Higher derivatives:-

Let $y = f(x)$

then its first, second, third, ... nth derivatives are denoted by

$\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, ..., $\frac{d^ny}{dx^n}$

y_1, y_2, y_3, ..., y_n

y', y'', y''', ..., y^n

$f'(x)$, $f''(x)$, $f'''(x)$, ..., $f^n(x)$

e.g., Let $y = x^4 + 2x^3 + 3x^2 + 7x + 5$

Diff. w.r.t. x successively

$y_1 = 4x^3 + 6x^2 + 6x + 7$

$y_2 = 12x^2 + 12x + 6$

$y_3 = 24x + 12$

$y_4 = 24$

Derivatives found above are called higher derivatives.

Some standard nth derivatives:-

① nth derivative of e^{ax}:-

Let $y = e^{ax}$

Diff. w.r.t. x successively

$y_1 = e^{ax} \cdot a = ae^{ax}$

$y_2 = ae^{ax} \cdot a = a^2e^{ax}$

$y_3 = a^2e^{ax} \cdot a = a^3e^{ax}$

$y_4 = a^3e^{ax} \cdot a = a^4e^{ax}$

$$\boxed{y_n = a^n e^{ax}}$$

② nth derivative of $(ax+b)^m$:-

Let $y = (ax+b)^m$

Diff. w.r.t. x successively

$$y_1 = m(ax+b)^{m-1} \cdot a = ma(ax+b)^{m-1}$$

$$y_2 = ma(m-1)(ax+b)^{m-2} \cdot a = m(m-1) \cdot a^2 (ax+b)^{m-2}$$

$$y_3 = m(m-1)a^2(m-2)(ax+b)^{m-3} \cdot a = m(m-1)(m-2) \cdot a^3 (ax+b)^{m-3}$$

$$y_n = m(m-1)(m-2) \ldots\ldots (m-(n-1)) \cdot a^n (ax+b)^{m-n}$$

$$= m(m-1)(m-2) \text{------} (m-n+1) \cdot (ax+b)^{m-n} \cdot a^n$$

$$= \frac{m(m-1)(m-2) \text{------} (m-n+1)(m-n)(m-n-1) \ldots 3 \cdot 2 \cdot 1 \cdot (ax+b)^{m-n} \cdot a^n}{(m-n)(m-n-1) \text{------} 3 \cdot 2 \cdot 1}$$

$$= \frac{m! \cdot (ax+b)^{m-n} \cdot a^n}{(m-n)!}$$

So $\boxed{y_n = \dfrac{m! \cdot a^n (ax+b)^{m-n}}{(m-n)!}}$

③ nth derivative of $\dfrac{1}{ax+b}$:-

Let $y = \dfrac{1}{ax+b}$

or $y = (ax+b)^{-1}$

Diff. w.r.t. x successively

$$y_1 = (-1)(ax+b)^{-2} \cdot a$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a \cdot a = (-1)(-2)(ax+b)^{-3} \cdot a^2$$

$$y_3 = (-1)(-2)(-3)(ax+b)^{-4}\cdot a^2\cdot a = (-1)(-2)(-3)(ax+b)^{-4}\cdot a^3$$ 49

$$y_n = (-1)(-2)(-3)\ldots\ldots(-n)(ax+b)^{-(n+1)}\cdot a^n$$

$$= (-1)^n\cdot n!\cdot a^n\cdot(ax+b)^{-(n+1)}$$

So $\boxed{y_n = \dfrac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}}$

④ <u>nth derivative of $\ln(ax+b)$</u>

Let $y = \ln(ax+b)$

Diff. w.r.t. x successively

$$y_1 = \frac{1}{ax+b}\cdot a = (ax+b)^{-1}\cdot a$$

$$y_2 = (-1)(ax+b)^{-2}\cdot a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3}\cdot a^3$$

$$y_n = (-1)(-2)\ldots\ldots(-(n-1))(ax+b)^{-n}\cdot a^n$$

$$= (-1)^{n-1}\cdot(n-1)!\cdot(ax+b)^{-n}\cdot a^n$$

So $\boxed{y_n = \dfrac{(-1)^{n-1}\cdot(n-1)!\cdot a^n}{(ax+b)^n}}$

<u>Note</u> ① $\sin(\pi/2+x) = \cos x$

② $\cos(\pi/2+x) = -\sin x$

(5) nth derivative of $\sin(ax+b)$:- 50

Let $y = \sin(ax+b)$

Diff. w.r.t. x

$y_1 = \cos(ax+b).a = a\cos(ax+b) = a\sin(ax+b+\pi/2)$

$y_2 = a\cos(ax+b+\pi/2)a = a^2\cos(ax+b+\pi/2) = a^2\sin(ax+b+2\pi/2)$

$y_3 = a^2\cos(ax+b+2\pi/2).a = a^3\cos(ax+b+2\frac{\pi}{2}) = a^3\sin(ax+b+3\pi/2)$

$$\boxed{y_n = a^n\sin(ax+b+n\pi/2)}$$

(6) nth derivative of $\cos(ax+b)$:-

Let $y = \cos(ax+b)$

Diff. w.r.t. x

$y_1 = -\sin(ax+b).a = a.-\sin(ax+b) = a\cos(ax+b+\pi/2)$

$y_2 = a.-\sin(ax+b+\pi/2).a = a^2.-\sin(ax+b+\pi/2) = a^2\cos(ax+b+2\pi/2)$

$y_3 = a^2.-\sin(ax+b+2\pi/2).a = a^3.-\sin(ax+b+2\pi/2) = a^3\cos(ax+b+3\frac{\pi}{2})$

$$\boxed{y_n = a^n\cos(ax+b+n\pi/2)}$$

(7) nth derivative of $e^{ax}.\sin(bx+c)$:-

Let $y = e^{ax}.\sin(bx+c)$

Diff. w.r.t. x

$y_1 = e^{ax}.\cos(bx+c).b + \sin(bx+c).e^{ax}.a$

$= e^{ax}\left[a\sin(bx+c) + b\cos(bx+c)\right]$

Put $a = r\cos\theta$ ——— (1)

$b = r\sin\theta$ ——— (2)

Sq. (1) & (2) & adding

$a^2+b^2 = r^2(\cos^2\theta+\sin^2\theta) \Rightarrow r^2 = a^2+b^2$ or $r = \sqrt{a^2+b^2}$

Dividing (2) by (1) $\tan\theta = \frac{b}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$

So above eq. becomes

$$y_1 = e^{ax}\left[r\cos\theta\cdot\sin(bx+c) + r\sin\theta\cdot\cos(bx+c)\right]$$

$$= re^{ax}\left[\sin(bx+c)\cdot\cos\theta + \cos(bx+c)\cdot\sin\theta\right]$$

$$y_1 = re^{ax}\cdot\sin(bx+c+\theta)$$

Again diff. w.r.t. x

$$y_2 = r\left[e^{ax}\cdot\cos(bx+c+\theta)\cdot b + \sin(bx+c+\theta)\cdot e^{ax}\cdot a\right]$$

$$= re^{ax}\left[a\sin(bx+c+\theta) + b\cos(bx+c+\theta)\right]$$

$$= re^{ax}\left[r\cos\theta\cdot\sin(bx+c+\theta) + r\sin\theta\cdot\cos(bx+c+\theta)\right]$$

$$= r^2e^{ax}\left[\sin(bx+c+\theta)\cdot\cos\theta + \cos(bx+c+\theta)\cdot\sin\theta\right]$$

$$y_2 = r^2e^{ax}\sin(bx+c+2\theta)$$

Diff. w.r.t. x

$$y_3 = r^2\left[e^{ax}\cdot\cos(bx+c+2\theta)\cdot b + \sin(bx+c+2\theta)\cdot e^{ax}\cdot a\right]$$

$$= r^2e^{ax}\left[a\sin(bx+c+2\theta) + b\cos(bx+c+2\theta)\right]$$

$$= r^2e^{ax}\left[r\cos\theta\cdot\sin(bx+c+2\theta) + r\sin\theta\cdot\cos(bx+c+2\theta)\right]$$

$$= r^3e^{ax}\left[\sin(bx+c+2\theta)\cdot\cos\theta + \cos(bx+c+2\theta)\cdot\sin\theta\right]$$

$$y_3 = r^3e^{ax}\cdot\sin(bx+c+3\theta)$$

$$y_n = r^ne^{ax}\cdot\sin(bx+c+n\theta)$$

or $y_n = \left[(a^2+b^2)^{1/2}\right]^n\cdot e^{ax}\cdot\sin(bx+c+n\theta)$

or $\boxed{y_n = (a^2+b^2)^{\frac{n}{2}}\cdot e^{ax}\cdot\sin(bx+c+n\tan^{-1}b/a)}$

$$y_3 = r^3 e^{ax}\left[\cos(bx+c+2\theta).\cos\theta - \sin(bx+c+2\theta).\sin\theta\right]$$ 53

$$= r^3 e^{ax}.\cos(bx+c+2\theta+\theta)$$

$$y_3 = r^3 e^{ax}\cos(bx+c+3\theta)$$

$$y_n = r^n e^{ax}.\cos(bx+c+n\theta)$$

$$= \left[(a^2+b^2)^{\frac{1}{2}}\right]^n \cdot e^{ax}.\cos(bx+c+n\theta)$$

So $$\boxed{y_n = (a^2+b^2)^{\frac{n}{2}} \cdot e^{ax}.\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)}$$

Leibniz's theorem:-

Statement:- If U & V are functions of x whose derivatives upto order n exist, then the nth derivative of their product is

$$[UV]^{(n)} = {}^nC_0 U^{(n)}V + {}^nC_1 U^{(n-1)}V' + {}^nC_2 U^{(n-2)}.V'' + \cdots\cdots + {}^nC_n UV^{(n)}$$

Proof:- We will prove this theorem by applying principle of mathematical induction.

Step① Put $n = 1$

$$L.H.S. = [UV]'$$

$$= U'V + UV'$$

& $$R.H.S. = {}^1C_0 U'V + {}^1C_1 UV'$$

$$= U'V + UV'$$

So $L.H.S. = R.H.S.$

Hence theorem is true for $n = 1$

So $C-1$ is satisfied.

Step② Suppose theorem is true for $n = r$

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i.e.,

$$[UV]^{(r)} = {}^{r}C_0 U^{(r)} V + {}^{r}C_1 U^{(r-1)} \cdot V' + {}^{r}C_2 U^{(r-2)} \cdot V'' + \cdots\cdots + {}^{r}C_r U V^{(r)}$$

<u>Step ③</u> Now we prove theorem for $n = r+1$

Diff. above eq. w.r.t. x

$$[UV]^{(r+1)} = {}^{r}C_0 \left[U^{(r+1)} \cdot V + U^{(r)} V' \right] + {}^{r}C_1 \left[U^{(r)} \cdot V' + U^{(r-1)} \cdot V'' \right] + {}^{r}C_2 \left[U^{(r-1)} \cdot V'' + U^{(r-2)} \cdot V''' \right] + \cdots\cdots + {}^{r}C_r \left[U' \cdot V^{(r)} + U \cdot V^{(r+1)} \right]$$

$$= {}^{r}C_0 U^{(r+1)} \cdot V + {}^{r}C_0 U^{(r)} \cdot V' + {}^{r}C_1 U^{(r)} \cdot V' + {}^{r}C_1 U^{(r-1)} \cdot V'' + {}^{r}C_2 U^{(r-1)} \cdot V'' + {}^{r}C_2 U^{(r-2)} \cdot V''' + \cdots\cdots + {}^{r}C_r U' V^{(r)} + {}^{r}C_r U V^{(r+1)}$$

$$= {}^{r}C_0 U^{(r+1)} \cdot V + \left({}^{r}C_0 + {}^{r}C_1 \right) U^{(r)} V' + \left({}^{r}C_1 + {}^{r}C_2 \right) U^{(r-1)} \cdot V'' + \left({}^{r}C_2 + {}^{r}C_3 \right) U^{(r-2)} V''' + \cdots\cdots + {}^{r}C_r U \cdot V^{(r+1)}$$

But $\quad {}^{n}C_0 = {}^{n}C_n = 1$

& $\quad {}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

So above eq. becomes

$$[UV]^{r+1} = {}^{r+1}C_0 U^{(r+1)} \cdot V + {}^{r+1}C_1 U^{(r)} \cdot V' + {}^{r+1}C_2 U^{(r-1)} \cdot V'' + {}^{r+1}C_3 U^{(r-2)} V''' + \cdots\cdots + {}^{r+1}C_{r+1} \cdot U V^{(r+1)}$$

Hence the theorem is true for $n = r+1$

So C-2 is satisfied.

Hence by principle of mathematical induction, the theorem is true for all +ve integers n.

<u>Note</u> By Leibniz's theorem

$$[UV]^{(n)} = {}^{n}C_0 U^{(n)} \cdot V + {}^{n}C_1 U^{(n-1)} V' + {}^{n}C_2 U^{(n-2)} \cdot V'' + \cdots\cdots + {}^{n}C_n U V^{(n)}$$

As ${}^{n}C_0 = {}^{n}C_n = 1$ & ${}^{n}C_1 = {}^{n}C_{n-1} = n$, ${}^{n}C_2 = {}^{n}C_{n-2} = \frac{n(n-1)}{2!}$

So above eq. becomes

$$[UV]^{(n)} = U^{(n)} \cdot V + n U^{(n-1)} \cdot V' + \frac{n(n-1)}{2!} U^{(n-2)} \cdot V'' + \cdots\cdots + U \cdot V^{(n)}.$$

EXERCISE 2.5 (NEW BOOK)

EXERCISE 2.4 (OLD BOOK)

In Problems 1 – 4, find the nth order derivative :

1. $\dfrac{x}{x^2-a^2}$

Sol.

Let $y = \dfrac{x}{x^2-a^2}$

$y = \dfrac{x}{(x+a)(x-a)}$ —— (1)

we resolve it into partial fraction

Now

$\dfrac{x}{(x+a)(x-a)} = \dfrac{A}{x+a} + \dfrac{B}{x-a}$

Multiplying both sides by $(x+a)(x-a)$

$x = A(x-a) + B(x+a)$ —— (2)

To find A put $x = -a$ in (2)

$-a = A(-a-a)$

$-a = -2aA$

$\boxed{A = \frac{1}{2}}$

To find B put $x = a$ in (2)

$a = B(a+a)$

$a = 2aB$

$\boxed{B = \frac{1}{2}}$

So

$\dfrac{x}{(x+a)(x-a)} = \dfrac{\frac{1}{2}}{x+a} + \dfrac{\frac{1}{2}}{x-a}$

$= \dfrac{1}{2(x+a)} + \dfrac{1}{2(x-a)}$

Put in (1)

$y = \dfrac{1}{2(x+a)} + \dfrac{1}{2(x-a)}$

Diff. w.r.t. x n times

$y^{(n)} = \dfrac{1}{2}\left[\dfrac{d^n}{dx^n}\left(\dfrac{1}{x+a}\right)\right] + \dfrac{1}{2}\left[\dfrac{d^n}{dx^n}\left(\dfrac{1}{x-a}\right)\right]$

$\because \dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n . n! . a^n}{(ax+b)^{n+1}}$

So $y^{(n)} = \dfrac{1}{2}\left[\dfrac{(-1)^n . n! . 1^n}{(x+a)^{n+1}}\right] + \dfrac{1}{2}\left[\dfrac{(-1)^n . n! . 1^n}{(x-a)^{n+1}}\right]$

$= \dfrac{(-1)^n . n!}{2}\left[\dfrac{1}{(x+a)^{n+1}} + \dfrac{1}{(x-a)^{n+1}}\right]$

2. $\dfrac{x^4}{(x-1)(x-2)}$

Sol.

Let $y = \dfrac{x^4}{(x-1)(x-2)}$

or $y = x^2+3x+7 + \dfrac{15x-14}{(x-1)(x-2)}$ —— (1)

$$\begin{array}{r|l} & x^2+3x+7 \\ \hline x^2-3x+2 & x^4 \\ & \cancel{x^4}-3x^3+2x^2 \\ \hline & 3x^3-2x^2 \\ & \cancel{3x^3}-9x^2+6x \\ \hline & 7x^2-6x \\ & \cancel{7x^2}-21x+14 \\ \hline & 15x-14 \end{array}$$

Now

$\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

Multiplying both sides by $(x-1)(x-2)$

$15x-14 = A(x-2) + B(x-1)$ —— (2)

To find A put $x=1$ in (2)

$15-14 = A(1-2)$

$1 = -A$

$\Rightarrow \boxed{A=-1}$

To find B put $x=2$ in (2)

$15(2)-14 = B(2-1)$

$30-14 = B$

$\boxed{B = 16}$

So

$\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{16}{x-2}$

Put in (1)

$y = (x^2+3x+7) + \dfrac{16}{x-2} - \dfrac{1}{x-1}$

Diff. w.r.t. x. n times

$y^{(n)} = 16 . \dfrac{d^n}{dx^n}\left(\dfrac{1}{x-2}\right) - \dfrac{d^n}{dx^n}\left(\dfrac{1}{x-1}\right)$

$$\because \frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}$$

So $$y^{(n)} = 16 \cdot \frac{(-1)^n \cdot n! \cdot 1^n}{(x-2)^{n+1}} - \frac{(-1)^n \cdot n! \cdot 1^n}{(x-1)^{n+1}}$$

or $$y^{(n)} = (-1)^n \cdot n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$ —— Ans.

3. $y = e^{ax}\sin(bx + c)$

Sol. It has already been solved.

4. $e^{ax}\cos^2 x \sin x$

Sol:-

Let $y = e^{ax}\cos^2 x \sin x$

$= \frac{1}{2}\left[e^{ax}.(2\cos^2 x).\sin x\right]$

$= \frac{1}{2}\left[e^{ax}(1+\cos 2x).\sin x\right]$

$= \frac{1}{2}\left[e^{ax}(\sin x + \cos 2x \cdot \sin x)\right]$

$= \frac{1}{2}\left[e^{ax}\sin x + e^{ax}.\cos 2x \sin x\right]$

$= \frac{1}{2}e^{ax}.\sin x + \frac{1}{2}(e^{ax}.\cos 2x \sin x)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}(2\cos 2x \sin x)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}(\sin(2x+x) - \sin(2x-x))$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}(\sin 3x - \sin x)$

$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x - \frac{1}{4}e^{ax}\sin x$

$= (\frac{1}{2} - \frac{1}{4})e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$= (\frac{2-1}{4})e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$= \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$

$y = \frac{1}{4}\left[e^{ax}\sin x + e^{ax}\sin 3x\right]$

Diff. w.r.t. x n times

$$y^{(n)} = \frac{1}{4}\left[\frac{d^n}{dx^n}(e^{ax}\sin x) + \frac{d^n}{dx^n}(e^{ax}\sin 3x)\right]$$

$$\because \frac{d^n}{dx^n}(e^{ax}\sin(bx+c)) = (a^2+b^2)^{\frac{n}{2}} \cdot e^{ax}\sin(bx+c+n\tan^{-1}\frac{b}{a})$$

So

$$y^{(n)} = \frac{1}{4}\left\{(a^2+1)^{\frac{n}{2}} \cdot e^{ax}\sin(x+n\tan^{-1}\frac{1}{a}) + (a^2+3^2)^{\frac{n}{2}} \cdot e^{ax}\sin(3x+n\tan^{-1}\frac{3}{a})\right\}$$

or

$$y^{(n)} = \frac{1}{4}\left[(a^2+1)^{\frac{n}{2}} \cdot e^{ax}\sin(x+n\tan^{-1}\frac{1}{a}) + (a^2+9)^{\frac{n}{2}}e^{ax}\sin(3x+n\tan^{-1}\frac{3}{a})\right]$$

5. If $y = \arctan x$, show that

$$(1+x^2)y^{(2)} + 2xy' = 0$$

Hence find the values of all derivatives of y when $x = 0$

Sol.

Let $y = \tan^{-1}x$

Diff. w.r.t. x

$$y' = \frac{1}{1+x^2} \text{ ——— } (1)$$

or $(1+x^2)y' = 1$

w.r.t. x

$$(1+x^2)y'' + 2xy' = 0 \text{ ——— } (2)$$

Diff. w.r.t. x n times

$$[y''\cdot(1+x^2)]^{(n)} + [2y'x]^{(n)} = 0$$

Using Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}\cdot 2x + \frac{n(n-1)}{2!}(y'')^{(n-2)}\cdot 2 + 2\left[(y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1\right] = 0$$

$$(1+x^2)y^{(n+2)} + 2nxy^{(n+1)} + (n^2-n)y^{(n)} + 2xy^{(n+1)} + 2ny^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + (n^2-n+2n)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + (n^2+n)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+2)xy^{(n+1)} + n(n+1)y^{(n)} = 0 \text{ ——— } (3)$$

Put $x=0$ in (1), (2) + (3)

$$\left.\begin{aligned} y'(0) &= 1 \\ y''(0) &= 0 \\ y^{(n+2)}(0) &= -n(n+1)y^{(n)}(0) \end{aligned}\right\}$$

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For $n=1$, $y^{(3)}(0) = -1.2y'(0) = -2.1 \Rightarrow y^{2(1)+1}(0) = (-1)^1 \cdot 2!$

For $n=2$, $y^{(4)}(0) = -2.3y''(0) = -2.3.0 = 0 \Rightarrow y^{2(2)}(0) = 0$

For $n=3$, $y^{(5)}(0) = -3.4.y^{(3)}(0) = -3.4.-2.1 \Rightarrow y^{2(2)+1}(0) = (-1)^2 \cdot 4!$

For $n=4$, $y^{(6)}(0) = -4.5.y^{(4)}(0) = -4.5.0 = 0 \Rightarrow y^{2(3)}(0) = 0$

For $n=5$, $y^{(7)}(0) = -5.6.y^{(5)}(0) = -5.6.(-1)^2.4! \Rightarrow y^{2(3)+1}(0) = (-1)^3 \cdot 6!$

For $n=6$, $y^{(8)}(0) = -6.7.y^{(6)}(0) = -6.7.0 = 0 \Rightarrow y^{2(4)}(0) = 0$

On generalizing we get

$$y^{2(n)+1}(0) = (-1)^n \cdot (2n)!$$

$$y^{2(n)}(0) = 0$$

---

6. If $y = \sin(a \text{ arc} \sin x)$, prove that

$$(1-x^2)\, y^{(n+2)} = (2n+1)\, xy^{(n+1)} - (n^2 - a^2)\, y^{(n)}$$

Sol. $y = \sin(\sin^{-1} x)$

Diff. w.r.t. x

$$y' = \cos(\sin^{-1}x) \cdot \frac{a}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y' = a\cos(\sin^{-1}x)$$

$$(1-x^2)y'^2 = a^2\cos^2(\sin^{-1}x)$$

$$(1-x^2)y'^2 = a^2(1-\sin^2(\sin^{-1}x))$$

or $(1-x^2)y'^2 = a^2(1-y^2)$

Diff. w.r.t. x

$$(1-x^2) \cdot 2y'y'' + (-2x)y'^2 = -2yy'a^2$$

Dividing both sides by $2y'$

$$(1-x^2)y'' - xy' = -a^2 y$$

Diff. w.r.t. x n times

$$[y''(1-x^2)]^{(n)} - [y'x]^{(n)} = -a^2 y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1-x^2) + n(y'')^{(n-1)}(-2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}(-2) - \left[(y')^{n} \cdot x + n(y')^{(n-1)} \cdot 1\right] = -a^2 y^{(n)}$$

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - (n^2-n)y^{(n)} - xy^{(n+1)} - ny^{(n)} + a^2y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2 - n + n - a^2)y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2-a^2)y^{(n)} = 0$$

7. If $y = e^{m \text{ arc} \sin x}$, show that

$$(1-x^2)\, y^{(n+2)} - (2n+1)\, xy^{(n+1)} - (n^2+m^2)\, y^{(n)} = 0.$$

Find the value of y^n at $x = 0$

Sol. $y = e^{m\sin^{-1}x}$

Diff. w.r.t. x

6.

$$y' = e^{m\sin^{-1}x} \cdot \frac{m}{\sqrt{1-x^2}} \quad \text{———} \; (1)$$

$$\sqrt{1-x^2}\, y' = m e^{m\sin^{-1}x}$$

$$\sqrt{1-x^2}\, y' = my$$

Sq. both sides

$$(1-x^2) y'^2 = m^2 y^2$$

Diff. w.r.t. x

$$(1-x^2) \cdot 2y' y'' + (-2x) y'^2 = m^2 (2yy')$$

Dividing both sides by $2y'$

$$(1-x^2) y'' - xy' = m^2 y \quad \text{———} \; (2)$$

Diff. w.r.t. x n times

$$\left[y''(1-x^2)\right]^{(n)} - \left[y' x\right]^{(n)} = m^2 y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1-x^2) + n(y'')^{(n-1)}(-2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}(-2) - \left[(y')^{(n)} \cdot x + n(y')^{(n-1)} \cdot 1\right] = m^2 y^{(n)}$$

$$(1-x^2) y^{(n+2)} - 2nx y^{(n+1)} - (n^2-n) y^{(n)} - x y^{(n+1)} - n y^{(n)} - m^2 y^{(n)} = 0$$

$$(1-x^2) y^{(n+1)} - (2n+1) x y^{(n+1)} - (n^2 - n + n + m^2) y^{(n)} = 0$$

$$(1-x^2) y^{(n+2)} - (2n+1) x y^{(n+1)} - (n^2+m^2) y^{(n)} = 0 \quad \text{———} \; (3)$$

Put $x = 0$ in (1), (2) & (3)

$$\left.\begin{aligned} y'(0) &= m \\ y''(0) &= m^2 \\ y^{(n+2)}(0) &= (n^2+m^2)\, y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = (1^2+m^2)y'(0) = (1^2+m^2).m$

For $n=2$, $y^{(4)}(0) = (2^2+m^2)y''(0) = (2^2+m^2)m^2$

For $n=3$, $y^{(5)}(0) = (3^2+m^2)y^{(3)}(0) = (3^2+m^2)(1^2+m^2).m$

For $n=4$, $y^{(6)}(0) = (4^2+m^2)y^{(4)}(0) = (4^2+m^2)(2^2+m^2).m^2$

For $n=5$, $y^{(7)}(0) = (5^2+m^2)y^{(5)}(0) = (5^2+m^2)(3^2+m^2)(1^2+m^2).m$

For $n=6$, $y^{(8)}(0) = (6^2+m^2)y^{(6)}(0) = (6^2+m^2)(4^2+m^2)(2^2+m^2).m^2$

on generalizing we have

$y_n(0) = [(n-2)^2+m^2]\text{--------}(4^2+m^2)(2^2+m^2).m^2$ if n is even

$y_n(0) = [(n-2)^2+m^2]\text{--------}(3^2+m^2)(1^2+m^2).m$ if n is odd

---

8. Find $y^{(n)}(0)$ if

(a) $y = \ln[x+\sqrt{1+x^2}]$

(b) $y = \ln(x+\sqrt{1+x^2})^m$

Sol. **(a)** $y = \ln(x+\sqrt{1+x^2})$

Diff. w.r.t. x

$$y' = \frac{1}{(x+\sqrt{1+x^2})}.\left(1+\frac{1}{2\sqrt{1+x^2}}.2x\right)$$

$$y' = \frac{1}{(x+\sqrt{1+x^2})}.\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

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$$y' = \frac{1}{\cancel{(x+\sqrt{1+x^2})}} \cdot \left(\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\right)$$

$$y' = \frac{1}{\sqrt{1+x^2}} \quad \text{——— (1)}$$

$$\sqrt{1+x^2}\, y' = 1$$

$$(1+x^2)y'^2 = 1$$

Diff. w.r.t. x

$$(1+x^2).2y'y'' + 2xy'^2 = 0$$

$$(1+x^2)y'' + xy' = 0$$

Diff. w.r.t. x n times

$$\left[y''(1+x^2)\right]^{(n)} + \left[y'x\right]^{(n)} = 0$$

By Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}.(2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}.(2) + (y')^{(n)}.x + n(y')^{(n-1)}.1 = 0$$

$$(1+x^2)\,y^{(n+2)} + 2nx\,y^{(n+1)} + (n^2-n)\,y^{(n)} + x\,y^{(n+1)} + n\,y^{(n)} = 0$$

$$(1+x^2)\,y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2-n+n)\,y^{(n)} = 0$$

$$(1+x^2)\,y^{(n+2)} + (2n+1)x\,y^{(n+1)} + n^2\,y^{(n)} = 0 \quad \text{——— (3)}$$

Put $x = 0$ in (1), (2) & (3)

$$\left.\begin{aligned} y'(0) &= 1 \\ y''(0) &= 0 \\ y^{(n+2)}(0) &= -n^2 y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = -(1)^2 y'(0) = -(1)^2.1 \Rightarrow y^{2(1)+1}(0) = (-1)^1.1^2$

For $n=2$, $y^{(4)}(0) = -(2)^2.y''(0) = -(2)^2.0 = 0 \Rightarrow y^{2(2)}(0) = 0$

For $n=3$, $y^{(5)}(0) = -(3)^2.y^{(3)}(0) = -(3)^2.-(1)^2 \Rightarrow y^{2(2)+1}(0) = (-1)^2.1^2.3^2$

For $n=4$, $y^{(6)}(0) = -(4)^2.y^{(4)}(0) = -(4)^2.0 = 0 \Rightarrow y^{2(3)}(0) = 0$

For $n=5$, $y^{(7)}(0) = -(5)^2.y^{(5)}(0) = -(5)^2.-(3)^2.-(1)^2 \Rightarrow y^{2(3)+1}(0) = (-1)^3.1^2.3^2.5^2$

For $n=6$, $y^{(8)}(0) = -(6)^2.y^{(6)}(0) = -(6)^2.0 = 0 \Rightarrow y^{2(4)}(0) = 0$

on generalizing we get

$$y^{(2n+1)}(0) = (-1)^n.1^2.3^2.5^2.\ldots\ldots(2n-1)^2$$

$$y^{(2n)}(0) = 0$$

Sol. (h) $y = (x+\sqrt{1+x^2})^m$

Diff. w.r.t. x

$$y' = m(x+\sqrt{1+x^2})^{m-1}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\cdot(x+\sqrt{1+x^2})\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y' = m(x+\sqrt{1+x^2})^{m}\cdot\frac{1}{\sqrt{1+x^2}}$$

$$y' = \frac{my}{\sqrt{1+x^2}} \quad \text{———} \; (1)$$

$$\sqrt{1+x^2}\,y' = my$$

Sq. both sides

$$(1+x^2)y'^2 = m^2y^2$$

Diff. w.r.t. x

$$(1+x^2)\cdot 2y'y'' + 2xy'^2 = m^2(2yy')$$

Dividing both sides by $2y'$

$$(1+x^2)y'' + xy' = m^2y \quad \text{———} \; (2)$$

Diff. w.r.t. x n times

$$\left[y''(1+x^2)\right]^{(n)} + \left[y'x\right]^{(n)} = m^2y^{(n)}$$

using Leibniz's theorem

$$(y'')^{(n)}(1+x^2) + n(y'')^{(n-1)}\cdot(2x) + \frac{n(n-1)}{2!}(y'')^{(n-2)}\cdot 2 + (y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1 = m^2y^{(n)}$$

$$(1+x^2)y^{(n+2)} + 2nx\,y^{(n+1)} + (n^2-n)y^{(n)} + xy^{(n+1)} + ny^{(n)} - m^2y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2-n+n-m^2)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0$$

Put $x = 0$ in ①, ②, ③

$$\left.\begin{aligned} y'(0) &= m \\ y''(0) &= m^2 \\ y^{(n+2)}(0) &= (m^2-n^2)\,y^{(n)}(0) \end{aligned}\right\}$$

For $n=1$, $y^{(3)}(0) = (m^2-1^2)\,y'(0) = (m^2-1^2).m \Rightarrow y^{2(1)+1}(0) = (m^2-1^2).m$

For $n=2$, $y^{(4)}(0) = (m^2-2^2)\,y''(0) = (m^2-2^2).m^2 \Rightarrow y^{2(2)}(0) = (m^2-2^2).m^2$

For $n=3$, $y^{(5)}(0) = (m^2-3^2).y^{(3)}(0) = (m^2-3^2)(m^2-1^2).m \Rightarrow y^{2(2)+1}(0) = (m^2-3^2)(m^2-1^2).m$

For $n=4$, $y^{(6)}(0) = (m^2-4^2).y^{(4)}(0) = (m^2-4^2)(m^2-2^2)m^2 \Rightarrow y^{2(3)} = (m^2-4^2)(m^2-2^2).m^2$

On generalizing we have

$$y^{2n+1}(0) = (m^2-(2n-1)^2)\,\text{------------}\,(m^2-3^2)(m^2-1^2).m$$

$$y^{2n}(0) = (m^2-(2n-2)^2)\,\text{------------}\,(m^2-4^2)(m^2-2^2).m^2$$

Q9 If $f(x) = \ln(1+\sqrt{1-x})$, Prove that

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

Sol:-

$$f(x) = \ln(1+\sqrt{1-x})$$

Diff. w.r.t. x

$$f'(x) = \frac{1}{(1+\sqrt{1-x})}\cdot\frac{1}{2\sqrt{1-x}}(-1)$$

or $$f'(x) = \frac{1}{(1+\sqrt{1-x})}\cdot\frac{-1}{2\sqrt{1-x}}$$

or $$2\sqrt{1-x}\,f'(x) = \frac{-1}{(1+\sqrt{1-x})}\times\frac{(1-\sqrt{1-x})}{(1-\sqrt{1-x})}$$

$$= \frac{-(1-\sqrt{1-x})}{1-(1-x)}$$

$$= \frac{-(1-\sqrt{1-x})}{1-1+x}$$

$$2\sqrt{1-x}\,f'(x) = \frac{-1+\sqrt{1-x}}{x}$$

$$2x\sqrt{1-x}\,f'(x) = -1+\sqrt{1-x}$$

Diff. w.r.t. x

$$2\left[x\sqrt{1-x}\,f''(x) + f'(x)\left(x\cdot\frac{1}{2\sqrt{1-x}}(-1) + \sqrt{1-x}\cdot 1\right)\right] = \frac{1}{2\sqrt{1-x}}(-1)$$

$$2x\sqrt{1-x}\,f''(x) + 2f'(x)\left(\frac{-x}{2\sqrt{1-x}} + \sqrt{1-x}\right) = \frac{-1}{2\sqrt{1-x}}$$

Multiplying both sides by $2\sqrt{1-x}$

$$4x(1-x)f''(x) + 4f'(x)\sqrt{1-x}\left(\frac{-x+2(1-x)}{2\sqrt{1-x}}\right) = -1$$

$$4x(1-x)f''(x) + 2f'(x)(-x+2-2x) + 1 = 0$$

$$4x(1-x)f''(x) + 2f'(x)(-3x+2) + 1 = 0$$

or $$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

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10. If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that

$$x^2 y^{(n+2)} + (2n+1)x y^{(n+1)} + (n^2+1) y^{(n)} = 0$$

Sol.: $y = a\cos(\ln x) + b\sin(\ln x)$

Diff. w.r.t. x

$$y' = -a\sin(\ln x)\cdot\frac{1}{x} + b\cos(\ln x)\cdot\frac{1}{x}$$

$$xy' = -a\sin(\ln x) + b\cos(\ln x)$$

Diff. w.r.t. x

$$xy'' + y'\cdot 1 = -a\cos(\ln x)\cdot\frac{1}{x} - b\sin(\ln x)\cdot\frac{1}{x}$$

$$x^2y'' + xy' = -\left(a\cos(\ln x) + b\sin(\ln x)\right)$$

$$x^2y'' + xy' = -y$$

$$x^2y'' + xy' + y = 0$$

Diff. w.r.t. x n times

$$[y''\cdot x^2]^{(n)} + [y'\cdot x]^{(n)} + y^{(n)} = 0$$

Using Leibniz's theorem

$$(y'')^{(n)}\cdot x^2 + n(y'')^{(n-1)}\cdot 2x + \frac{n(n-1)}{2!}(y'')^{(n-2)}\cdot 2 + (y')^{(n)}\cdot x + n(y')^{(n-1)}\cdot 1 + y^{(n)} = 0$$

$$x^2 y^{(n+2)} + 2nx\,y^{(n+1)} + (n^2-n)\,y^{(n)} + x\,y^{(n+1)} + n\,y^{(n)} + y^{(n)} = 0$$

$$x^2 y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2-n+n+1)\,y^{(n)} = 0$$

$$x^2 y^{(n+2)} + (2n+1)x\,y^{(n+1)} + (n^2+1)\,y^{(n)} = 0$$

11. If $x^y = e^{x-y}$, find $\dfrac{d^n y}{dx^n}$.

Sol. $x^y = e^{x-y}$

taking ln on both sides

$$\ln x^y = \ln e^{x-y}$$

$$y\ln x = (x-y)\ln e$$

$$y\ln x = x-y$$

$$y + y\ln x = x$$

$$y(1+\ln x) = x$$

Diff. w.r.t. x n times

$$[y(1+\ln x)]^{(n)} = 0$$

$$y^{(n)}(1+\ln x) + n\,y^{(n-1)}\cdot\frac{1}{x} + \frac{n(n-1)}{2!}\cdot y^{(n-2)}\cdot\left(\frac{-1}{x^2}\right) + \cdots\cdots + y\frac{(-1)^{n-1}\cdot(n-1)!\cdot 1^n}{x^n} = 0$$

$$y^{(n)}(1+\ln x) + \frac{n}{x}\,y^{(n-1)} - \frac{n(n-1)}{2x^2}\,y^{(n-2)} + \cdots\cdots + y\cdot\frac{(-1)^{n-1}\cdot(n-1)!}{x^n} = 0$$

12. Show that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3}\cdots - \frac{1}{n}\right]$$

Sol.-

Let $U = \frac{1}{x}$ & $V = \ln x$

As we know that $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n\cdot n!\cdot a^n}{(ax+b)^{n+1}}$ & $\dfrac{d^n}{dx^n}\left(\ln(ax+b)\right) = \dfrac{(-1)^{n-1}\cdot(n-1)!\cdot a^n}{(ax+b)^n}$

So $U^{(n)} = \dfrac{(-1)^n\cdot n!}{x^{n+1}}$ & $V^{(n)} = \dfrac{(-1)^{n-1}\cdot(n-1)!}{x^n}$

By Leibniz's theorem, we have

$$[UV]^{(n)} = U^{(n)}\cdot V + n\,U^{(n-1)}\cdot V' + \frac{n(n-1)}{2!}U^{(n-2)}V'' + \cdots\cdots + n\,U'V^{(n-1)} + UV^{(n)}$$

Putting values we get

$$\left[\frac{1}{x}\cdot\ln x\right]^{(n)} = \frac{(-1)^n\cdot n!}{x^{n+1}}\cdot\ln x + n\cdot\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}\cdot\frac{1}{x} + \frac{n(n-1)}{2!}\cdot\frac{(-1)^{n-2}\cdot(n-2)!}{x^{n-1}}\cdot\frac{-1}{x^2}$$

$$+ \frac{n(n-1)(n-2)}{3!}\cdot\frac{(-1)^{n-3}\cdot(n-3)!}{x^{n-2}}\cdot\frac{2}{x^3} + \cdots + \frac{1}{x}\cdot\frac{(-1)^{n-1}\cdot(n-1)!}{x^n}$$

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$$\left(\frac{\ln x}{x}\right)^{(n)} = \frac{(-1)^n \cdot n!}{x^{n+1}} \cdot \ln x + \frac{(-1)^{n-1} \cdot n(n-1)!}{x^{n+1}} - \frac{(-1)^{n-2} \cdot n(n-1)(n-2)!}{2! \cdot x^{n+1}} + \frac{(-1)^{n-3} 2n(n-1)(n-1)(n-3)!}{3! \cdot x^{n+1}} + \cdots\cdots + \frac{(-1)^{n-1} \cdot (n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n \cdot n! \cdot \ln x}{x^{n+1}} + \frac{(-1)^n \cdot (-1)^{-1} \cdot n!}{x^{n+1}} - \frac{(-1)^n \cdot (-1)^{-2} \cdot n!}{2x^{n+1}} + \frac{(-1)^n \cdot (-1)^{-3} \cdot n!}{3x^{n+1}} + \cdots\cdots + \frac{(-1)^n \cdot (-1)^{-1} \cdot n(n-1)!}{n x^{n+1}}$$

$$= \frac{(-1)^n \cdot n! \cdot \ln x}{x^{n+1}} - \frac{(-1)^n \cdot n!}{x^{n+1}} - \frac{(-1)^n \cdot n!}{2x^{n+1}} - \frac{(-1)^n \cdot n!}{3x^{n+1}} - \cdots\cdots - \frac{(-1)^n \cdot n!}{n x^{n+1}}$$

$$\left(\frac{\ln x}{x}\right)^{(n)} = \frac{(-1)^n \cdot n!}{x^{n+1}} \left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots\cdots - \frac{1}{n}\right]$$