}{2}\frac{(1-x^2)^{-1/2}}{-1/2} + c$$

$$\frac{1}{\sqrt{1-x^2}} + c \quad \text{Ans.}$$

$$\int \frac{x^2-3}{x\sqrt{x^2+4}}\,dx$$

$$\int \frac{x^2}{x\sqrt{x^2+4}}\,dx - 3\int \frac{dx}{x\sqrt{x^2+4}}$$

$$\int \frac{x}{\sqrt{x^2+4}}\,dx - 3\int \frac{dx}{x\sqrt{x^2+4}}$$

$$= \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+4}} - 3\int \frac{dx}{x\sqrt{x^2+4}}$$

$$= \frac{1}{2}\int (x^2+4)^{-1/2}(2x\,dx) - 3\int \frac{dx}{x\sqrt{x^2+4}}$$

$$\sqrt{x^2+4} - 3I_1 \quad \text{——— (i)}$$

$$I_1 = \int \frac{dx}{x\sqrt{x^2+4}}$$

$$x = 2\tan\theta$$

$$dx = 2\sec^2\theta\,d\theta$$

$$\int \frac{dx}{x\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\tan\theta\sqrt{(2\tan\theta)^2+(2)^2}}$$

$$= \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta\sqrt{4(\tan^2\theta+1)}} \Rightarrow \int \frac{2\sec^2\theta\,d\theta}{2\tan\theta\cdot 2\sec\theta}$$

$$= \int \frac{\sec\theta}{2\tan\theta}\,d\theta \Rightarrow \frac{1}{2}\int \frac{\sec\theta\,d\theta}{\tan\theta}$$

$$= \frac{1}{2}\int \frac{1/\cos\theta}{\sin\theta/\cos\theta}\,d\theta$$

$$\frac{1}{2}\int\left(\frac{1}{\cos\theta}-\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$\frac{1}{2}\int \operatorname{cosec}\theta\, d\theta \Rightarrow \frac{1}{2}\ln|\operatorname{cosec}\theta-\cot\theta|+c$$

$$\frac{1}{2}\ln\left|\frac{1}{\sin\theta}-\frac{\cos\theta}{\sin\theta}\right|+c \Rightarrow \frac{1}{2}\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|+c$$

$$\frac{1}{2}\ln\left|\frac{1-2/\sqrt{x^2+4}}{x/\sqrt{x^2+4}}\right|+c$$

$$\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}\ +2}{x}\right|+c$$

$$\int\frac{dx}{x\sqrt{x^2+4}}=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}\ +2}{x}\right|+c$$

using this value in equation (i)

$$\int\frac{x^2-3}{x\sqrt{x^2+4}}\,dx=\sqrt{x^2+4}-\frac{3}{2}\ln\left|\frac{\sqrt{x^2+4}\ +2}{x}\right|+c \text{ Ans.}$$

(6)
$$\int\sqrt{3x^2-4x+1}\,dx$$

$$=\int\sqrt{3\left(x^2-4/3x+1/3\right)}dx \Rightarrow \sqrt{3}\sqrt{\left(x^2-4/3x+1/3\right)}dx$$

$$=\sqrt{3}\int\sqrt{x^2-2(x)(2/3)+(2/3)^2-(2/3)^2+1/3}\ dx$$

$$=\sqrt{3}\int\sqrt{(x-2/3)^2-(1/3)^2}\,dx$$

$$\because \sqrt{x^2-a^2}$$

$$\because \sqrt{x^2-a^2}=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|$$

$$\sqrt{3}\ \frac{(x-2/3)\sqrt{(x-2/3)^2-(1/3)^2}}{2}-\frac{(1/3)^2}{2}\ln\frac{(x-2/3)+\sqrt{(x-2/3)^2-(}}{1/3}$$

$$= \sqrt{3}\left[\frac{(3x-2/3)\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{2} - \frac{1/9}{2}\ln\left|\frac{(3x-2/3)+\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{1/3}\right|\right]$$

$$= \sqrt{3}\ \frac{3x-2}{6}\sqrt{\frac{9x^2+4-12x}{9}-\frac{1}{9}} - \frac{1}{18}\ln\left|\frac{\left(\frac{3x-2}{3}\right)+\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{1/3}\right|$$

$$= \sqrt{3}\ \frac{3x-2}{6}\sqrt{\frac{9x^2+4-12x-1}{9}} - \frac{1}{18}\ln\left|\frac{\left(\frac{3x-2}{3}\right)+\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{1/3}\right|$$

$$= \sqrt{3}\ \frac{3x-2}{6}\sqrt{\frac{3(3x^2+1-4x)}{9}} - \frac{1}{18}\ln\left|\frac{\left(\frac{3x-2}{3}\right)+\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{1/3}\right|$$

$$= \left(\frac{3x-2}{6}\right)\sqrt{3}\sqrt{\frac{3x^2-4x+1}{3}} - \frac{\sqrt{3}}{18}\ln\left|\frac{(3x-2/3)+\sqrt{\left(\frac{3x-2}{3}\right)^2-(1/3)^2}}{1/3}\right|$$

$$= \left(\frac{3x-2}{6}\right)\sqrt{3}\ \frac{\sqrt{3x^2-4x+1}}{\sqrt{3}} - \frac{\sqrt{3}}{18}\ \text{Cosh}^{-1}\left[\frac{(3x-2/3)}{1/3}\right] + c$$

$$= \left(\frac{3x-2}{6}\right)\sqrt{3x^2-4x+1} - \frac{\sqrt{3}}{18}\ \text{Cosh}^{-1}(3x-2) + c \text{ Ans.}$$

7 $$\int\sqrt{x^2+2x+3}\ dx$$

$$= \int\sqrt{(x)^2+2(x)(1)+(1)^2-(1)^2+3}\ dx$$

$$= \int\sqrt{(x+1)^2+(\sqrt{2})^2}\ dx$$

$$= \frac{(x+1)\sqrt{(x+1)^2+(\sqrt{2})^2}}{2} + \frac{(\sqrt{2})^2}{2}\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

$$= \frac{(x+1)\sqrt{x^2+2x+2+1}}{2} + \frac{2}{2}\sinh^{-1}\left(\frac{x+1}{2}\right) + c$$

$$= \frac{(x+1)\sqrt{x^2+2x+3}}{2} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$ Ans.

$$\int \frac{x}{\sqrt{4+3x-2x^2}}\,dx$$

$$= \frac{1}{-4}\int \frac{-4x}{\sqrt{4+3x-2x^2}}\,dx$$

$$= \frac{-1}{4}\int \frac{-4x+3-3}{\sqrt{4+3x-2x^2}}\,dx$$

$$\frac{-1}{4}\int \frac{3-4x}{\sqrt{4+3x-2x^2}}\,dx - \int \frac{3}{\sqrt{4+3x-2x^2}}\,dx$$

$$= \frac{-1}{4}\,\frac{(4+3x-2x^2)^{1/2}}{1/2} + \frac{3}{4}\,I$$

$$= \frac{-2}{4}(4+3x-2x^2)^{1/2} + \frac{3}{4}\,I$$

where

$$I = \int \frac{dx}{\sqrt{4+3x-2x^2}}$$

$$= \int \frac{dx}{\sqrt{4-2(x^2-3x/2)}}$$

$$= \int \frac{dx}{\sqrt{4-2(x^2-2(x)(3/4)+(3/4)^2-(3/4)^2}}$$

$$= \int \frac{dx}{\sqrt{4-2[(x-3/4)^2-9/16}} \Rightarrow \int \frac{dx}{\sqrt{2(2-(x-4/4)^2-9/16)}}$$

$$= \int \frac{dx}{\sqrt{2(41/16-(x-3/4)^2}}$$

$$\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(\sqrt{41}/4)^2-(x-3/4)^2}}$$

$$\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{(\sqrt{41}/4)^2-(x-3/4)^2}}$$

$$\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x-3/4}{\sqrt{41}/4}\right)+c$$

$$\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$$

use this value in equation (i)

$$=\frac{-1}{2}\sqrt{4+3x-2x^2}+\frac{3}{4\sqrt{2}}\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c \text{ Ans.}$$

9

$$\int\frac{dx}{\sqrt{3x^2-4x+1}}$$

$$=\int\frac{dx}{\sqrt{3(x^2-4/3x+1/3)}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2-2(x)(2/3)+(2/3)^2-(2/3)^2+1/3}}$$

$$=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x-2/3)^2-(1/3)^2}}$$

$$=\frac{1}{\sqrt{3}}\cosh^{-1}\left(\frac{3x-2/3}{1/3}\right)$$

$$=\frac{1}{\sqrt{3}}\cosh^{-1}(3x-2)$$

$$\because \quad x - \frac{2}{3} = \frac{1}{3}\cosh\theta$$

$$dx = \frac{1}{3}\sinh\theta\, d\theta$$

$$\int \frac{1/3 \sinh\theta\, d\theta}{\sqrt{1/9\cosh^2\theta - 1/9}}$$

$$= \int \frac{1/3 \sinh\theta\, d\theta}{1/3 \sinh\theta}$$

$$\int 1\, d\theta$$

$$\theta + c$$

$$\because \quad x - \frac{2}{3} = \frac{1}{3}\cosh\theta$$

ns.

$$\theta = \cosh^{-1}\left(\frac{x-2/3}{1/3}\right)$$

$$\Rightarrow \theta = \cosh^{-1}(3x-2) \text{ A1}$$

$$= \frac{1}{\sqrt{3}}\cosh^{-1}(3x-2) + c \text{ Ans.}$$

10

$$\int \frac{x+1}{\sqrt{x^2+2x+3}}\, dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\, dx$$

$$= \frac{1}{2}\frac{(x^2+2x+3)^{-1/2+1}}{-1/2+1} + c$$

$$= \sqrt{x^2+2x+3} + c \text{ Ans.}$$

11

$$\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\, dx$$

$$= \int \frac{x^2 + x + x + 2 + 1}{\sqrt{x^2 + x + 1}}\, dx$$

$$\int \frac{x^2+x+1}{\sqrt{x^2+x+1}}\, dx + \int \frac{x+2}{\sqrt{x^2+x+1}}\, dx$$

$$= \int \sqrt{x^2+x+1}\, dx + \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+x+1}}\, dx$$

$$= \int \sqrt{x^2+2(x)(1/2)+(1/2)^2-(1/2)^2+1} + \frac{1}{2}\left[\int \frac{2x+1}{\sqrt{x^2+x+1}}\, dx + \int \frac{3}{\sqrt{x^2+x+1}}\, dx\right]$$

$$\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}\, dx + \frac{1}{2}(x^2+x+1)^{-1/2}(2x+1)\, dx + \frac{3}{2}\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$\frac{(x+1/2)\sqrt{(x+1/2)^2-(\sqrt{3}/2)^2}}{2} + \frac{(\sqrt{3}/2)^2}{2}\sinh^{-1}\left[\frac{(x+1/2)}{\sqrt{3}/2}\right]$$

$$+\frac{1}{2}\,\frac{(x^2+x+1)^{-1/2+1}}{-1/2+1} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2+2(x)(1/2)+(1/2)^2-(1/2)^2+1}}$$

$$\frac{(x+1/2)\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}}{2} + \frac{(\sqrt{3}/2)^2}{2}\sinh^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)$$

$$+\sqrt{x^2+x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}}$$

$$= (2x+1/2)\sqrt{(2x+1/2)^2+(\sqrt{3}/2)^2} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1/2}{\sqrt{3}/2}\right)$$

$$+\sqrt{x^2+x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x+1/2}{\sqrt{3}/2}\right) + C$$

$$= \left(\frac{2x+1}{4}\right)\sqrt{\frac{4x^2+1+4x}{4}+\frac{3}{4}} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$+\sqrt{x^2+x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$=\frac{(2x+1)}{4}\sqrt{\frac{4(x^2+x+1)}{4}} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \sqrt{x^2+x+1}$$
$$+ \frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$\left(\frac{2x+1}{4}\right)\sqrt{x^2+x+1} + \sqrt{x^2+x+1} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$
$$+\frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

$$\sqrt{x^2+x+1}\left(\frac{2x+1}{4}+1\right) + \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\left[\frac{3}{8}+\frac{3}{2}\right]+c$$

$$\sqrt{x^2+x+1}\left(\frac{2x+1+4}{4}\right) + \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\left(\frac{3+12}{8}\right)+c$$

$$\sqrt{x^2+x+1}\left(\frac{2x+5}{4}\right) + \sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\left(\frac{15}{8}\right)+c$$

$\frac{1}{2})^2+1$

$$\int\frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \frac{2x+5}{4}\sqrt{x^2+x+1}+\frac{15}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$=\left(\frac{2x+5}{4}\right)\sqrt{x^2+x+1}+\frac{15}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$ Ans.

12 $$\int\frac{1}{(2x+3)\sqrt{x+5}}\,dx$$

Type _ I

$$\int\frac{dx}{\text{linear}\sqrt{\text{linear}}}$$

$\sqrt{\text{linear}}=t$

put

$$\sqrt{x+5}=t$$

squaring on both sides

$$x+5=t^2$$

$$x=t^2-5$$

$$dx=2t\,dt$$

$$= \int \frac{2t\,dt}{(2(t^2-5)+3)t}$$

$$= \int \frac{2\,dt}{2t^2-10+3} \Rightarrow \int \frac{2\,dt}{2t^2-7}$$

$$= \int \frac{2\,dt}{2(t^2-7/2)} \Rightarrow \int \frac{dt}{(t^2-7/2)}$$

$$= \int \frac{dt}{(t)^2-(\sqrt{7/2})^2}$$

$$\frac{1}{2\sqrt{7/2}} \ln\left|\frac{t-\sqrt{7/2}}{t+\sqrt{7/2}}\right| + c$$

$$= \frac{\sqrt{2}}{2\sqrt{7}} \ln\left|\frac{\sqrt{x+5}-\sqrt{7/2}}{\sqrt{x+5}+\sqrt{7/2}}\right| + c$$

$$= \frac{1}{\sqrt{2}\sqrt{7}} \ln\left|\frac{\sqrt{x+5}-\sqrt{7/2}}{\sqrt{x+5}+\sqrt{7/2}}\right| + c$$ 14

$$= \frac{1}{\sqrt{14}} \ln\left|\frac{\sqrt{x+5}-\sqrt{7/2}}{\sqrt{x+5}+\sqrt{7/2}}\right| + c \text{ Ans.}$$

13 $$\int \frac{1\quad dx}{(1-2x)\sqrt{1+4x}}$$

put

$$\sqrt{1+4x}$$

$$1+4x = t^2$$

$$4x = t^2-1$$

$$x = \frac{t^2-1}{4}$$

$$dx = \frac{1}{4}(2t)dt$$

$$dx = \frac{1}{2}t\,dt$$

$$1-2x = 1-\frac{2(t^2-1)}{4}$$

$$= \frac{2 - t^2 + 1}{2} = \frac{3 - t^2}{2}$$

$$\int \frac{1\, dx}{(1-2x)\sqrt{1+4x}} = \int \frac{1/2\, t\, dt}{1/2\,(3-t^2)t}$$

$$\int \frac{dt}{3-t^2} \Rightarrow -\int \frac{dt}{t^2-(\sqrt{3})^2}$$

$$= \because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$= \frac{-1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| \Rightarrow \frac{-1}{2\sqrt{3}}\ln\left|\frac{\sqrt{1+4x}-\sqrt{3}}{\sqrt{1+4x}+\sqrt{3}}\right|$$

$$= \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{1+4x}}{\sqrt{3}-\sqrt{1+4x}}\right| + c \text{ Ans.}$$

14 $$\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx$$

$$= \int \frac{x\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}} \Rightarrow \int \frac{x(\sqrt{1+x})^2}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{x(1+x)}{\sqrt{1-x^2}}\,dx \Rightarrow \int \frac{x+x^2}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{1-x^2}}\,dx + \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$-\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx + \int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$-(1-x^2)^{1/2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx$$

$$-(1-x^2)^{1/2} - \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= (1-x^2)^{1/2} - \int (1-x^2)^{1/2}\,dx + \sin^{-1}x$$

$$= -(1-x^2)^{1/2} - \frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1}x + \sin^{-1}x$$

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$$= -\sqrt{1-x^2} - \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x \quad \text{Ans.}$$

15 $$\int \frac{x^4}{(x-1)}\,dx$$ 16

$$\int \frac{x^4}{(x-1)} \frac{dx}{\sqrt{x+2}}$$

$$\begin{array}{r|l} & x^3+x^2+x+1 \\ x-1 & \cancel{x^4} \\ & \cancel{-x^4} \mp x^3 \\ & \cancel{x^3} \\ & \cancel{-x^3} \mp x^2 \\ & \cancel{x^2} \\ & \pm \cancel{x^2} \mp x \\ & \cancel{x} \\ & \cancel{-x} \mp 1 \\ & 1 \end{array}$$

$$= \int\left(\left(x^3+x^2+x+1\right) + \frac{1}{x-1}\right)\frac{dx}{\sqrt{x+2}}$$

put $\sqrt{x+2} = t$

$x+2 = t^2$

$dx = 2t\,dt$

$x = t^2-2$

$$\int\left[(t^2-2)^3 + (t^2-2)^2 + (t^2-2) + 1 + \frac{1}{(t^2-2)-1}\right]\frac{2t\,dt}{t}$$

$$\int (t^2)^3 - (2)^3 - 3(t^2)^2(2) + 3(t^2)(2)^2 + (t^2)^2 + (2)^2 - 2(t^2)(2) + \frac{1}{t^2-3}\,2\,dt$$

$$\int\left[(t^6 - 5t^4 + 9t^2 - 5) + \frac{1}{t^2-3}\right]2\,dt$$

$$= 2\left[\int t^6\,dt - 5\int t^4\,dt + 9\int t^2\,dt - 5\int 1\,dt + \int\frac{dt}{t^2-(\sqrt{3})^2}\right.$$

$$2\left[\frac{t^7}{7} - \frac{5t^5}{5} + \frac{9t^3}{3} - 5t + \frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|\right]$$

$$\frac{2t^7}{7} - 2t^5 + 6t^3 - 10t + \frac{2}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|$$

$$\frac{2t^7}{7} - 2t^5 + 6t^3 - 10t + \frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|$$

Putting value of 't'

$t = \sqrt{x+2}$

$$= \frac{2}{7}(x+2)^{7/2} - 2(x+2)^{5/2} + 6(x+2)^{3/2} - 10(x+2)^{1/2} + \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{x+2} - \sqrt{3}}{\sqrt{x+2} + \sqrt{3}}\right| + c$$ Ans.

16 $$\int \frac{1 \, dx}{(x^2-2x+2)\sqrt{x-1}}$$

$$\int \frac{f(x)\, dx}{\text{quadretic}\sqrt{\text{linear}}}$$

$\sqrt{\text{linear}} = t$

$$\sqrt{x-1} = t$$

$$x - 1 = t^2$$

$$x = t^2 + 1$$

$$dx = 2t\, dt$$

$$\Rightarrow x^2 - 2x + 2 = (t^2+1)^2 - 2(t^2+1) + 2$$

$$= t^4 + 1 + 2t^2 - 2t^2 - 2 + 2$$

$$= t^4 + 1$$

$$= \int \frac{2t\, dt}{(1+t^4)t} \Rightarrow \int \frac{2}{1+t^4} dt$$

$2dt$

adding and subtracting t^2

$$\int \frac{t^2 - t^2 + 2\, dt}{t^4+1} \Rightarrow \int \frac{t^2 + 1 - (t^2-1)\, dt}{t^4+1}$$

$$\int \frac{t^2+1}{t^4+1} dt - \int \frac{t^2-1}{t^4+1} dt$$

$(\sqrt{3})^2$

$$\int \frac{t^2(1+1/t^2)}{t^2(t^2+1/t^2)} dt - \int \frac{t^2(1-1/t^2)}{t^2(t^2+1/t^2)} dt$$

$$\int \frac{1 + 1/t^2\, dt}{t^2 + 1/t^4} - \int \frac{(1-1/t^2)\, dt}{(t^2+1/t^2)}$$

$$= I_1 - I_2$$

$$I_1 = \int \frac{1 + 1/t^2\, dt}{t^2 + 1/t^2}$$

$$t - \frac{1}{t} = u$$

$$1 + \frac{1}{t^2} = du$$

$$t^2 + \frac{1}{t^2} - 2 = u^2$$

$$t^2 + \frac{1}{t^2} = u^2 + 2$$

$$I = \int \frac{du}{u^2 + 2}$$

$$\int \frac{du}{(u)^2 + (\sqrt{2})^2} \Rightarrow \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - 1/t}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right)$$

$$I_2 = \int \frac{(1 - 1/t^2)dt}{t^2 + 1/t^2}$$

$$t + \frac{1}{t} = u$$

$$\left(1 - \frac{1}{t^2}\right)dt = du$$

$$t^2 + \frac{1}{t^2} + 2 = u^2$$

$$t^2 + \frac{1}{t^2} = u^2 - 2$$

$$I_2 = \int \frac{du}{u^2 - 2} \Rightarrow \int \frac{du}{(u)^2 - (\sqrt{2})^2}$$

$$\frac{1}{2\sqrt{2}} \ln\left|\frac{u - \sqrt{2}}{u + \sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}} \ln\left|\frac{t + 1/t - \sqrt{2}}{t + 1/t + \sqrt{2}}\right|$$

$$\frac{1}{2\sqrt{2}} \ln\left|\frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t}\right|$$

putting the value of I_1 and I_2

(111) 4.5

$$\frac{1}{\sqrt{2}}\tan^{-1}\left|\frac{t^2-1}{\sqrt{2}t}\right| - \frac{1}{2\sqrt{2}}\ln\left|\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right|$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right|$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-2}{\sqrt{2}\sqrt{x-1}}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2(x-1)}}{x+\sqrt{2(x-1)}}\right| \text{ Ans.}$$

17 $$\int \frac{1}{(x^2+4x+5)\sqrt{x+2}}\,dx$$

$$\sqrt{x+2} = t$$
$$x+2 = t^2$$
$$dx = 2t\,dt$$
$$x^2+4x+5 = (t^2-2)^2+4(t^2-2)+5$$
$$= t^4+4-4t^2+4t^2-8+5$$
$$= t^4+1$$

$$= \int \frac{2t\,dt}{(t^4+1)(t)} \Rightarrow \int \frac{2\,dt}{1+t^4}$$

adding and subtracting t^2

$$\int \frac{t^2-t^2+2}{t^4+1}\,dt \Rightarrow \int \frac{(t^2+1)-(t^2-1)}{t^4+1}\,dt$$

$$\int \frac{t^2+1}{t^4+1}\,dt - \int \frac{t^2-1}{t^4+1}\,dt$$

$$\int \frac{t^2(1+1/t^2)}{t^2(t^2+1/t^2)}\,dt - \int \frac{t^2(1-1/t^2)}{t^2(t^2+1/t^2)}\,dt$$

$$\int \frac{(1+1/t^2)}{(t^2+1/t^2)}\,dt - \int \frac{t^2(1-1/t^2)}{t^2(t^2+1/t^2)}\,dt$$

$$\int \frac{(1+1/t^2)}{(t^2+1/t^2)}\,dt - \int \frac{(1-1/t^2)}{(t^2+1/t^2)}\,dt$$

$$I_1 - I_2$$

$$I_1 = \int \frac{(1+1/t^2)}{(t^2+1/t^2)}\,dt$$

(18)

$$t - \frac{1}{t} = u$$

$$\left(1 + \frac{1}{t^2}\right) dt = du$$

$$t^2 + \frac{1}{t^2} - 2 = du$$

$$t^2 + \frac{1}{t^2} = u^2 + 2$$

$$I = \int \frac{du}{u^2 + 2}$$

$$\int \frac{du}{u^2 + (\sqrt{2})^2} \Rightarrow \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - 1/t}{\sqrt{2}}\right) \Rightarrow \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right)$$

$$I_2 = \int \frac{(1 - 1/t^2)\, dt}{t^2 + 1/t^2}$$

$$t + \frac{1}{t} = u$$

$$\left(1 - \frac{1}{t^2}\right) dt = du$$

$$t^2 + \frac{1}{t^2} + 2 = u^2$$

$$t^2 + \frac{1}{t^2} = u^2 - 2$$

$$I_2 = \int \frac{du}{u^2 - 2} \Rightarrow \int \frac{du}{u^2 - (\sqrt{2})^2}$$

$$\frac{1}{2\sqrt{2}} \ln \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| \Rightarrow \frac{1}{2\sqrt{2}} \ln \left| \frac{t + 1/t - \sqrt{2}}{t + 1/t + \sqrt{2}} \right|$$

$$\frac{1}{2\sqrt{2}} \ln \left| \frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t} \right|$$

putting the value of I_1 and I_2

$$\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) - \frac{1}{2\sqrt{2}} \ln \left| \frac{t^2 + 1 - \sqrt{2}t}{t^2 + 1 + \sqrt{2}t} \right|$$

$$I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2(x+2)}}\right) - \frac{1}{2\sqrt{2}} \ln \left| \frac{x+3 - \sqrt{2(x+2)}}{x+3 + \sqrt{2(x+2)}} \right|$$ Ans

(8) $$\int \frac{1}{(x-1)\sqrt{x^2+1}} dx$$

$$x - 1 = \frac{1}{t}$$

$$\int \frac{dx}{\text{linear}\sqrt{\text{quadretic}}}$$

$$\text{linear} = \frac{1}{t}$$

$$x = \frac{1}{t} + 1$$

$$= \frac{1+t}{t}$$

$$dx = \frac{-1}{t^2} dt$$

$$\Rightarrow x^2+1 = \left(\frac{1+t}{t}\right)^2 + 1$$

$$= \frac{1+t^2+2t}{t^2} + 1 \Rightarrow \frac{1+2t^2+2t}{t^2}$$

putting the value

$$\int \frac{1/t^2 \, dt}{1/t \frac{\sqrt{2t^2+2t+1}}{t}} \Rightarrow \int \frac{-1/t^2 \, dt}{1/t^2 \sqrt{2t^2+2t+1}}$$

$$= -\int \frac{1 \, dt}{\sqrt{2t^2+2t+1}} \Rightarrow \frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2+t+1/2}}$$

$$\frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2+(1/2)t+1/4-1/4+1/2}} \Rightarrow -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{(t+1/2)^2+1/4}}$$

$$\because t + 1/2 = \frac{1}{2} \sinh\theta$$

$$dt = \frac{1}{2} \cosh\theta \, d\theta$$

$$\frac{2t+1}{2} = \frac{1}{2} \sinh\theta \Rightarrow \theta = \sinh^{-1}(2t+1)$$

$$= \frac{-1}{\sqrt{2}} \int \frac{1/2 \cosh\theta \, d\theta}{1/2 \cosh\theta} \Rightarrow \frac{-1}{\sqrt{2}} \theta + c$$

$$= \frac{-1}{\sqrt{2}} \sinh^{-1}(2t+1) + c$$

$$= \frac{-1}{\sqrt{2}} \sinh^{-1}\left[2\left(\frac{1}{x-1}\right) + 1\right] + c$$

$$= \frac{-1}{\sqrt{2}} \sinh^{-1}\left[\frac{2+x-1}{x-1}\right] + c$$

$$= \frac{-1}{\sqrt{2}} \sinh^{-1}\left(\frac{x+1}{x-1}\right) + c \text{ Ans.}$$

19 $$\int \frac{1}{(x+1)\sqrt{x^2-1}} dx$$

$$x + 1 = \frac{1}{t}$$

$$x = \frac{1}{t} - 1$$

$$dx = \frac{-1}{t^2} \Rightarrow t = \frac{1}{1+x}$$

$$I = \int \frac{(-1/t^2)\, dt}{(1/t)(1/t - 1)^2 + 1} \Rightarrow \int \frac{(-1/t^2)\, dt}{1/t \sqrt{1/t^2 + 1 - 2/t - 1}}$$

$$\int \frac{(-1/t^2)\, dt}{(1/t)\sqrt{1/t^2 - 2/t}} \Rightarrow \int \frac{(-1/t^2)\, dt}{(1/t)\sqrt{\frac{1-2t}{t^2}}}$$

$$\int \frac{(-1/t^2)\, dt}{(1/t^2)\sqrt{1-2t}} \Rightarrow \int \frac{dt}{\sqrt{1-2t}}$$

$$= \frac{1}{2} \int \frac{-2dt}{\sqrt{1-2t}} \Rightarrow \frac{1}{2} \int (1-2t)^{-1/2}(-2dt)$$

$$= \frac{1}{2} \frac{(1-2t)^{1/2}}{1/2} + c$$

$$I = \sqrt{(1-2t)} + c$$

$$I = \sqrt{1 - \frac{2}{x+1}} + c$$

$$I = \sqrt{\frac{x+1-2}{x+1}} + c$$

$$I = \sqrt{\frac{x-1}{x+1}} + c \text{ Ans.}$$

20 $$\int \frac{1}{ax^n + bx} dx$$

$$= \int \frac{dx}{x^n(a + bx^{1-n})} \Rightarrow \int \frac{x^{-n}}{a + bx^{1-n}} dx$$

$$a + bx^{1-n} = t$$

$$b(1-n)x^{1-n-1} dx = dt$$

$$x^{-n} dx = \frac{dt}{b(1-n)}$$

$$= \frac{1}{b(1-n)} \int \frac{dt}{t} \Rightarrow \frac{1}{b(1-n)} \ln|t| + c$$

$$\frac{1}{b(1-n)} \ln|a + bx^{1-n}| + c \text{ Ans.}$$

21 $$\int \frac{x^2 + 2x + 3}{(x+2)\sqrt{x^2+1}} dx$$

$$= \int \left(\frac{x^2 + 2x + 3}{x+2}\right) \frac{1}{\sqrt{x^2+1}} dx$$

$$\begin{array}{r} x \\ x+2 \overline{\smash{)}\, \cancel{x^2} + \cancel{2x} + 3} \\ \underline{-\cancel{x^2} \mp \cancel{2x}} \\ 3 \end{array}$$

$$I = \int \left(\frac{x^2 + 2x + 3}{x+2}\right) \frac{1}{\sqrt{x^2+1}} dx$$

$$= \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{3}{(x+2)\sqrt{x^2+1}} dx$$

$$\frac{1}{2}\int \frac{2x}{\sqrt{x^2+1}}\,dx + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$\frac{1}{2}\int (x^2+1)^{-1/2}\,2x\,dx + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$I = \sqrt{x^2+1} + 3\int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \sqrt{x^2+1} + 3I_1$$

where

$$I_1 = \int \frac{dx}{(x+2)\sqrt{x^2+1}} \Rightarrow \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{1/t^2 - 4/t + 5}}$$

$$I_1 = -\int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{\frac{1-4t+5t^2}{t^2}}} \Rightarrow \int \frac{(-1/t^2)\,dt}{(1/t^2)\sqrt{1-4t+5t^2}}$$

$$I_1 = -\int \frac{dt}{\sqrt{1-4t+5t^2}} \Rightarrow -\int \frac{dt}{\sqrt{5(t^2-4/5\,t+1/5)}}$$

$$\frac{-1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2-4/5\,t+1/5}} \Rightarrow \frac{-1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2-2(t)(2/5)+(2/5)^2-(2/5)+1/5}}$$

$$I_1 = \frac{-1}{\sqrt{5}}\int \frac{dt}{\sqrt{(t-2/5)^2+(1/5)^2}}$$

$$\frac{-1}{\sqrt{5}}\sinh^{-1}\left(\frac{t-2/5}{1/5}\right)$$

$$= -\frac{1}{\sqrt{5}}\sinh^{-1}\left(\frac{5t-2}{1}\right) \Rightarrow \frac{-1}{\sqrt{5}}\sinh^{-1}\left[5\left(\frac{1}{x+2}\right)-2\right]$$

$$\frac{-1}{\sqrt{5}}\sinh^{-1}\left(\frac{5}{x+2}-2\right) \Rightarrow \frac{-1}{\sqrt{5}}\sinh^{-1}\left(\frac{5-2x-4}{x+2}\right)$$

$$= -\frac{1}{\sqrt{5}}\sinh^{-1}\left(\frac{1-2x}{x+2}\right)$$

putting the value of I_1

$$I = \sqrt{x^2+1} + 3\left[\frac{-1}{\sqrt{5}}\sinh^{-1}\left(\frac{1-2x}{x+2}\right)\right] + c$$

putting value of I, we get

$$\int \frac{x^2+2x+3}{(x+2)\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + 3\left(\frac{-1}{\sqrt{5}}\sinh^{-1}\left(\frac{1-2x}{x+2}\right)\right) + c$$

22
$$\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$$

Type IV

$$\int \frac{dx}{\text{Pure quad}\sqrt{\text{pure qua}}}$$

standrad qua

$ax^2+bx+c=0$

pure quad

$ax^2+c=0$

put $x=\frac{1}{t}$

$$x = \frac{1}{t}$$

$$dx = \frac{-1}{t^2}\,dt$$

$$x^2+1 = \frac{1}{t^2}+1$$

$$= \frac{1+t^2}{t^2}$$

$$= \int \frac{-1/t^2\,dt}{1/t^2\frac{\sqrt{1+t^2}}{t}} \Rightarrow \int \frac{-t\,dt}{\sqrt{1+t^2}}$$

$-(2/5)+1/5$

$$= \frac{-1}{2}\int (1+t^2)^{-1/2}(2t)\,dt$$

$$= \frac{-1}{2}\frac{(1+t^2)^{1/2}}{1/2} \Rightarrow -(1+t^2)^{1/2}+c$$

$-2]$

$$= -\sqrt{1+1/x^2}+c$$

$$= -\frac{\sqrt{1+x^2}}{x} + c \text{ Ans.}$$

3
$$\int \frac{1\,dx}{(1+x^2)\sqrt{1-x^2}}$$

$$x = \frac{1}{t}$$

$$dx = \frac{-1}{t^2}\,dt$$

$$1+x^2 = 1+\frac{1}{t^2}$$

$$= \frac{1+t^2}{t^2}$$

$$1-x^2 = 1-\frac{1}{t^2} = \frac{t^2-1}{t^2}$$

$$= \int \frac{-1/t^2}{\frac{1+t^2}{t^2}\frac{\sqrt{t^2-1}}{t}} \Rightarrow \int \frac{-t\,dt}{1+t^2\sqrt{t^2-1}}$$

$$\sqrt{t^2-1} = z$$

$$t^2-1 = z^2$$

$$t^2 = z^2+1$$

$$2t\,dt = 2z\,dz$$

$$t\,dt = z\,dz$$

$$1+t^2 = 1+z^2+1$$

$$\text{" "} \quad = z^2+2$$

$$= \int \frac{-z\,dz}{(z^2+2)z} \Rightarrow -\int \frac{dz}{z^2+2}$$

$$-\int \frac{dz}{z^2+(\sqrt{2})^2} \Rightarrow \frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)$$

$$= \frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right) \Rightarrow \frac{-1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{1/x^2-1}}{\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right) \text{ Ans.}$$

.4

$$\int \frac{1}{(1-2x^2)\sqrt{1-x^2}}\,dx$$

$$x = \frac{1}{t}$$

$$dx = \frac{-1}{t^2}\,dt$$

$$I = \int \frac{-1/t^2 \quad dt}{\left[1-2(1/t)^2\right]\sqrt{1-(1/t)^2}}$$

$$= \int \frac{(-1/t^2)\,dt}{(t^2-2/t^2)\sqrt{\frac{t^2-1}{t^2}}}$$

$$= \int \frac{-t\,dt}{(t^2-2)\sqrt{t^2-1}}$$

put $\sqrt{t^2-1} = u$

$$t^2 - 1 = u^2$$

$$t^2 = 1+u^2$$

$$2t\,dt = 2u\,du$$

$$I = -\int \frac{u\,du}{(u^2+1-2)u} \Rightarrow -\int \frac{du}{-1+u^2}$$

$$= \int \frac{du}{-u^2+1} \Rightarrow \int \frac{du}{(1)^2-(u)^2}$$

$$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{1+\sqrt{t^2-1}}{1-\sqrt{t^2-1}}\right|$$

$$\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^2}/x}{1-\sqrt{1-x^2}/x}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right|$$

$$\because x = \cos\theta$$

$$I = \frac{1}{2}\ln\left|\frac{\cos\theta+\sqrt{1-\cos^2\theta}}{\cos\theta-\sqrt{1-\cos^2\theta}}\right| \Rightarrow \frac{1}{2}\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right|$$

dividing by $\cos\theta$

$$I = \frac{1}{2}\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\tan\pi/4+\tan\theta}{1-\tan\pi/4\tan\theta}\right|$$

$$= \frac{1}{2}\ln\left|\tan(\pi/4+\theta)\right|$$

$$\frac{1}{2}\ln\left|\tan\left(\pi/4 + \cos^{-1}x\right)\right| + c$$ Ans.

25 $$\int \frac{1}{(2x^2-3x+1)\sqrt{3x^2-2x+1}}\,dx$$

$$\frac{1}{2x^2-3x+1} = \frac{A}{x-1} + \frac{B}{2x-1}$$

$$1 = A(2x-1) + B(x-1)$$

put $x-1=0 \Rightarrow x=1$

$2x^2-3x+1$

$2x^2-2x-x+1$

$2x(x-1)+(x-1)$

$(x-1)(2x-1)$

$$1 = A(2-1)$$

$$A = 1$$

put $2x-1=0 \Rightarrow x=1/2$

$$1 = 0 + B(-1/2)$$

$$B = -2$$

$$\frac{1}{2x^2-3x+1} = \frac{1}{x-1} - \frac{2}{2x-1} \quad \text{---(iii)}$$

$$= \left(\frac{1}{x-1} - \frac{2}{2x-1}\right)\frac{1}{\sqrt{3x^2-2x+1}}$$

$$= \int \frac{dx}{(x-1)\sqrt{3x^2-2x+1}} - \int \frac{2\,dx}{(2x-1)\sqrt{3x^2-2x+1}}$$

$$= I_1 - I_2$$

$$I_1 = \int \frac{dx}{(x-1)\sqrt{3x^2-2x+1}}$$

$$x - 1 = \frac{1}{t}$$

$$x = \frac{1}{t} + 1$$

4.5

. + 1

- x + 1

1) + (x-1)

. x - 1)

$$I_1 = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{3(1/t+1)^2-2(1/t+1)+1}}$$

$$= \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{3(1/t^2+1+2/t)-2/t^{-2}+1}}$$

$$I_1 = \int \frac{(-1/t^2)\,dt}{(1/t^2)\sqrt{3/t^2+3+6/t-2/t-1}}$$

$$= \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{\frac{3+4t+2t^2}{t^2}}}$$

$$\int \frac{(-1/t^2)\,dt}{(1/t^2)\sqrt{2t^2+4t+3}}$$

$$= \int \frac{-dt}{\sqrt{2\,(t^2+2t+3/2)}}$$

$$\frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{(t)^2+2(1)(1)+(1)^2-(1)^2+3/2}}$$

$$= \frac{-1}{\sqrt{2}} \int \frac{dt}{\sqrt{(t+1)^2+(1/\sqrt{2})^2}} \Rightarrow \frac{-1}{\sqrt{2}} \sinh^{-1}\left(\frac{t+1}{1/\sqrt{2}}\right)$$

$$I_1 = \frac{-1}{\sqrt{2}} \sinh^{-1}\left(\frac{1/x-1+1}{1/\sqrt{2}}\right)$$

$$\frac{-1}{\sqrt{2}} \sinh^{-1}\left(\frac{\sqrt{2}\,x}{x-1}\right) \text{------ (iii)}$$

$$I_2 = \int \frac{2\,dx}{(2x-1)\sqrt{3x^2-2x+1}}$$

$$2x-1 = \frac{1}{t}$$

$$2\,dx = \frac{-1}{t^2}\,dt$$

$$2\,dx = \frac{-1}{t^2}\,dt$$

$$2x = \frac{1}{t} + 1$$

$$x = \frac{1+t}{2t}$$

$$I_2 = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{3\left(1+t/2t\right)^2 - 2\left(1+t/2t\right)+1}}$$

$$\int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{3\left(\frac{1+t^2+2t}{4t^2}\right) - \frac{1-t}{t} + 1}}$$

$$\frac{(-1/t^2)\,dt}{(1/t)\sqrt{3+3t^2-6t-4t-4t^2+4t^2/4t^2}}$$

26

$$= \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{3+3t^2+2t/2t}}$$

$$\int \frac{(-1/t^2)\,dt}{(1/t^2)\sqrt{3+3t^2+2t/2}}$$

$$= \int \frac{-2\,dt}{\sqrt{3\left(t^2+2/3\,t+1\right)}}$$

$$-\frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{(t)^2+2(1/3)(t)+(1/3)^2-(1/3)^2+1}}$$

$$-\frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{(t+1/3)^2+(\sqrt{8}/3)^2}}$$

$$-\frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{t+1/3}{\sqrt{8}/3}\right)$$

$$I_2 = -\frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{t+1}{\sqrt{8}}\right)$$

$$-\frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{3/2x-1+1}{\sqrt{8}}\right)$$

$$-\frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{3+2x-1}{2\sqrt{2}\,(2x+1)}\right) \Rightarrow -\frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{2(x+1)}{2\sqrt{2}\,(2x+1)}\right)$$

$$I_2 = -\frac{2}{\sqrt{3}}\sinh^{-1}\left(\frac{x+1}{\sqrt{2}(2x+1)}\right)$$

26 $$\int \frac{x^{1/2}}{1+x^{1/3}}\,dx$$

put $x^{1/6} = t$

$x = t^6$

$dx = 6t^5\,dt$

$$= \int \frac{t^3}{1+t^2} \times 6t^5\,dt$$

$$6\int \frac{t^8}{t^2+1}\,dt$$

$$t^2+1 \,\big)\, t^8 \quad (t^6)$$

$t^8 + t^6$

$-t^6$

$\underset{+}{-}t^6 \ \underset{+}{-}t^4$

t^4

$-t^4 \pm t^2$

$-t^2$

$\underset{+}{-}t^2 \ \underset{+}{-}1$

1

$$= 6\left[\int t^6 - t^4 + t^2 - 1 + \int \frac{d}{1+t^2}\right]$$

$$= 6\left[\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \tan^{-1}(t)\right]$$

$$6\left[\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{x^{3/6}}{3} - x^{1/6} + \tan^{-1}(x^{1/6})\right] + c$$

$$6\left[\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{x^{3/6}}{3} - x^{1/6} + \tan^{-1}(x^{1/6})\right] + c \text{ Ans.}$$ =

27 $$\int \frac{z^3}{\sqrt{1+x^2}}\,dz$$

put $\sqrt{1+x^2} = t$

$1 + x^2 = t^2$

$\Rightarrow x^2 = t^2 - 1$

$2x\,dx = 2t\,dt$

$$= \int \frac{x^2 \cdot x\,dx}{\sqrt{1+x^2}}$$

$$= \int \frac{(t^2-1)t}{t}\,dt$$

$$= \int (t^2-1)\,dt$$

$$= \frac{t^3}{3} - t + c$$

$$= \frac{(1+x^2)^{3/2}}{3} - \sqrt{1+x^2} + c \text{ Ans.}$$

28 $$\int \frac{1}{\sqrt{x} + 2x^{1/3}}\,dx$$

$x^{1/6} = t$ (1)

$x = t^6$

$dx = 6t^5\,dt$

$$\Rightarrow \frac{1}{\sqrt{x}+2x^{1/3}} \Rightarrow \int \frac{1}{t^3+2t^2}\,6t^5dt$$

$$\int \frac{6t^5}{t^3+2t^2}dt \Rightarrow 6\int \frac{t^5}{t^2(t+2)}dt$$

$$= 6\int \frac{t^3}{t+2}dt$$

$$= 6\left\{\int (t^2-2t+4)dt + \int \frac{-8}{t+2}dt\right.$$

$$\begin{array}{r|l} & t^2-2t+4 \\ \hline t+2 & t^3 \\ & \underline{-t^3 \pm 2t^2} \\ & -2t^2 \\ & \underline{\mp 2t^2 \mp 4t} \\ & 4t \\ & \underline{-4t \pm 8} \\ & -8 \end{array}$$

$$6\left\{\frac{t^3}{3} - \frac{2t^2}{2} + 4t - 8\int \frac{1}{t+2}dt\right\}$$

$$6\left\{\frac{t^3}{3} - t^2 + 4t - 8\ln|t+2|\right\} + c$$

$$= 6\left\{\frac{x^{3/6}}{3} - x^{2/6} + 4x^{1/6} - 8\ln|x^{1/6}+2|\right\} + c$$

$$= \left\{6\frac{x^{3/6}}{3} - 6x^{2/6} + 24x^{1/6} - 48\ln|x^{1/6}+2|\right\} + c$$

$$= 2x^{3/6} - 6x^{2/6} + 24x^{1/6} - 48\ln|x^{1/6}+2| + c \text{ Ans.}$$

Reduction formula.

$$\int \sin^n x\,dx = -\frac{\cos x \sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

where n an integ

Exercise NO 4.6

(1) $\int \sin^5 x\,dx$

$$= \int \sin^4 x \cdot \sin x\,dx$$