$$= \int \frac{2t \, dt}{3t^2 + 4t}$$

$$= \int \frac{2t \, dt}{t(3t+4)}$$

$$= \frac{2}{3} \int \frac{3}{(3t+4)} \, dt$$

$$= \frac{2}{3} \ln|3t + 4| + c$$

$$= \frac{2}{3} \ln|3\sqrt{\sin x} + 4| + c \text{ Answer}$$

3

Exercise No. 4.3

Evaluate

1. $$\int x . \sec^2 x \, dx$$

Itegration by parts

$$= x \int \sec^2 x \, dx - \int \frac{d}{dx}(x) \int \sec^2 x \, dx$$

$$= x . \tan x - \int 1 (\tan x) dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x} dx$$

$$= x . \tan x + \int \frac{-\sin x}{\cos x} dx$$

$$= x \tan x + \ln|\cos x| + c \text{ Ans.}$$

2. $$\int x \operatorname{cosec}^2 x \, dx$$

$$= x \int \operatorname{cosec}^2 x - \int \frac{d}{dx}(x) \int \operatorname{cosec}^2 x \, dx$$

$$= -x\cot x - \int (-\cot x)(1)\,dx$$

$$= -x\cot x + \int \frac{\cos x}{\sin x}\,dx$$

$$= -x\cot x + \ln|\sin x| + c \quad \text{Ans.}$$

3 $$\int x^n \ln x\,dx$$

$$\ln x \int x^n dx - \int \frac{d}{dx}(\ln x) \int x^n dx$$

$$= \ln x \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \frac{x^{n+1}}{n+1} dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^{n+1-1} dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + c \quad \text{Ans.}$$

$$\int x^2 \tan^{-1}x\,dx$$

$$= \tan^{-1}x \int x^2 dx - \int \frac{d}{dx}(\tan^{-1}x) \int x^2 dx$$

$$= \tan^{-1}x \frac{x^3}{3} - \int \frac{1}{(1+x^2)} \frac{x^3}{3} dx$$

$$= \frac{x^3 \tan^{-1}x}{3} - \frac{1}{3} \int \left(x - \frac{x}{1+x^2}\right) dx \quad \text{by division}$$

$$= \frac{x^3 \tan^{-1}x}{3} - \frac{1}{3} \int x\,dx + \frac{1}{3} \int \frac{x}{1+x^2} dx$$

$$= \frac{x^3 \tan^{-1}x}{3} - \frac{1}{3} \frac{x^2}{2} + \frac{1}{6} \int \frac{2x}{1+x^2} dx$$

$$= \frac{x^3 \tan^{-1}x}{3} - \frac{x^2}{6} + \frac{1}{6} \ln|1+x^2| + c$$

5 $$\int \sec^3 x \, dx$$

$$I = \int \sec x \cdot \sec^2 x \, dx$$

$$= \sec x \int \sec^2 x \, dx - \int \frac{d}{dx}(\sec x) \int \sec^2 x \, dx$$

$$= \sec x \tan x - \int (\sec x \tan x) \tan x \, dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$ 7

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$I = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$I + I = \sec x \tan x + \int \sec x \, dx$$

1

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

$$I = \frac{1}{2}\left[\sec x \tan x + \ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]\right] + c \quad \text{Ans.}$$

6 $$\int \operatorname{cosec}^3 x \, dx$$

$$I = \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x \, dx$$

$$= \operatorname{cosec} x \int \operatorname{cosec}^2 x \, dx - \int \frac{d}{dx}(\operatorname{cosec} x) \int \operatorname{cosec}^2 x \, dx$$ 8

$$= \operatorname{cosec} x(-\cot x) - \int (-\operatorname{cosec} x \cot x)(-\cot x) \, dx$$

2 $$= -\operatorname{cosec} x \cot x + \int -(\operatorname{cosec} x \cot^2 x) \, dx$$

$$= -\cot x \operatorname{cosec} x \, dx - \int \operatorname{cosec} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= -\operatorname{cosec}x\cot x - \int \operatorname{cosec}^3 x + \int \operatorname{cosec}x\,dx$$

$$I + I = -\operatorname{cosec}x\cot x + \ln|\operatorname{cosec}x - \cot x|$$

$$2I = -\operatorname{cosec}x\cot x + \ln|\tan(x/2)| + c$$

$$I = -\frac{1}{2}\operatorname{cosec}x\cot x + \frac{1}{2}\ln|\tan(x/2)| + c$$

7 $$\int \frac{x - \sin x}{1 - \cos x}\,dx$$

$$I = \int \frac{x - 2\sin x/2\cos x/2}{2\sin^2 x/2}\,dx$$

$$= \int \frac{x}{2\sin^2 x/2}\,dx - \int \frac{\sin x/2\cos x/2}{\sin^2 x/2}\,dx$$

$$= \frac{1}{2}\int x\operatorname{cosec}^2 x/2 - \int \frac{\cos x/2}{\sin x/2}\,dx$$

$$= \frac{1}{2}\left[x\left(-\frac{\cot x/2}{1/2}\right) - \int (1)\left(-\frac{\cot x/2}{1/2}\right)\right] - \int \cot\left(\frac{x}{2}\right)$$

Ans. $$= \frac{\cancel{2}}{\cancel{2}}\left[-x\cot\left(\frac{x}{2}\right) + \int \cot\left(\frac{x}{2}\right)\right] - \int \cot\left(\frac{x}{2}\right)$$

$$= -x\cot\left(\frac{x}{2}\right) + \int \cot\left(\frac{x}{2}\right) - \int \cot\left(\frac{x}{2}\right)$$

$$= -x\cot\left(\frac{x}{2}\right) + c \text{ Ans.}$$

8 $$\int x\sin^{-1}x\,dx$$

$\sec^2 x dx$

$$I = \sin^{-1}x\int x\,dx - \int \frac{d}{dx}(\sin^{-1}x)\int x\,dx$$

$\cot x)dx$

$$= \sin^{-1}x\,\frac{x^2}{2} - \int \frac{x^2}{2}\,\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

x

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2}\,dx - \frac{1}{2}(\sin^{-1}x) \text{-----(i)}$$

Now

let $I_1 = \int\sqrt{1-x^2}\,dx$

$x = a\sin\theta$

$x = (1)\sin\theta$

$dx = \cos\theta\, d\theta$

$$= \int\sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= \int \cos\theta\cdot\cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\left[\int 1\, d\theta + \int \cos 2\theta\, d\theta\right.$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

$$= \frac{1}{2}\left[\sin^{-1}(x) + \sin\theta\cos\theta\right] \quad \because \sin\theta = \sqrt{1-\cos^2\theta}$$

$\cos\theta = \sqrt{1-\sin^2\theta}$

$$= \frac{1}{2}\left[\sin^{-1}x + x\sqrt{1-x^2}\right]$$

$$I_1 = \frac{1}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{2}$$

put in (i)

$$I = \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{1}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{2}\right] - \frac{1}{2}\sin^{-1}x$$

$$= \frac{x^2}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{2}\sin^{-1}x$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4}$$

$$= \sin^{-1}x\left[\frac{x^2}{2} - \frac{1}{4}\right] + \frac{x\sqrt{1-x^2}}{4}$$

$$= \sin^{-1}x\left[\frac{2x^2-1}{4}\right] + \frac{x\sqrt{1-x^2}}{4} + c \text{ Ans.}$$

9 $$\int \sqrt{x^2+1}\; x^3\,dx$$

$$I = \int \sqrt{x^2+1}\; x^2 \cdot x\,dx$$

$$= \int \sqrt{x^2+1}\;(x^2+1-1)\,x\,dx$$

$$= (x^2+1)^{3/2} x\,dx - \int \sqrt{x^2+1}\; x\,dx$$

$$= \frac{1}{2}\int (x^2+1)^{3/2} 2x - \frac{1}{2}\int (x^2+1)^{1/2} 2x\,dx$$

$$= \frac{1}{2}\frac{(x^2+1)^{5/2}}{5/2} - \frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2} + c$$

$$= \frac{2}{5}\times\frac{1}{2}(x^2+1)^{5/2} - \frac{1}{2}\times\frac{2}{3}(x^2+1)^{3/2} + c$$

$$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + c \quad \text{Ans.}$$

10 $$\int e^x\,\frac{1+x\ln x}{x}\,dx$$

formula:-

$$e^x[f(x)+f'(x)]dx = e^x f(x)+c$$

$$\int e^x\left[\frac{1}{x}+\ln x\right]dx$$

$$\because f(x) = \ln x$$ 12

$$= e^x \ln x + c \text{ Ans.}$$

OR

$$= \int e^x\left(\frac{1}{x}+\ln x\right)dx$$

$$= \int e^x \frac{1}{x}dx + \int e^x \ln x\, dx$$

$$= \int e^x \frac{1}{x}dx + \left[\ln x\, e^x - \int \frac{1}{x} e^x dx\right]$$

$$= e^x \frac{1}{x}dx + \ln x\, e^x - \int e^x \frac{1}{x}dx$$

$$= e^x \ln x + c \text{ Ans.}$$

11 $$\int e^x \frac{1-\sin x}{1-\cos x}dx$$

$$= \int e^x\left[\frac{1-2\sin x/2 \cos x/2}{2\sin^2 x/2}\right]dx$$

$$= \int e^x\left[\frac{1}{2\sin^2 x/2} - \frac{2\sin x/2 \cos x/2}{2\sin^2 x/2}\right]dx$$

$$= \int e^x\left(\frac{1}{2}\operatorname{cosec}^2 x/2 - \cot x/2\right)dx$$

$$= \frac{1}{2}\int e^x \operatorname{cosec}^2 x/2\, dx - \int e^x \cot\frac{x}{2}dx$$

$$= \frac{1}{2}\, e^x\left[\frac{-\cot x/2}{1/2}\right] - \int e^x\left(\frac{-\cot x/2}{1/2}\right)\Bigg] - \int e^x \cot\frac{x}{2}$$

$$= -e^x\cot\frac{x}{2} + \int e^x\cot\frac{x}{2}\,dx - \int e^x\cot\frac{x}{2}\,dx$$

$$= -e^x\cot\left(\frac{x}{2}\right) + c \quad \text{Ans.}$$

12 $$\int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

$$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx$$

$$x = \cos\theta$$

$$\Rightarrow \theta = \cos^{-1}x$$

$$x = \cos\theta$$

$$dx = -\sin\theta\,d\theta$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-x^2}$$

$$= \int \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,(-\sin\theta\,d\theta)$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\,(-\sin\theta\,d\theta)$$

$$= \int \tan^{-1}\left(\tan\frac{\theta}{2}\right)(-\sin\theta\,d\theta)$$

$$= \int -\frac{\theta}{2}\sin\theta\,d\theta$$

$$= \frac{-1}{2}\int \theta\sin\theta\,d\theta$$

$$= \frac{-1}{2}\left\{\theta(-\cos\theta) - \int (1)(-\cos\theta)\,d\theta\right.$$

$$= \frac{-1}{2}\left\{-\theta\cos\theta + \int \cos\theta\,d\theta\right.$$

$$= \frac{-1}{2}\int \cos\theta\,d\theta \cancel{+ \frac{1}{2}(\theta\cos\theta)}$$

$$= \frac{-1}{2}\sin\theta + \frac{1}{2}\theta\cos\theta$$

$$= -\frac{\sqrt{1-x^2}}{2} + \frac{1}{2}x\cos^{-1}x$$

$$= \frac{x\cos^{-1}x}{2} - \frac{\sqrt{1-x^2}}{2} + c \quad \text{Ans.}$$

13 Let $I = \int \sin^{-1}\sqrt{\frac{x}{x+a}}\,dx$

$$x = a\tan^2\theta$$

$$dx = 2a\tan\theta\sec^2\theta\,d\theta \qquad \because (\sqrt{x})^2 + (\sqrt{a})^2$$

$$x = a\tan^2\theta$$

$$= \int \sin^{-1}\left(\frac{a\tan^2\theta}{a\tan^2\theta + a}\right)^{1/2} \cdot 2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\left(\frac{\tan^2\theta}{\sec^2\theta}\right)^{1/2} 2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\left(\frac{\sin^2\theta/\cos^2\theta}{1/\cos^2\theta}\right)^{1/2} 2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int \sin^{-1}\sin\theta \cdot 2a\tan\theta\sec^2\theta\,d\theta$$

$$= 2a\int \theta\tan\theta \cdot \sec^2\theta\,d\theta$$

$$= 2a\left[\theta \cdot \frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta\,d\theta(1)}{2}\right.$$

14

$$= 2a\left[\frac{\theta\tan^2\theta}{2} - \frac{1}{2}\int(\sec^2\theta - 1)\,d\theta\right]$$

$$= 2a\left[\frac{\theta\tan^2\theta}{2} - \frac{1}{2}(\tan\theta - \theta)\right]$$

$$= a\theta\tan^2\theta - a(\tan\theta - \theta) + c$$

$$\tan^2\theta = \frac{x}{a} \Rightarrow \tan\theta = \sqrt{\frac{x}{a}}$$

$$\theta = \tan^{-1}\sqrt{\frac{x}{a}}$$

$$= a\left(\frac{x}{a}\right)\tan^{-1}\sqrt{\frac{x}{a}} - a\left[\sqrt{\frac{x}{a}} - \tan^{-1}\sqrt{\frac{x}{a}}\right]$$

$$x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + c$$

$$I = (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c \quad \text{Ans.}$$

14 $$\int e^{ax}\sin(bx+c)\,dx$$

Let

$$I = \int e^{ax}\sin(bx+c)\,dx$$

$$= e^{ax}\left(\frac{-\cos(bx+c)}{b}\right) - \int ae^{ax}\left(\frac{-\cos(bx+c)}{b}\right)dx$$

$$= \frac{-e^{ax}\cos(bx+c)}{b} + \frac{a}{b}\int e^{ax}\cos(bx+c)\,dx$$

$$= \frac{-e^{ax}\cos(bx+c)}{b} + \frac{a}{b}\left[e^{ax}\left(\frac{+\sin(bx+c)}{b}\right) - \int ae^{ax}\frac{\sin(bx+c)}{b}\right.$$

$$= \frac{-e^{ax}\cos(bx+c)}{b} + \frac{a}{b^2}e^{ax}\sin(bx+c) - \frac{a^2}{b^2}\int e^{ax}\sin(bx+c)$$

$$I + \frac{a^2}{b^2}I = \frac{e^{ax}}{b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$I\left[1 + \frac{a^2}{b^2}\right] = \frac{e^{ax}}{b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$= \frac{b^2}{a^2+b^2} \frac{e^{ax}}{b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$$ 16

let $a = r\cos\theta$; $b = r\sin\theta$

by suquaring and adding, we get

$$r = \sqrt{a^2+b^2}$$

and $\theta = \tan^{-1}(b/a)$

so

$$I = \frac{e^{ax}}{a^2+b^2}\left[r\cos\theta\sin(bx+c) - r\sin\theta\cos(bx+c)\right]$$

$$= \frac{re^{ax}}{a^2+b^2}\left[\sin(bx+c-\theta)\right] + c$$

$$= \frac{\sqrt{a^2+b^2}\,e^{ax}}{a^2+b^2}\left[\sin(bx+c - \tan^{-1}(b/a))\right]$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\left[\sin(bx+c-\tan^{-1}(b/a))\right] + c$$

15 $$I = \int \ln(x+\sqrt{1+x^2})\,dx$$

$$= \ln(x+\sqrt{1+x^2})(x) - \int\left(\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\right)x\,dx$$

$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{\sqrt{1+x^2}+x/\sqrt{1+x^2}}{x+\sqrt{1+x^2}}\,x\,dx$$ 17

$$= x\ln(x+\sqrt{1+x^2}) - \int\frac{1}{\sqrt{1+x^2}}\,x\,dx$$

$$= x\ln(x+\sqrt{1+x^2}) - \frac{1}{2}\int\frac{1}{\sqrt{1+x^2}}\,2x\,dx$$

$$= x\ln(x+\sqrt{1+x^2}) - \frac{1}{2}\,\frac{(1+x^2)^{-1/2+1}}{-1/2+1} + c$$

$I = x\ln(x+\sqrt{1+x^2}) - (1+x^2)^{1/2} + c$ Ans.

16 $$\int \frac{x^2+1}{(x+1)^2} e^x dx$$

$$= \int e^x \left[\frac{x^2+1+2x-2x}{(x+1)^2}\right] dx$$

$$= \int e^x \left[\frac{(x+1)^2 - 2x}{(x+1)^2}\right] dx$$

$$= \int e^x \left[1 - \frac{2x}{(x+1)^2}\right] dx$$

$$= \int e^x dx - 2\int e^x \frac{x}{(x+1)^2} dx$$

$$= e^x - 2\int e^x \left[\frac{x+1-1}{(x+1)^2}\right] dx$$

$$= e^x - 2\int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] dx$$

Now

$$f(x) = \frac{1}{x+1} \quad ; \quad f'(x) = \frac{-1}{(x+1)^2}$$

$$= e^x - 2e^x\left(\frac{1}{x+1}\right)$$

$$= e^x\left[1 - \frac{2}{x+1}\right] \Rightarrow e^x\left\{\frac{x+1-2}{x+1}\right\}$$

$$= e^x\left(\frac{x-1}{x+1}\right) + c \text{ Ans:}$$

17 $$\int \cos(\ln x)\, dx$$

Let

$$I = \int \cos(\ln x)\, dx$$

$$\ln x = t$$

$$x = e^t$$

$$dx = e^t dt$$

$$\Rightarrow \int \cos t\, e^t dt$$

$$\Rightarrow \int \cos t \cdot e^{t} dt$$

$$e^{t} \times \sin t - \int \sin t \, e^{t} \Rightarrow e^{t} \sin t - \int e^{t} \sin t$$

$$e^{t} \sin t - \left[e^{t}(-\cos t) - \int (-\cos t) e^{t}\right]$$

$$= e^{t} \sin t + e^{t} \cos t - \int e^{t} \cos t$$

$$I + I = e^{t} \sin t + e^{t} \cos t$$

$$2I = e^{t} \sin t + e^{t} \cos t$$

$$I = \frac{1}{2}\left[e^{t} \sin t + e^{t} \cos t\right].$$

$$= \frac{e^{t}(\sin t + \cos t)}{2}$$

$$I = \frac{x}{2}\left[\sin(\ln x) + \cos(\ln x)\right] + c \text{ Ans.}$$

18 $$\int \sqrt{x}\, e^{-\sqrt{x}} dx$$ 19

let $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$dx = 2\sqrt{x}\, dt$$

$$dx = 2t\, dt$$

$$\int \sqrt{x}\, e^{-\sqrt{x}} dx = \int t e^{-t} 2t\, dt$$

$$= 2\int t^{2} e^{-t} dt \cdot$$

$$= 2\int t^{2} e^{-t} dt$$

$$= 2\left[t^{2} \int e^{-t} dt - \int \frac{d}{dt}(t^{2}) \int e^{-t} dt) dt)\right]$$

$= 2\left[t^2(-e^{-t}) - \int 2t \cdot (e^{-t})\,dt\right]$

$= 2\left[-t^2e^{-t} + \int (2te^{-t})\,dt\right]$

$2\left[-e^{-t}t^2 + 2\left[t\int e^{-t}dt - \int \frac{d}{dt}(t)\int e^{-t}dt\right)\right.$

$= -2t^2e^{-t} + 4\left[t(-e^{-t}) - \int -e^{-t}dt\right]$

$= -2t^2e^{-t} + 4\left[-te^{-t} + \int e^{-t}dt\right]$

$= -2t^2e^{-t} - 4te^{-t} + 4(-e^{-t})$

$= -2t^2e^{-t} - 4te^{-t} - 4e^{-t}$

$= -2e^{-t}(+t^2 + 2t + 2) + c$

$= -2e^{-\sqrt{x}}(x + 2\sqrt{x} + 2) + c$ Ans.

19 $\int x^3 e^{2x} dx$

$\frac{x^3e^{2x}}{2} - \int \left(\frac{e^{2x}}{2} 3x^2\right)dx$

$\frac{x^3}{2}e^{2x} - \frac{3}{2}\int x^2 e^{2x} dx$

$\frac{x^3}{2}e^{2x} - \frac{3}{2}\left[x^2\int e^{2x}dx - \int\left(\frac{d}{dx}(x^2)\int e^{2x}dx\right)dx\right.$

$= \frac{x^3}{2}e^{2x} - \frac{3}{2}\frac{x^2e^{2x}}{2} + \frac{3}{2}\left[x\int e^{2x}dx - \int\left(\frac{d}{dx}(x)\int e^{2x}dx\right)dx\right.$

$\frac{x^3}{2}e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{2}\left[x\frac{e^{2x}}{2} - \int 1\,\frac{e^{2x}}{2}dx\right]$

$\frac{x^3}{2}e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{2}\frac{1}{2}\int e^{2x}dx$

$\frac{x^3}{2}e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{4}\int e^{2x}dx$

$$= e^{2x}\left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8}\right) + c$$

$$= e^{2x}\left(\frac{4x^3 - 6x^2 + 6x - 3}{8}\right) + c$$

20 $$\int x^5 e^{x^3} dx$$ (b)

$$= \int x^3 e^{x^3} x^2 dx$$

$$= \frac{1}{3}\int x^3 e^{x^3} (3x^2) dx$$

let $x^3 = t$

$3x^2 dx = dt$

$$\Rightarrow \frac{1}{3}\int x^3 e^{x^3}(3x^2)dx = \frac{1}{3}\int t \cdot e^t dt$$

$$= \frac{1}{3}\left[t\int e^t dt - \int\left(\frac{d}{dt}(t)\int e^t dt\right)dt\right.$$

$$= \frac{1}{3}\left[te^t - \int e^t dt\right]$$

$$= \frac{1}{3}(te^t - e^t) + c$$

$$= \frac{1}{3}(x^3 e^{x^3} - e^{x^3}) + c$$ 22

$$= \frac{1}{3}(x^3 e^{x^3} - e^{x^3}) + c \text{ Ans.}$$

$$\int x^n \tan^{-1}x \, dx$$

$$I = \tan^{-1}x \int x^n dx - \int\left(\frac{d}{dx}(\tan^{-1}x)\int x^n dx\right)dx$$

$$= \tan^{-1}x \frac{x^{n+1}}{n+1} - \int\left[\left(\frac{1}{1+x^2}\right)\frac{x^{n+1}}{n+1}dx\right]$$

$$I = \frac{x^{n+1}}{n+1}\tan^{-1}x - \frac{1}{n+1}\int \frac{x^{n+1}}{n+1}\,dx$$

Hence proved.

(b) Now evaluate

$$\int x^3 \tan^{-1}x\,dx \qquad \text{put } n = 3$$

$$= \frac{x^{3+1}}{3+1}\tan^{-1}x - \frac{1}{3+1}\int \frac{x^{3+1}}{1+x^2}\,dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int\left[(x^2-1) + \frac{1}{x^2+1}\right]dx$$

$$\begin{array}{r|l} & x^2-1 \\ 1+x^2 & x^4 \\ & -x^4 \pm x^2 \\ & -x^2 \\ & \mp x^2 \mp 1 \\ & 1 \end{array}$$

$$= \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1}x\right] + c$$

$$= \left(\frac{x^4}{4} - \frac{1}{4}\right)\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c$$

$$= \left[\frac{x^4-1}{4}\right]\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} + c$$

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$$\int x^n e^{ax}\,dx$$

$$= x^n\int e^{ax}\,dx - \int\left(\frac{d}{dx}(x^n)\int e^{ax}\,dx\right)dx$$

$$= x^n\frac{e^{ax}}{a} - \int\left(nx^{n-1}\frac{e^{ax}}{a}\right)dx$$

Now

$$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \quad \text{(Reduction formula)}$$

putting $n = 3$

$$\int x^3 e^{ax}\,dx = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

Putting $n=2$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[\frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x^1 e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^2}\left[\frac{x e^{ax}}{a} - \frac{1}{a}\int x^0 e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^3}x e^{ax} - \frac{6}{a^3}\int e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^3}x e^{ax} - \frac{6}{a^3}\frac{e^{ax}}{a} + c$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^3}x e^{ax} - \frac{6e^{ax}}{a^4} + c$$ 24

$$\int x^3 e^{ax}\,dx = \frac{e^{ax}}{a^4}\left(a^3x^3 - 3a^2x^2 + 6ax - 6\right) + c \quad \text{Ans.}$$

Find reduction formulas for

23 $$\int \sin^n x\,dx$$

$$I = \int \sin^{n-1+1} x\,dx$$

$$= \int \sin^{n-1}x\,\sin x\,dx$$

$$= \sin^{n-1}x\int \sin x\,dx - \int\left[\frac{d}{dx}(\sin^{n-1}x)\int \sin x\,dx\right]dx$$

$$= -\sin^{n-1}x\cos x - \int (n-1)\sin^{n-2}x\,.\cos x(-\cos x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,\cos^2 x\,dx$$

$$= -\sin^{n-1}\cos x + (n-1)\int \sin^{n-2}x\,(1-\sin^2 x)\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)I$$

$$I + (n-1)I = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$I(1+n-1) = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$I = \frac{-\sin^{n-1}x\cos x}{n} + \frac{(n-1)}{n}\int \sin^{n-2}x\,dx$$

This is Reduction formula.

24 Find a reduction formula $\int x^n \sin ax\,dx$ where $n>1$ is integer. Evaluate $\int x^4 \sin 4x\,dx$

$$I = \int \cos^{n-1+1}x\,dx$$

$$= \int (\cos^{n-1}x\cos x)\,dx$$

$$= \cos^{n-1}x\int \cos x\,dx - \int\left(\frac{d}{dx}(\cos^{n-1}x)\int \cos x\,dx\right)dx$$

$$= \cos^{n-1}x\sin x + \int (n-1)\cos^{n-2}x(-\sin x)(\sin x)\,dx$$

$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx$$

$$= \cos^{n-1}\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}\sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$I + (n-1)I = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$I(1+n-1) = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$I = \frac{\cos^{n-1}x\sin x}{n} + \frac{(n-1)}{n}\int \cos^{n-2}x\,dx.$$

So this is reduction formula.

24 Find a reductio formula for $\int x^n \sin ax\,dx$, where $n \geq 1$ is integar. Hence evaluate $\int x^4 \sin 4x\,dx$

$$= x^n \int \sin ax\,dx - \int\left(\frac{d}{dx}(x^n)\int \sin ax\,dx\right)dx$$

$$= x^n\left(-\frac{\cos ax}{a}\right) - \int\left(nx^{n-1}\left(-\frac{\cos ax}{a}\right)\right)dx$$

$$= -\frac{x^n \cos ax}{a} + \frac{n}{a}\int x^{n-1}\cos ax\,dx$$

$$= -\frac{x^n \cos ax}{a} + \frac{n}{a}\left[x^{n-1}\frac{\sin ax}{a} - \int (n-1)x^{n-2}\frac{\sin ax}{a}\,dx\right]$$

$$= -\frac{x^n \cos ax}{a} + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx$$

This is reduction formula.

putting $n = 4$; $a = 4$

$$\int x^4 \sin 4x\,dx = -\frac{x^4 \cos 4x}{4} + \frac{4}{16}x^{4-1}\sin 4x - \frac{4(4-1)}{16}\int x^{4-2}\sin 4x\,dx$$

$$= -\frac{x^4 \cos 4x}{4} + \frac{1}{4}x^3 \sin 4x - \frac{12}{16}\int x^2 \sin 4x\,dx$$

$$= -\frac{x^4 \cos 4x}{4} + \frac{1}{4}x^3 \sin 4x - \frac{12}{16}\left[-\frac{x^2\cos 4x}{4} + \frac{2}{16}x^{2-1}\sin 4x - \frac{2(1)}{16}\int x^{2-2}\sin 4x\,dx\right]$$

$$= -\frac{x^4 \cos 4x}{4} + \frac{1}{4}x^3 \sin 4x + \frac{3}{16}x^2\cos 4x - \frac{24}{256}x \sin 4x + \frac{24}{256}\left(-\frac{\cos 4x}{4}\right) + c$$

$$= -\frac{x^4 \cos 4x}{4} + \frac{x^3}{4}\sin 4x + \frac{3}{16}x^2 \cos 4x - \frac{3x}{32}\sin 4x - \frac{3}{128}\cos 4x + c$$

$$\int x^4 \sin 4x\,dx = -\frac{x^4}{4}\cos 4x + \frac{x^3}{4}\sin 4x + \frac{3x^2}{16}\cos 4x - \frac{3x}{32}\sin 4x - \frac{3}{128}\cos 4x + C$$ Ans

25 Find a reduction formula for $\int x^m (\ln x)^n dx$
Hence evaluate $\int x^3 (\ln x)^2 dx$

$$I = (\ln x)^n \int x^m dx - \int\left(\frac{d}{dx}(\ln x)^n \int x^m dx\right) dx$$

$$= (\ln x)^n \frac{x^{m+1}}{m+1} - \int\left(n(\ln x)^{n-1} \cdot \frac{1}{x} \frac{x^{m+1}}{m+1}\right) dx$$

$$= \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int (\ln x)^{n-1} x^m dx$$

$$\int x^m (\ln x)^n dx = \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1}\int (\ln x)^{n-1} x^m dx \quad (R.F)$$

putting $m = 3$; $n = 2$

$$\int x^3 (\ln x)^2 dx = \frac{(\ln x)^2 x^{3+1}}{3+1} - \frac{2}{3+1}\int (\ln x)^{2-1} x^3 dx$$

$$= \frac{(\ln x)^2 x^4}{4} - \frac{2}{4}\int (\ln x)^1 x^3 dx$$

Now $n = 1$, $m = 3$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{1}{2}\left[\frac{(\ln x) x^{3+1}}{3+1} - \frac{1}{3+1}\int (\ln x)^{1-1} x^3 dx\right]$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{(\ln x) x^4}{2\times 4} + \frac{1}{2\times 4}\int x^3 (\ln x)^0 dx$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{8}\int x^3 dx$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{8} \times \frac{x^4}{4} + c$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{x^4}{32} + c$$

$$\int x^3 (\ln x)^2 dx = \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{x^4}{32} + c \quad \text{Ans.}$$