(58) 4.4

(2)

# Exercise No 4.4

Integrate each of the following with respect to '$x$'.

(1) $$\int \frac{x}{(x-1)(x-2)}\,dx$$

let

$$\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} \quad \text{(i)}$$

multiply equation (i) by $(x-1)(x-2)$

$$x = A(x-2) + B(x-1) \quad \text{(ii)}$$

put $x-1=0 \Rightarrow x=1$ in (ii)

$$1 = A(1-2) + B(0)$$

$$1 = -A$$

$$A = -1$$

$$\boxed{A = -1}$$

put $x-2=0 \Rightarrow x=2$ in (ii)

$$2 = B(2-1) + A(0)$$

$$\boxed{B = 2}$$

Putting $A$, $B$ in (i) then integrate

$$\int \frac{x}{(x-1)(x-2)}\,dx = -\int \frac{1}{x-1}\,dx + 2\int \frac{1}{x-2}\,dx$$

$$= -\ln|x-1| + 2\ln|x-2| + c$$

$$= \ln|x-1|^{-1} + 2\ln|x-2| + c$$

$$= \ln(x-2)^2 \cdot (x-1)^{-1}$$

$$= \ln\left|\frac{(x-2)^2}{(x-1)}\right| + c \text{ Ans.}$$

59) 4.4

(2) $$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$

$$\frac{2x-3}{(x^2-1)(2x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(2x+3)} \quad \text{(i)}$$

Multiplying equation (i) by $(x^2-1)(2x+3)$

$$2x-3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x^2-1) \quad \text{(ii)}$$

putting $x-1=0 \Rightarrow x = 1$ in (ii)

$$2(1)-3 = A(1+1)(2+3) + B(0) + C(0)$$

$$-1 = 10A$$

$$\boxed{A = -1/10}$$

putting $x+1=0 \Rightarrow x = -1$ in (ii)

$$2(-1)-3 = A(0) + B(-1-1)(2(-1)+3) + C(0)$$

$$-5 = -2B$$

$$\boxed{B = 5/2}$$

putting $2x+3=0 \Rightarrow x = \frac{-3}{2}$ in (ii)

$$2\left(\frac{-3}{2}\right) - 3 = A(0) + B(0) + C\left[(-3/2)^2 - 1\right]$$

$$-6 = C(9/4 - 1)$$

$$-6 = C\,9-4/4$$

$$-6 = C\,5/4$$

$$C = \frac{-6 \times 4}{5}$$

$$\boxed{C = -24/5}$$

putting A, B and C in (i), then integrate

$$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx = -\int \frac{1}{10(x-1)}\,dx + \int \frac{5}{2(x+1)} - \int \frac{24}{5(2x+3)}\,dx$$

$$= -\frac{1}{10}\int \frac{1}{x-1}\,dx + \frac{5}{2}\int \frac{1}{x+1}\,dx - \frac{\cancel{24}^{12}}{\cancel{2}\times 5}\int \frac{1}{2x+3}\,2dx$$

$$= \frac{5}{2}\ln|x+1| - \frac{1}{10}\ln|x-1| - \frac{12}{5}\ln|2x+3| + C \text{ Ans.}$$

3 $$\int \frac{x+1}{x^2+4x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+4x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4-2}{x^2+4x+5}\,dx$$



$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \frac{2}{2}\int \frac{dx}{x^2+4x+5}$$

$$= \frac{1}{2}\ln|x^2+4x+5| - \int \frac{dx}{(x^2+4x+4+1)}$$

$$= \frac{1}{2}\ln|x^2+4x+5| - \int \frac{dx}{(x+2)^2+(1)^2}$$

$$= \frac{1}{2}\ln|x^2+4x+5| - \tan^{-1}(x+2) + c \quad \text{Ans.}$$

4 $$\int \frac{2x^2+3x+1}{x^2+2x+2}\,dx$$

First we change it in proper fraction
So

$$\begin{array}{r|l} & \quad 2 \\ x^2+2x+2 & 2x^2+3x+1 \\ & -2x^2 \pm 4x \pm 4 \\ \hline & \quad -x-3 \end{array}$$

$$\text{so} \int 2\,dx - \int \frac{x+3}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{2(x+3)}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{2x+6}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - \frac{1}{2}\times 4\int \frac{1}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\ln|x^2+2x+2| - 2\int \frac{dx}{x^2+2(x)(1)+(1)^2+(1)^2}$$

$$= 2x - \frac{1}{2}\ln|x^2+2x+2| - 2\int \frac{dx}{(x+1)^2+(1)^2}$$

$$= 2x - \frac{1}{2}\ln|x^2+2x+2| - 2\tan^{-1}(x+1) + C \text{ Ans.}$$

5 $$\int \frac{x^2}{(x-1)^3(x+1)}\,dx$$

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Multiply by L.C.M

$$x^2 = A(x-1)^2 + B(x-1)^2(x+1) + C(x-1)(x+1) + D(x+1) \quad \text{(ii)}$$

putting $x+1=0 \Rightarrow x=-1$ in (ii)

$$(-1)^2 = A(-1-1)^3$$

$$1 = A(-2)^3$$

$$\boxed{A = -1/8}$$

putting $\Rightarrow x = 1$ in (ii)

$$(1)^2 = A(0) + B(0) + C(0) + D(1+1)$$

$$1 = 2D$$

$$\boxed{D = 1/2}$$

Simplyfiy (i)

$$x^2 = A(x^3-3x^2+3x-1) + B(x^3-x^2-x+1) + C(x^2-1) + D(x+1)$$

$$= (A+B)x^3 + (-3A-B+C)x^2 + (3A-B+D)x - A+B-C+D$$

by Compareing coffients

$$x^3 \Rightarrow \quad 0 = A + B$$

$$0 = \frac{-1}{8} + B \qquad \boxed{B = 1/8}$$

$x^2 \Rightarrow \quad 1 = -3A - B + C$

$$1 = -3(-1/8) - \frac{1}{8} + C$$

$$1 = \frac{3}{8} - \frac{1}{8} + C$$

$$1 - \frac{1}{4} = C$$

$$\boxed{C = 3/4}$$

Putting A, B, C and D in equation (i)

$$= \frac{-1}{8}\int \frac{1}{x+1}dx + \frac{1}{8}\int \frac{1}{x-1}dx + \frac{3}{4}\int (x-1)^{-2}dx + \frac{1}{2}\int (x-1)^{-3}dx$$

$$= \frac{-1}{8}\ln|x+1| + \frac{1}{8}\ln|x-1| + \frac{3}{4}\frac{(x-1)^{-1}}{-1} + \frac{1}{2}\frac{(x-1)^{-3+1}}{-3+1} + c$$

$$= \frac{-1}{8}\ln|x+1| + \frac{1}{8}\ln|x-1| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2}$$

$$= \frac{-1}{4(x-1)^2} - \frac{3}{4(x-1)} + \frac{1}{8}\ln\left|\frac{x-1}{x+1}\right| \text{ Ans.}$$



6 $$\int \frac{1}{x(x+1)^3}dx$$

$$\int \frac{1}{x(x+1)^3}dx = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

Multiplying by L.C.M.

$$1 = A(x+1)^3 + B(x)(x+1)^2 + C(x)(x+1) + D(x)$$

putting $x = 0$ in (ii)

$$1 = A(0+1)^3 + 0 + 0 + 0$$

$$\boxed{A = 1}$$

putting $x+1 = 0 \Rightarrow x = -1$ in (ii)

$$+1 = A(0) + B(0) + C(0) + D(-1)$$

$$+1 = -D$$

$$\boxed{D = -1}$$

Simplyfiy equation (ii)

$$1 = A(x^3 + 1 + 3x(x+1)) + B(x^3 + x + 2x^2) + C(x^2 + x) + D(x)$$

$$= (A+B)x^3 + (3A + 2B + C)x^2 + (3A + B + C + D)x + A$$

$x^3 \Rightarrow$

$$A + B = 0$$

$$1 + B = 0$$

$$\boxed{B = -1}$$

$x^2 \Rightarrow \quad 3A + 2B + C = 0$

$$3(1) + 2(-1) + C = 0 \Rightarrow 3 - 2 + C = 0$$

$$\boxed{C = -1}$$

putting A, B, C and D in equation (i)

$$\int \frac{1}{x(x+1)^3}dx = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$$

$$〃 \quad 〃 = \int \frac{1}{x}dx - \int \frac{1}{x+1}dx - \int \frac{1}{(x+1)^2}dx - \int \frac{1}{(x+1)^3}dx$$

$$= \ln|x| - \ln|x+1| - \frac{(x+1)^{-2+1}}{-2+1} - \frac{(x+1)^{-3+1}}{-3+1} + C$$

$$= \ln|x| - \ln|x+1| + \frac{1}{(x+1)} + \frac{1}{2(x+1)^2} + C \text{ Ans.}$$

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$$\int \frac{x+1}{(x-1)^2(x+2)^2}dx$$

$$\frac{x+1}{(x-1)^2(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

Multiply by L.C.M

$$x+1 = A(x+2)^2(x-1) + B(x+2)^2 + C(x-1)^2(x+2) + D(x-1)^2$$

putting $x-1=0$ in equation (ii) $x=1$

$$1+1 = A(0) + B(1+2)^2 + C(0) + D(0)$$

$$2 = 9B$$

$$\boxed{B = 2/9}$$

putting $x+2=0 \Rightarrow x=-2$ in equation (ii)

$$-2+1 = A(0) + B(0) + C(0) + D(-2-1)^2$$

$$-1 = D9$$

$$\boxed{D = -1/9}$$

Simplyfy equation (ii).

$$x+1 = A(x^2+4+4x)(x-1) + B(x^2+4+4x) + C(x^2-1-2x)(x+2) + D(x^2+1-2x)$$

$$= A(x^3+\cancel{4x}+4x^2-x^2-4-\cancel{4x}) + B(x^2+4x+4) + C(x^3-x-\cancel{2x^2}+\cancel{2x^2}-2-4x) + D(x^2-2x+1)$$

$$= A(x^3+3x^2-4) + B(x^2+4x+4) + C(x^3-5x-2) + D(x^2-2x+1)$$

$$x+1 = (A+C)x^3 + (3A+B+D)x^2 + (4B-5C-2D)x - 4A + 4B - 2C + D$$

Now Comparing Coffeients

$$x^3 \Rightarrow A + C = 0$$

$$x^2 \Rightarrow 3A + B + D = 0$$

$$3A + \frac{2}{9} + \left(-\frac{1}{9}\right) = 0$$

$$3A = \frac{1}{9} - \frac{2}{9}$$

$$= \frac{1-2}{9}$$

$$3A = \frac{-1}{9}$$

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$$\boxed{A = -1/27}$$

$$x^3 \to \quad A + C = 0$$

$$\frac{-1}{27} + C = 0$$

$$\boxed{C = 1/27}$$

putting A, B, C and in equation (i)

$$\frac{x+1}{(x-1)^2(x+2)^2} = \frac{-1/27}{x-1} + \frac{2/9}{(x-1)^2} + \frac{1/27}{(x+2)} + \frac{-1/9}{(x+2)^2}$$

by Integration.

$$\frac{-1}{27}\int\frac{1}{(x-1)}dx + \frac{2}{9}\int\frac{1}{(x-1)^2}dx + \frac{1}{27}\int\frac{dx}{(x+2)} \frac{-1}{9}\int\frac{dx}{(x+2)^2}$$

$$= \frac{-1}{27}\ln|x-1| + \frac{2}{9}\frac{(x-1)^{-2+1}}{-2+1} + \frac{1}{27}\ln|x+2| - \frac{1}{9}\frac{(x+2)^{-2+1}}{-2+1} + C$$

$$= \frac{-2}{9(x-1)} + \frac{1}{9(x+2)} + \frac{1}{27}\ln\left|\frac{x+2}{x-1}\right| + C \text{ Ans.}$$

-2)

8 $$\int\frac{1}{(1-x^3)}dx$$

$$\frac{1}{(1-x^3)} = \frac{1}{(1-x)(1+x+x^2)}$$

$$= \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2} \quad \text{------ (i)}$$

Multiply by L.C.M.

$$1 = A(x^2+x+1) + Bx + C(1-x) \quad \text{------ (ii)}$$

putting $1-x = 0 \Rightarrow 1 = x$ in (ii)

$$1 = A(1+1+1) + 0$$

$$\boxed{A = 1/3}$$

simplyfiy equation (ii)

$1 = Ax^2 + Ax + A + Bx - Bx^2 + c - cx$

$1 = (A - B)x^2 + (A + B - C)x + A + C$

comparing coffients

$x^2 \Rightarrow \quad A - B = 0$

$\frac{+1}{3} - B = 0$

$\boxed{B = 1/3}$

$x^0 \Rightarrow \quad A + C = 1$

$C = 1 - A$

$C = 1 - \frac{1}{3}$

$= \frac{3 - 1}{3} = \frac{2}{3}$

$\boxed{C = 2/3}$

putting A, B and C in equation i)

$\frac{1}{(1-x^3)} = \frac{A}{1-x} + \frac{Bx + C}{x^2 + x + 1}$

$\frac{1}{3}\int (1-x)^{-1} dx + \frac{1}{3}\int \frac{x + 2}{x^2 + x + 1} dx$

$-\frac{1}{3}\int \frac{1}{(1-x)}(-dx) + \frac{1}{6}\int \frac{2x + 4}{x^2 + x + 1} dx$

$-\frac{1}{3}\ln|1 - x| + \frac{1}{6}\int \frac{2x + 1 + 3}{x^2 + x + 1}$

$-\frac{1}{3}\ln|1 - x| + \frac{1}{6}\int \frac{2x + 1}{x^2 + x + 1} dx + \frac{1}{6}\int \frac{3}{x^2 + x + 1} dx$

$-\frac{1}{3}\ln|-x + 1| + \frac{1}{6}\ln|x^2 + x + 1| + \frac{1}{2}\int \frac{1}{x^2 + x + 1} dx.$

let

$I_1 = \int \frac{1}{x^2 + x + 1} dx.$

$\int \frac{1}{x^2 + x + (1/2)^2 - (1/2) + 1} dx$

(9)

$$= \int \frac{1}{(x+1/2)^2 + (\sqrt{3}/2)^2}$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)$$

$$I_1 = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

putting in (iii)



$$-\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \frac{1}{2} \times \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$-\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \text{ Ans.}$$

(9) $$\int \frac{x^2+1}{x^3+1}\,dx$$

$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)}$$

$$= \frac{A}{(x+1)} + \frac{Bx+C}{x^2-x+1} \quad \text{--- (i)}$$

Multiplying by L.C.M.

$$x^2+1 = A(x^2-x+1) + Bx+C(x+1) \quad \text{--- (ii)}$$

putting $x+1=0 \Rightarrow x=-1$

$$(-1)^2+1 = A(1-(-1)+1) + 0$$

$$2 = 3A$$

$$\boxed{A = 2/3}$$

Simplyfiy equation (ii)

$$x^2+1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$x^2+1 = (A+B)x^2 + (-A+B+C)x + A + C$$

Comparing coffients of

$x^2 \Rightarrow A + B = 1$

$$B = 1 - A$$

$$B = 1 - \frac{2}{3}$$

$$= \frac{3-2}{3} = \frac{1}{3}$$

$$\boxed{B = 1/3}$$

$x^0 \Rightarrow \quad A + C = 1$

$$-C = A - 1$$

$$-C = \frac{2}{3} - 1$$

$$-C = \frac{2-3}{3}$$

$$-C = \frac{-1}{3}$$

$$\boxed{C = 1/3}$$

putting A, B and C in equation (i)

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$$\frac{x^2+1}{x^3+1} = \frac{2}{3(x+1)} + \frac{x+1}{3(x^2-x+1)}$$

by integration.

$$\int \frac{x^2+1}{x^3+1}\,dx = \frac{2}{3}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{2x+2}{x^2-x+1}\,dx$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1+2+1}{x^2-x+1}\,dx$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1} + \frac{1}{6}\int \frac{3}{x^2-x+1}$$

$$\frac{2}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{1}{2}\int \frac{dx}{x^2-x+1} \quad \text{(iii)}$$

Now let

$$I_1 = \int \frac{dx}{x^2-x+1}$$

$$\int \frac{dx}{x^2-x+1+(1/2)^2-(1/2)^2}$$

$$\int \frac{dx}{(x-1/2)^2+3/4}$$

$$\int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

putting $I_1$ in (iii)

$$\frac{2}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

$$= \frac{2}{3}\ln|x+1| + \frac{1}{6}\ln|x^2-x+1| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) \text{ Ans.}$$

10

$$\int \frac{1}{(x-1)(x^2+4)}\,dx$$

$$\frac{1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \quad \text{---(i)}$$

Multiply by L.C.M.

$$1 = A(x^2+4) + Bx+C(x-1) \quad \text{----(ii)}$$

putting $x-1=0 \Rightarrow x=1$ in equation (ii)

$$1 = A(1+4) + 0$$

$$1 = 5A$$



$$\boxed{A = 1/5}$$

Simplyfify equation (ii)

$$1 = Ax^2 + 4A + Bx^2 - Bx + Cx - C$$

$$1 = (A+B)x^2 + (-B+C)x + 4A - C$$

Comparing coffients of

$$x^2 \Rightarrow A+B = 0$$

$$B = -A$$

$$\boxed{B = -1/5}$$

$x \Rightarrow \quad -B + C = 0$

$$C = B$$

$$C = -1/5$$

$$\boxed{C = -1/5}$$

Putting A, B and C in equation (i)

$$\frac{1}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$= \frac{1/5}{x-1} + \frac{x(-1/5)+(-1/5)}{x^2+4}$$

$$= \frac{1}{5(x-1)} - \frac{x+1}{5(x^2+4)}.$$

by integration.

$$\int \frac{1}{(x-1)(x^2+4)} = \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{5}\int \frac{x+1}{x^2+4}\,dx$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\int \frac{2x+2}{x^2+4}\,dx$$

$$-\frac{1}{10}\int \frac{2x}{x^2+4} - \frac{1}{10}\int \frac{2}{x^2+4}\,dx.$$

$$-\frac{1}{10}\ln|x^2+4| - \frac{1}{5}\int \frac{1}{(x)^2+(2)^2}\,dx$$

$$-\frac{1}{10}\ln|x^2+4| - \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{5}\ln|x-1| - \frac{1}{10}\ln|x^2+4| - \frac{1}{10}\tan^{-1}\left(\frac{x}{2}\right) + C \text{ Ans.}$$

$$\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}\,dx$$

$$\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-2x+5)}$$

Multiplying by L.C.M.

$$2x^2-3x-3 = A(x^2-2x+5) + Bx + C(x-1)$$

putting $x-1=0 \Rightarrow x=1$

$$2(1)^2-3(1)-3 = A(1-2+5)+B(0)$$
$$-4 = A(4)$$

$$A=-1$$

Simplyfiy equation (ii)

$$2x^2-3x-3 = Ax^2+Ax+A+Bx+Bx^2+C+Cx$$
$$2x^2-3x-3 = (A+B)x^2+(A+B+C)x+A+C$$

Camparing coffients

$$x^2 \Rightarrow \quad A+B=2$$
$$-1+B=2$$
$$+B=2+1$$

$$B=+3$$

$$x \Rightarrow \quad A+B-C=-3$$
$$-1-3-C=-3$$
$$-4-C=-3$$
$$-C=-3+4$$
$$C=-1$$

$$C=-1$$

Putting A, B and C in (i)

$$\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)} = \frac{-1}{(x-1)} + \frac{(+3)x+(-1)}{x^2-2x+5}$$

by integration

$$\int \frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx = -\int \frac{dx}{x-1} - \int \frac{3x+1}{x^2-2x+5}dx$$

$$\because 3x-2 \Rightarrow 3(x-2/3) \Rightarrow \frac{3}{2}(2x-4/3)$$

$$3x-2 = \frac{3}{2}\left[(2x-2)+(2-4/3)\right]$$ (by adding & subtracting 2)

$$=\frac{3}{2}(2x-2)+\frac{3}{2}\left(2-\frac{4}{3}\right)$$

$$3x-2 = \frac{3}{2}(2x-2)+\frac{3}{2}\left(\frac{6-4}{3}\right) \Rightarrow \frac{3}{2}(2x-2)+\frac{3}{2}\times\frac{2}{3}$$

$$3x-2 = \frac{3}{2}(2x-2)+1$$

$$\int\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx = -\int\frac{1}{x-1}dx + \int\frac{3/2(2x-2)+1}{x^2-2x+5}dx$$

$$= -\ln|x-1| + \frac{3}{2}\int\frac{2x-2}{x^2-2x+5} + 1\int\frac{dx}{x^2-2x+5}$$

$$= -\ln|x-1| + \frac{3}{2}\ln|x^2-2x+5| + \int\frac{dx}{x^2-2(x)(1)+(1)^2-(1)^2+5}$$

$$= -\ln|x-1| + \frac{3}{2}\ln|x^2-2x+5| + \int\frac{dx}{(x-1)^2+(2)^2}$$

$$= \frac{3}{2}\ln|x^2-2x+5| - \ln|x-1| + \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C \text{ Ans.}$$

12 $$\int\frac{1}{x^4+1}dx$$

Multipiying and dividing by 2.

$$\frac{1}{2}\int\frac{2}{x^4+1}dx$$

Now adding and subtracting '$x^2$'

$$= \frac{1}{2}\int\frac{x^2-x^2+2}{x^4+1}dx$$

$$= \frac{1}{2}\int\frac{(x^2+1)-(x^2-1)}{x^4+1}dx$$

$$= \frac{1}{2}\int\frac{x^2+1}{x^4+1}dx - \frac{1}{2}\int\frac{x^2-1}{x^4+1}dx$$

$$\frac{1}{2}\int\frac{x^2(1+1/x^2)}{x^2(x^2+1/x^2)}dx - \frac{1}{2}\int\frac{x^2(1-1/x^2)}{x^2(x^2+1/x^2)}dx$$

$$\frac{1}{2}\int\frac{1+1/x^2}{x^2+1/x^2}dx - \frac{1}{2}\int\frac{1-1/x^2}{x^2+1/x^2}dx$$

let $\frac{1}{2}I_1 - \frac{1}{2}I_2$

Now.

$$I_1 = \int \frac{1 + 1/x^2}{x^2 + 1/x^2}\,dx$$

let $\quad x - \frac{1}{x} = t \text{ -----(i)}$

$$(1 + 1/x^2)dx = dt \text{ -----(ii)}$$

suquaring (i)

$$x^2 + \frac{1}{x^2} - 2 = t^2$$

$$x^2 + \frac{1}{x^2} = t^2 + 2$$

$$I_1 = \int \frac{dt}{t^2 + 2}$$

$$= \int \frac{dt}{(t)^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) \Rightarrow \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x - 1/x}{\sqrt{2}}\right)$$

$$I_1 = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{2}\,x}\right) \text{ -----(iii)}$$

Now

$$I_2 = \int \frac{(1 - 1/x^2)}{(x^2 + 1/x^2)}\,dx$$

$$x + \frac{1}{x} = t \text{ -----(iv)}$$

$$1 - \frac{1}{x^2}\,dx = dt$$

suquaring (iv)

$$x^2 + \frac{1}{x^2} + 2 = t^2$$

$$x^2 + \frac{1}{x^2} = t^2 - 2$$

$$I_2 = \int \frac{dt}{t^2 - 2}$$

$$I_2 = \int \frac{dt}{(t)^2-(\sqrt{2})^2}$$

by formula

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right|$$

Now putting $I_1$ and $I_2$ in $I$

$$I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| \text{ Ans.}$$



13 $$\int \frac{x^4}{x^4+2x^2+1}\,dx$$

It is an improper fraction, so we convert it into proper fraction.

$$\begin{array}{r|l} & \;\;1 \\ x^4+2x^2+1 & \;x^4 \\ & -x^4 \pm 2x^2 \pm 1 \\ & \;\;\;-2x^2-1 \end{array}$$

$$= 1 - \frac{2x^2+1}{x^4+2x^2+1}$$

$$\int \frac{x^4}{x^4+2x^2+1}\,dx = \int 1\,dx - \int \frac{2x^2+1}{(x^2+1)^2}\,dx$$

$$= x - \int \frac{2x^2+1}{(x^2+1)^2}\,dx \qquad (i)$$

Now by partial fraction

$$\frac{2x^2+1}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} \quad (i)$$

$$2x^2+1 = Ax+B(x^2+1) + Cx + D$$
$$2x^2+1 = Ax^3 + Ax + Bx^2 + B + Cx + D$$
$$2x^2+1 = Ax^3 + Bx^2 + (A+C)x + B + D$$

comparing coffients

$x^3 \Rightarrow \quad A = 0$

$$\boxed{A = 0}$$

$x^2 \Rightarrow \quad B = 2$

$$\boxed{B = 2}$$

$x \Rightarrow \quad 0 = A + C$

$$\boxed{C = 0}$$

$x^0 \Rightarrow \quad 1 = B + D \Rightarrow 1 - 2 = D$

$$\boxed{D = -1}$$

putting A, B, C and D in (i)

$$\frac{2x^2+1}{(x^2+1)^2} = \int \frac{2}{x^2+1}\,dx + \int \frac{-1}{(x^2+1)^2}\,dx$$

$$= 2\tan^{-1}x + I_1$$

let

$$I_1 = -\int \frac{1}{(x^2+1)^2}\,dx$$

$$x = \tan\theta$$
$$dx = \sec^2\theta\,d\theta$$

$$I_1 = -\int \frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}$$

$$= -\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= -\int \frac{1}{\sec^2\theta}\,d\theta \Rightarrow -\int \cos^2\theta\,d\theta$$

$$= -\int \frac{1+\cos 2\theta}{2}\,d\theta \Rightarrow \frac{-1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$= \frac{-1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] \Rightarrow \frac{-1}{2}\left[\theta + \sin\theta\cos\theta\right]$$

$$\because x = \tan\theta$$

$$\theta = \tan^{-1}(x)$$

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$\cos\theta = \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{-1}{2}\left[\tan^{-1}(x) + \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right]$$

$$I_1 = \frac{-1}{2}\left[\tan^{-1}(x) + \frac{x}{1+x^2}\right]$$

by putting $I_1$

$$\int \frac{2x^2+1}{(x^2+1)^2}\,dx = 2\tan^{-1}(x) - \frac{1}{2}\tan^{-1}(x) - \frac{x}{2(1+x^2)}$$

$$\text{" " "} = \frac{3}{2}\tan^{-1}(x) - \frac{x}{2(1+x^2)}$$

putting in equation (i)

$$\int \frac{x^4}{x^4+2x^2+1}\,dx = x - \frac{3}{2}\tan^{-1}(x) + \frac{x}{2(1+x^2)} + C \text{ Ans.}$$

$$\int \frac{x^2+1}{x^4-x^2+1}\,dx$$

by taking $x^2$ common from numerator

and denomenator.

$$\int \frac{x^2(1+1/x^2)}{x^2(x^2-1+1/x^2)}\,dx$$

$$\int \frac{(1+1/x^2)}{(x^2-1+1/x^2)}\,dx \Rightarrow \int \frac{(1+1/x^2)}{(x^2+1/x^2-1)}\,dx$$

let $\quad x - \frac{1}{x} = t \quad$ --- (i)

$$\left(1+\frac{1}{x^2}\right)dx = dt$$

NOW squaring (i)

$$x^2 + \frac{1}{x^2} - 2 = t^2$$

$$x^2 + \frac{1}{x^2} - 1 - 1 = t^2$$

$$x^2 + \frac{1}{x^2} - 1 = t^2 + 1$$

NOW

$$\int \frac{(1+1/x^2)\,dx}{(x^2+1/x^2-1)} \Rightarrow \int \frac{dt}{t^2+1}$$

$$= \tan^{-1}(t) + c$$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) + c$$

$$= \tan^{-1}\left(\frac{x^2-1}{x}\right) + c \text{ Ans.}$$

15 $\quad \int \frac{1}{(e^x-1)^2}\,dx$

Ans.

$$e^x = t$$

$$e^x dx = dt$$

$$dx = \frac{1}{e^x}\,dt$$

$$dx = \frac{dt}{t}$$

$$= \int \frac{1}{(t-1)^2}\frac{dt}{t} \Rightarrow \int \frac{1}{t(t-1)^2}\,dt$$

let

$$\frac{1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{(t-1)} + \frac{C}{(t-1)^2} \quad \text{---(i)}$$

Multiplying by L.C.M.

$$1 = A(t-1)^2 + B(t)(t-1) + C(t) \quad \text{---(ii)}$$ 16

put $t=0$ in equation (ii)

$$1 = A(0-1)^2 + 0 + 0$$

$$1 = A(-1)^2$$

$$A = 1$$

put $t-1=0 \Rightarrow t=1$ in equation (2)

$$1 = 0 + 0 + C(1)$$

$$C = 1$$

from (ii)

$$1 = A(t^2-t+1) + B(t^2-t) + Ct$$

$$1 = (A+B)t^2 + (-A-B)t + A$$

by comparing coffients

$$t^2 \Rightarrow \quad A + B = 0$$

$$1 + B = 0$$

$$B = -1$$

putting A, B and C in equation (i)

$$\frac{1}{t(t-1)^2} = \frac{1}{t} + \frac{(-1)}{t-1} + \frac{1}{(t-1)^2}$$

Now by integration.

$$\int \frac{1}{t(t-1)^2}dt = \int \frac{1}{t}dt - \int \frac{1}{t-1}dt + \int \frac{1}{(t-1)^2}dt$$

$$= \ln|t| - \ln|t-1| + \frac{(t-1)^{-2+1}}{-2+1} + c$$

$$= \ln|e^x| - \ln|e^x-1| - (e^x-1)^{-1} + c$$

$$= x - \ln|e^x-1| - \frac{1}{e^x-1} + c \quad \text{Ans.}$$

16 $$\int \frac{dx}{(1+e^x)(1+e^{-x})}$$

$$= \int \frac{dx}{(1+e^x)(1+1/e^x)} \Rightarrow \int \frac{e^x\,dx}{(1+e^x)(1+e^x)}$$

$$\int \frac{e^x}{(1+e^x)^2}\,dx \Rightarrow \int (1+e^x)^{-2} e^x\,dx$$

$$= \frac{(1+e^x)^{-2+1}}{-2+1} + c$$

$$= -\left(\frac{1}{e^x+1}\right) + c \quad \text{Ans.}$$

$$\int \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)(3+\sin x)}$$

let $\sin x = t$

$\cos x\,dx = dt$

$$\int \frac{dt}{(1+t)(2+t)(3+t)}$$

let

$$\frac{1}{(1+t)(2+t)(3+t)} = \frac{A}{1+t} + \frac{B}{2+t} + \frac{C}{3+t} \quad (i)$$

Multiplying by L.C.M.

$1 = A(2+t)(3+t) + B(1+t)(3+t) + C(1+t)(2+t)$

put $1+t = 0 \Rightarrow t = -1$ in equation (ii)

$1 = A(2-1)(3-1) + 0 + C$

$1 = A(2)$

$A = \frac{1}{2}$

$$A = 1/2$$

put $t+2 = 0 \Rightarrow t = -2$

$$1 = 0 + B(-2+1)(3-2) + 0$$

$$1 = B(-1)(1)$$

$$B = -1$$

put $t+3 = 0 \Rightarrow t = -3$

$$1 = 0 + 0 + c(1-3)(2-3)$$

$$1 = c(-2)(-1)$$



$$c = 1/2$$

putting A, B and in equation (i)

$$\frac{1}{(1+t)(2+t)(3+t)} = \frac{1}{2(1+t)} - \frac{1}{2+t} + \frac{1}{2(3+t)}$$

by integration.

$$\int \frac{1}{(1+t)(2+t)(3+t)}\,dt = \frac{1}{2}\int \frac{1}{1+t}\,dt - \int \frac{1}{2+t}\,dt + \frac{1}{2}\int \frac{1}{3+t}\,dt$$

$$= \frac{1}{2}\ln|1+t| - \ln|2+t| + \frac{1}{2}\ln|3+t|$$

$$= \frac{1}{2}\ln|1+\sin x| - \ln|2+\sin x| + \frac{1}{2}\ln|3+\sin x| \text{ Ans.}$$

18

$$\int \frac{\sec x}{1+\operatorname{cosec} x}\,dx$$

$$= \int \frac{1/\cos x}{1 + 1/\sin x} \Rightarrow \int \frac{\sin x}{\cos x(1+\sin x)}\,dx$$

Multipiying and dividing by 'cosx'

$$= \int \frac{\sin x \cos x}{\cos^2 x(1+\sin x)}\,dx$$

$$= \int \frac{\sin x \cos x}{(1-\sin^2 x)(1+\sin x)} dx$$

$$= \int \frac{\sin x \cos x}{(1-\sin x)(1+\sin x)^2} dx$$

$$\Rightarrow \quad \sin x = t$$

$$\cos x \, dx = dt$$

$$\int \frac{t \, dt}{(1+t^2)(1-t)}$$

let

$$\frac{t}{(1-t)(1+t)^2} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{(1+t)^2} \quad \text{---(i)}$$

Multiplying by L.C.M

$$t = A(1+t^2)^2 + B(1+t)(1-t) + C(1-t) \text{ ---(ii)}$$

put $1-t=0 \Rightarrow t=1$

$$1 = A(1+1)^2$$

$$A = 1/4$$

It

put $1+t = 0 \Rightarrow t = -1$

$$-1 = 0 + 0 + C(1-(-1))$$

$$-1 = C(1+1)$$

$$C = -1/2$$

From (ii)

$$t = A + At^2 + B(1-t^2) + C - Ct$$

$$t = A + At^2 + B - Bt^2 + C - Ct$$

$$t = (A-B)t^2 - Ct + (A+B+C)$$

by comparing coffients of $t^2$

$$t^2 \Rightarrow \quad A - B = 0$$

$$\frac{1}{4} = B$$

$$B = 1/4$$

putting the values of A, B and C in equation (1)

$$\frac{t}{(1-t)(1+t)^2} = \frac{1}{4(1-t)} + \frac{1}{4(1+t)} - \frac{1}{2(1+t)^2}$$

by integration.

$$= \frac{-1}{4}\int \frac{-1}{1-t}\,dt + \frac{1}{4}\int \frac{1}{1+t}\,dt - \frac{1}{2}\int \frac{1}{(1+t)^2}\,dt$$

$$= \frac{-1}{4}\ln|1-t| + \frac{1}{4}\ln|1+t| - \frac{1}{2}\frac{(1+t)^{-2+1}}{-2+1}$$

$$= \frac{-1}{4}\ln|1-t| + \frac{1}{4}\ln|1+t| + \frac{1}{2(1+t)} + c$$

$$= \frac{-1}{4}\ln|1-\sin x| + \frac{1}{4}\ln|1+\sin x| + \frac{1}{2(1+\sin x)} + c$$

$$\frac{1}{2(1+\sin x)} + \frac{1}{4}\left[\ln|1+\sin x| - \ln|1-\sin x|\right]$$

$$\frac{1}{2(1+\sin x)} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c \text{ Ans.}$$



9 $$\int \frac{\sin x\,dx}{\sin 3x}$$

$\because \sin 3x = 3\sin x - 4\sin^3 x$

$$= \int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx$$

$$\int \frac{\sin x}{\sin x(3-4\sin^2 x)}\,dx$$

$$= \int \frac{dx}{3-4\sin^2 x}$$

$$= \int \frac{dx}{3(\sin^2 x + \cos^2 x) - 4\sin^2 x}$$

$$= \int \frac{dx}{3\cos^2 x + 3\sin^2 x - 4\sin^2 x}$$

$$= \int \frac{dx}{3\cos^2 x - \sin^2 x}$$

Dividing numinator and demoninator by $\cos^2 x$

$$= \int \frac{1/\cos^2 x \; dx}{3 - \tan^2 x}$$

$$= \int \frac{\sec^2 x \, dx}{3 - \tan^2 x}$$

$$\because \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|$$

let $\tan x = t$

$\sec^2 x \, dx = dt$

$$= \int \frac{dt}{3 - t^2} \Rightarrow \int \frac{dt}{(\sqrt{3})^2 - (t)^2}$$

$$= \frac{1}{2\sqrt{3}} \ln\left|\frac{\sqrt{3} + t}{\sqrt{3} - t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \ln\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + c \quad \text{Ans.}$$

20 $$\int \frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x} dx$$

First we slove

$$\because \tan 3x = \tan(2x + x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{2\tan x/1-\tan^2 x + \tan x}{1 - 2\tan x/1-\tan^2 x \cdot \tan x}$$

$$= \frac{2\tan x + \tan x - \tan^3 x}{1 - \tan^2 x - 2\tan^2 x}$$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

putting in question

$$= \int \frac{\cot x - 3\left(1-3\tan^2 x/3\tan x - \tan^3 x\right)}{3\left(3\tan x - \tan^3 x/1-3\tan^2 x\right) - \tan x}\, dx$$

$$= \frac{\frac{1}{\tan x} - \frac{3(1-3\tan^2 x)}{3\tan x - \tan^3 x}}{\frac{3(3\tan x - \tan^3 x)}{1-3\tan^2 x} - \tan x}$$

$$= \frac{3-\tan^2 x - 3 + 9\tan^2 x / \tan x(3-\tan^2 x)}{9\tan x - 3\tan^3 x - \tan x + 3\tan^3 x / 1 - 3\tan^2 x}$$

$$= \frac{8\tan^2 x}{\tan x(3-\tan^2 x)} \times \frac{1-3\tan^2 x}{8\tan x}$$

$$= \frac{1-3\tan^2 x}{3-\tan^2 x}$$

$$\begin{array}{r|l} & 3 \\ -\tan^2 x + 3 & -3\tan^2 x + 1 \\ & {}_{+}-3\tan^2 x \pm 9 \\ & -8 \end{array}$$

$$= \int 3\, dx - \int \frac{8}{3-\tan^2 x}\, dx$$

19

$$= 3x - 8\int \frac{dx}{(3-\tan^2 x)}$$

$$\tan x = t$$

$$\sec^2 x\, dx = dt$$

$$dx = \frac{dt}{\sec^2 x}$$

$$= \frac{dt}{1+\tan^2 x}$$

$$dx = \frac{dt}{1+t^2}$$

$dx$ $\quad 3x - 8\int \frac{dt}{(3-t^2)(1+t^2)}$

let

$$\frac{1}{(\sqrt{3}-t)(\sqrt{3}+t)(1+t^2)} = \frac{A}{(\sqrt{3}-t)} + \frac{B}{(\sqrt{3}+t)} + \frac{ct+D}{1+t^2}$$

Multiplying by L.C.M.

$$1 = A(\sqrt{3}+t)(1+t^2) + B(\sqrt{3}-t)(1+t^2) + ct + D(\sqrt{3}-t)(\sqrt{3}+t)$$

put $\sqrt{3} - t = 0$

$$t = \sqrt{3}$$

$$1 = A(\sqrt{3}+\sqrt{3})(1+3)$$

$$1 = A(2\sqrt{3})(4)$$

$$\boxed{A = 1/8\sqrt{3}}$$

put $\sqrt{3} + t = 0 \Rightarrow t = -\sqrt{3}$

$$1 = B(\sqrt{3}+\sqrt{3})(1+\sqrt{3})$$

$$1 = B(2\sqrt{3})(4)$$

$$\boxed{B = 1/8\sqrt{3}}$$

From (i)

$$1 = A(\sqrt{3}+\sqrt{3}t^2+t+t^3) + B(\sqrt{3}+\sqrt{3}t^2-t-t^3)$$
$$+ c(3t-t^3) + D(3-t^2)$$

$$1 = (A-B-c)t^3 + (\sqrt{3}A+\sqrt{3}B-D)t^2 + (A-B+3c)t$$
$$+ (\sqrt{3}A+\sqrt{3}B+3D)$$

comparing coffients of $t^3$

$t^3 \Rightarrow$

$$0 = A - B - C$$

$$0 = \frac{1}{8\sqrt{3}} - \frac{1}{8\sqrt{3}} - c$$

$$\boxed{C = 0}$$

$t^2 \Rightarrow \quad 0 = \sqrt{3}A + \sqrt{3}B - D$

$$0 = \sqrt{3}\,(1/8\sqrt{3}) + \sqrt{3}\,(1/8\sqrt{3}) - D$$

$$0 = \frac{2}{8} - D$$

$$\boxed{D = 1/4}$$

$$\int \frac{1}{(\sqrt{3}-t)(\sqrt{3}+t)(1+t^2)}\,dt = \frac{-1}{8\sqrt{3}}\int \frac{-1}{\sqrt{3}-t}\,dt + \frac{1}{8\sqrt{3}}\int \frac{1}{\sqrt{3}+t}\,dt$$

$$+\int \frac{1/4}{1+t^2}$$

$$= \frac{-1}{8\sqrt{3}} \ln|\sqrt{3}-t| + \frac{1}{8\sqrt{3}} \ln|\sqrt{3}+t| + \frac{1}{4}\tan^{-1}(t)$$

$$= \frac{1}{8\sqrt{3}} \ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + \frac{1}{4}\tan^{-1}(t)$$

$$= \frac{1}{8\sqrt{3}} \ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}\tan^{-1}(\tan x)$$

$$= \frac{1}{8\sqrt{3}} \ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}x + c \quad \text{Ans.}$$

$$\int \frac{\cos x}{\sin^2 x + 4\sin x - 5}\,dx$$

19 $\sin x = t$

$\cos x\,dx = dt$

$$= \int \frac{dx}{t^2+4t-5}$$

$\because t^2+4t-5$

$t^2+5t-t-5$

$t(t+5)-1(t+5)$

$(t+5)(t-1)$

$$= \int \frac{dt}{(t+5)(t-1)}$$

$$\frac{1}{(t+5)(t-1)} = \frac{A}{t+5} + \frac{B}{t-1}$$

Multiplying by L.C.M

$$1 = A(t-1) + B(t+5)$$

put $t+5=0 \Rightarrow t=-5$ in (ii)

$1 = A(-5-1) \Rightarrow 1 = -6A$

$$\boxed{A = -1/6}$$

put $t-1=0 \Rightarrow t=1$

$1 = B(1+5) \Rightarrow 1 = 6B$

$$\boxed{B = 1/6}$$

$$\frac{1}{(t+5)(t-1)} = \frac{A}{t+5} + \frac{B}{t-1}$$

$$= \frac{-1}{6(t+5)} + \frac{1}{6(t-1)}$$

by integration

$$\int \frac{1}{(t+5)(t-1)}dt = -\frac{1}{6}\int \frac{dt}{(t+5)} + \frac{1}{6}\int \frac{dt}{(t-1)}$$

$$= \frac{-1}{6}\ln|t+5| + \frac{1}{6}\ln|t-1|$$

$$= \frac{1}{6}\ln\left|\frac{t-1}{t+5}\right| \Rightarrow \frac{1}{6}\ln\left|\frac{\sin x-1}{\sin x+1}\right|$$

$$= \frac{1}{6}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + c \text{ Ans.}$$

$$\int \frac{\sec^2 x}{\tan^3 x - \tan^2 x}dx$$

let

$$\tan x = t$$

$$\sec^2 x\, dx = dt$$

$$= \int \frac{dt}{t^3 - t^2} \Rightarrow \int \frac{dt}{t^2(t-1)}$$

$$\frac{1}{t^2(t-1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-1} \quad \text{---(i)}$$

Multiplying by L.C.M

$$1 = A(t)(t-1) + B(t-1) + C(t^2) \quad \text{---(ii)}$$

put $t-1 = 0 \Rightarrow t = 1$ in (ii)

$$1 = 0 + 0 + C(1)^2$$

$$\boxed{C = 1}$$

From (ii)

$$1 = At^2 - At + Bt - B + Ct^2$$

$$= (A+C)t^2 + (-A+B)t - B$$

Comparing coffients of $t^2$

$t^2 \Rightarrow$

$$0 = A + C$$

$$0 = A + 1$$

$$\boxed{A = -1}$$

$t \Rightarrow$

$$0 = -A + B$$

$$0 = -(-1) + B$$

$$\boxed{B = -1}$$

putting A, B and C in equation (i)

$$\frac{1}{t^2(t-1)} = \frac{-1}{t} + \frac{(-1)}{t^2} + \frac{1}{t-1}$$

by integration

$$\int \frac{dt}{t^2(t-1)} = -\int \frac{1}{t}\,dt - \int \frac{1}{t^2}\,dt + \int \frac{1}{t-1}\,dt$$

$$= -\ln|t| - \frac{t^{-2+1}}{-2+1} + \ln|t-1| + c$$

$$= -\ln|t| - \frac{t^{-1}}{-1} + \ln|t-1| + c$$

$$= -\ln|\tan x| + \frac{1}{\tan x} + \ln|\tan x - 1| + c$$

$$= \cot x + \ln|\tan x - 1| - \ln|\tan x| + c$$

$$= \cot x + \ln\left|\frac{\tan x - 1}{\tan x}\right| + c \text{ Ans.}$$



23 $$\int \frac{x^2+1}{(x^2+2x+3)^2}\,dx$$

let

$$\frac{x^2+1}{(x^2+2x+3)^2} = \frac{Ax+B}{x^2+2x+3} + \frac{Cx+D}{(x^2+2x+3)^2} \quad \text{---(i)}$$

Multiplying by L.C.M.

$$x^2+1 = Ax+B(x^2+2x+3) + Cx+D$$

$$x^2+1 = A(x^3+2x^2+3x) + Bx^2 + 2Bx + 3B + Cx + D$$

$$= Ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx + D$$

$$x^2+1 = Ax^3 + (2A+B)x^2 + (3A+2B+C)x + 3A+3B+D$$

by comparing coffients

$x^3 \Rightarrow \quad 0 = A$

$$\boxed{A=0}$$

$x^2 \Rightarrow$

$$2A + B = 1$$

$$0 + B = 1$$

$$\boxed{B=1}$$

$x \Rightarrow \quad 3A + 2B + C = 0$

$0 + 2(1) + C = 0$

$\boxed{C = -2}$

$x^0 \Rightarrow \quad 1 = 3A + 3B + D$

$1 = 0 + 3(1) + D$

$D = -2$

$$\int \frac{x^2+1}{(x^2+2x+3)^2}dx = \int \frac{dx}{x^2+2x+3} + \int \frac{-2x-2}{(x^2+2x+3)^2}$$

$$= \int \frac{dx}{\cancel{x^2+2x+1}+\sqrt{2}^2} - \int \frac{2x+2}{(x^2+2x+3)^2}dx$$

$$= \int \frac{dx}{(x+1)^2+(\sqrt{2})^2} - \int (2x+2)(x^2+2x+3)^{-2}dx$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) - \frac{(x^2+2x+3)^{-1}}{-1} + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{x^2+2x+3} + c \text{ Ans.}$$

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24 $$\int \frac{x^3+2x^2-3}{(x^2+9)^2}dx$$

$$\frac{x^3+2x^2-3}{(x^2+9)^2} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2}$$

$x^3+2x^2-3 = Ax + B(x^2+9) + Cx + D$

" " " $= Ax^3 + 9Ax + Bx^2 + 9B + Cx + D$

$x^3+2x^2-3 = Ax^3 + Bx^2 + (9A+C)x + 9B + D$

Comparing coffients

$x^3 \Rightarrow \quad 1 = A$

$1 = A$

$x^2 \Rightarrow$

$2 = B$

$x \Rightarrow \quad 0 = 9A + C$

$0 = 9 + C$

$C = -9$

$\therefore \quad C = -9$

$x^0 \Rightarrow \quad -3 = 9B + D$

$9(2) + D = -3$

$18 + D = -3$

$D = -3 - 18$

$D = -21$

$$\frac{x^3 + 2x^2 - 3}{(x^2+9)^2} = \frac{x+2}{(x^2+9)} + \frac{-9x - 21}{(x^2+9)^2}$$

$$\text{" " "} = \frac{x+2}{(x^2+9)} - \frac{9x+21}{(x^2+9)^2}$$

by integration.

1/4/2013

$$\int \frac{x^3+2x^2-3}{(x^2+9)^2}\,dx = \int \frac{x+2}{x^2+9}\,dx - \int \frac{9x+21}{(x^2+9)^2}\,dx$$

$$= \int \frac{x}{x^2+9}\,dx + 2\int \frac{1}{x^2+9}\,dx - \int \frac{9x\,dx}{(x^2+9)^2} - \int \frac{21}{x^2+9}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{(x^2+9)}\,dx + 2\int \frac{1}{x^2+(3)^2}\,dx - \frac{9}{2}\int \frac{2x}{(x^2+9)^2}\,dx - 21\int \frac{1}{x^2+9}$$

$$= \frac{1}{2}\ln|x^2+9| + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\,\frac{(x^2+9)^{-1}}{-1} - 21\int \frac{1}{(x^2+9)^2}\,dx$$

$$\ln\sqrt{x^2+9} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{9}{2(x^2+9)} - 21 I_1 \text{ --- (i)}$$

let

$$I_1 = \int \frac{1}{(x^2+9)^2}\,dx$$

$$x = 3\tan\theta$$
$$dx = 3\sec^2\theta\, d\theta$$
$$\frac{x}{3} = \tan\theta$$

$$I_1 = \int \frac{3\sec^2\theta\, d\theta}{(9\tan^2\theta + 9)^2}$$

$$= \int \frac{3\sec^2\theta\ d\theta}{9^2(\tan^2\theta + 1)^2}$$

$$= \frac{3}{81}\int \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$\frac{1}{27}\int \frac{1}{\sec^2\theta}\, d\theta \Rightarrow \frac{1}{27}\int \cos^2\theta\, d\theta$$

$$\frac{1}{27}\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta \Rightarrow \frac{1}{54}\int (\cos 2\theta + 1)\, d\theta$$

$$\frac{1}{54}\left[\int \cos 2\theta\, d\theta + \int 1\, d\theta\right]$$

$$\frac{1}{54}\left[\frac{\sin 2\theta}{2} + \theta\right] \Rightarrow \frac{1}{54}\left[\frac{2\sin\theta\cos\theta}{2} + \theta\right]$$

$$\frac{1}{54}\left[\sin\theta\cos\theta + \theta\right] \Rightarrow \frac{1}{54}\left[\frac{x}{\sqrt{x^2+9}} \cdot \frac{3}{\sqrt{x^2+9}} + \tan^{-1}\left(\frac{x}{3}\right)\right]$$

$$= \frac{1}{54}\left[\frac{3x}{x^2+9} + \tan^{-1}\left(\frac{x}{3}\right)\right].$$

put $I_1$ in

$$= \ln(x^2+9)^{1/2} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{9}{2(x^2+9)} - \left[\frac{21}{54}\left(\frac{3x}{x^2+9}\right) - \frac{21}{54}\tan^{-1}\left(\frac{x}{3}\right)\right]$$

$$\ln\sqrt{x^2+9} + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + \frac{9}{2(x^2+9)} - \frac{7x}{6(x^2+9)} - \frac{7}{18}\tan^{-1}\left(\frac{x}{3}\right)$$

$$\ln\sqrt{x^2+9} + \left(\frac{2}{3} - \frac{7}{18}\right)\tan^{-1}\left(\frac{x}{3}\right) + \frac{27 - 7x}{6(x^2+9)} + C$$

$$\ln\sqrt{x^2+9} + \frac{5}{18}\tan^{-1}\left(\frac{x}{3}\right) + \frac{27-7x}{6(x^2+9)} + C \quad \text{Ans.}$$

$$\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$$

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

Multiplying by L.C.M

$$3x^4+4x^3+16x^2+20x+9 = A(x^2+3)^2 + Bx+C(x+2)(x^2+3)$$
$$+Dx+E(x+2) \quad \text{---(ii)}$$

put $x+2=0 \Rightarrow x=-2$

$$3(-2)^4+4(-2)^3+16(-2)^2+20(-2)+9 = A((-2)^2+3)^2$$
$$48-32+64-40+9 = A(49)$$
$$49 = A(49)$$

$$A = 1$$



From (ii)

$$3x^4+4x^3+16x^2+20x+9 = A(x^4+9+6x^2) + B(x^3+3x)(x+2)$$
$$+C(x^2+3)(x+2) + Dx + E(x+2)$$
$$A(x^4+9+6x^2) + B(x^4+2x^3+3x^2+6x) + C(x^3$$
$$+2x^2+3x+6) + Dx^2 + 2Dx + Ex + 2E$$
$$= A(x^4+9+6x^2) + B(x^4+2x^3+3x^2+6x) + Cx^3+2Cx^2$$
$$+3Cx+6Cx^2+Dx^2+2Dx+Ex+2E$$
$$= (A+B)x^4 + (2B+C)x^3 + (6A+3B+2C+D)x^2$$
$$+(6B+3C+2D+E)x$$

comparing coffients of $x^4$

$$A+B = 3$$
$$1+B = 3$$
$$B = 2$$

$$B = 2$$

$x^3 \Rightarrow \quad 4 = 2B + C$

$4 = 2(2) + C$

$C = 0$

$x^2 \Rightarrow \quad 16 = 6A + 3B + 2C + D$

$16 = 6(1) + 3(2) + 2(0) + D$

$D = 4$

$x \Rightarrow \quad 20 = 6B + 3C + 2D + E$

$20 = 6(2) + 3(0) + 2(4) + E$

$E = 0$

putting, A, B, C, D and E in equation (i)

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x+0}{x^2+3} + \frac{4x+0}{(x^2+3)^2}$$

$$= \int \frac{1}{x+2}\,dx + \int \frac{2x}{x^2+3}\,dx + \int \frac{4x}{(x^2+3)}\,dx$$

$$= \ln|x+2| + \ln|x^2+3| + \frac{2\,(x^2+3)^{-1}}{-1} + c$$

$$= \ln|x+2| + \ln|x^2+3| - \frac{2}{x^2+3} + c$$