BSc Mathematics
Calculus- Chapter 02

2.1-1

Ch-2 (Derivatives)

Derivative of a function:

Let f be a function from R to R then derivative of f at x is defined by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

The above limit will exist only left & right hand limits

$$Lf'(x) = \lim_{h\to 0-0} \frac{f(x+h)-f(x)}{h}$$

& $Rf'(x) = \lim\limits_{h\to 0+0} \dfrac{f(x+h)-f(x)}{h}$ exist & are equal

In this case we say f is differentiable or derivable at x.

Note The derivative of a function f at a pt. $x=a$ is defined as

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Theorem If f is differentiable at a pt. $x=a \in D_f$ then f is Continuous at $x=a$.

Proof: Given that f is differentiable at $x=a$.
To show that f is Continuous at $x=a$ we have to show that

$$\lim_{x\to a} f(x) = f(a)$$

As f is differentiable at $x=a$

So $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists.

Consider
$$\lim_{x\to a}\left(f(x)-f(a)\right) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\times(x-a)\right]$$
$$=\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)$$
$$= f'(a)\cdot(a-a)$$
$$= f'(a).(0)$$

$$\Rightarrow \lim_{x\to a}\left(f(x)-f(a)\right) = 0$$

$$\Rightarrow \lim_{x\to a} f(x) = f(a)$$

Which shows that f is Continuous at $x=a$.

Note The Converse of this theorem does not hold. i.e., a Continuous function may not be differentiable. We give an example to prove it.

Example 3 Let $f: R \to R$ be defined by

$$f(x) = |x|$$

then prove that f is continuous at $x=0$ but is not differentiable at $x=0$

Sol:- Given function is

$$f(x) = |x|$$

Here $f(0) = 0$

$$f(0-0) = \lim_{x \to 0-0} |x| = \lim_{x \to 0-0} (-x) = 0$$

& $$f(0+0) = \lim_{x \to 0+0} |x| = \lim_{x \to 0+0} (x) = 0$$

Since $f(0-0) = f(0+0) = f(0)$

So f is continuous at $x=0$

Now we check the derivability of f

As $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

So $$f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h|-0}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

Now $$Lf'(0) = \lim_{h \to 0-0} \frac{|h|}{h} = \lim_{h \to 0-0} \frac{-h}{h} = -1$$

& $$Rf'(0) = \lim_{h \to 0+0} \frac{|h|}{h} = \lim_{h \to 0+0} \frac{h}{h} = 1$$

Since $Lf'(0) \neq Rf'(0)$. So f is not derivable at $x=0$

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(Exercise No. 2.1)

Q1 Show that the function $f(x) = |x| + |x-1|$ is Continuous for every value of x but is not differentiable at $x = 0$ & $x = 1$

Sol:- Given function is

$$f(x) = |x| + |x-1|$$

First we discuss the Continuity of f

let x_0 be an arbitrary real no. then

$$f(x_0) = |x_0| + |x_0 - 1|$$

Now $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} \left(|x| + |x-1| \right)$

$$= |x_0| + |x_0 - 1|$$

Since $\lim\limits_{x \to x_0} f(x) = f(x_0)$

So f is Continuous at $x = x_0$. But x_0 is any real no. So f is Continuous for every real value of x.

Now we will show that f is not differentiable at $x = 0$ & $x = 1$

Since $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

So $f'(0) = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h}$

Now $Lf'(0) = \lim\limits_{h \to 0-0} \dfrac{|h| + |h-1| - |-1|}{h}$

$$= \lim_{h \to 0-0} \frac{-h - (h-1) - 1}{h}$$

$$= \lim_{h \to 0-0} \frac{-h - h + 1 - 1}{h}$$

$$= \lim_{h \to 0-0} \frac{-2h}{h}$$

$$= \lim_{h \to 0-0} (-2)$$

$$Lf'(0) = -2$$

$$\& \quad Rf'(0) = \lim_{h \to 0+0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0+0} \frac{|h| + |h-1| - |-1|}{h}$$

$$= \lim_{h \to 0+0} \frac{h - (h-1) - 1}{h}$$

$$= \lim_{h \to 0+0} \frac{h - h + 1 - 1}{h}$$

$$= \lim_{h \to 0+0} \frac{0}{h}$$

$$= \lim_{h \to 0+0} (0)$$

$$Rf'(0) = 0$$

Since $Lf'(0) \neq Rf'(0)$

So f is not differentiable at $x = 0$

Now $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$Lf'(1) = \lim_{h \to 0-0} \frac{(|1+h| + |1+h-1|) - (|1| + |1-1|)}{h}$$

$$= \lim_{h \to 0-0} \frac{1+h-h-1}{h}$$

$$= \lim_{h \to 0-0} (0)$$

So $Lf'(1) = 0$

& $Rf'(1) = \lim\limits_{h\to 0+0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{(|1+h|+|1+h-1|)-(|1|+|1-1|)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{1+h+h-1}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{2h}{h}$

$= \lim\limits_{h\to 0+0} (2)$

$Rf'(1) = 2$

Since $Lf'(1) \neq Rf'(1)$

So f is not derivable at $x=1$

Q2 Let $f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x-1 & 1 < x \le 2 \end{cases}$

Discuss the Continuity & differentiability of f at $x=1$

Sol: Given eq. is

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2x-1 & 1 < x \le 2 \end{cases}$$

First we will discuss the Continuity of f

Here $f(1) = 1$

& $f(1-0) = \lim\limits_{x\to 1-0} f(x)$

$= \lim\limits_{x\to 1-0} (x)$

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$f(1-0) = 1$

& $f(1+0) = \lim\limits_{x\to 1+0} f(x)$

$= \lim\limits_{x\to 1+0} (2x-1)$

$= 2(1)-1$

$= 1$

Since $f(1-0) = f(1+0) = f(1)$

So f is continuous at $x=1$

As $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

So $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$Lf'(1) = \lim\limits_{h\to 0-0} \dfrac{1+h-1}{h}$

$= \lim\limits_{h\to 0-0} \dfrac{h}{h}$

$= \lim\limits_{h\to 0-0} (1)$

$Lf'(1) = 1$

& $Rf'(1) = \lim\limits_{h\to 0+0} \dfrac{f(1+h)-f(1)}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{[2(1+h)-1]-1}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{2+2h-2}{h}$

$= \lim\limits_{h\to 0+0} \dfrac{2h}{h}$

$= \lim\limits_{h\to 0+0} (2)$

$Rf'(1) = 2$

Since $Lf'(1) \neq Rf'(1)$

So f is not differentiable at $x = 1$

Q3 If $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show that f is Continuous & differentiable at $x=0$

Sol:- Given function is

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here $f(0) = 0$

Now $f(0-0) = \lim_{x\to 0-0} f(x)$

$= \lim_{x\to 0-0} x^2 \sin\frac{1}{x}$

Put $x = 0-h$
where $h>0$ & $h\to 0$

$= \lim_{h\to 0} (-h)^2 \sin\left(\frac{1}{-h}\right)$

$= -\lim_{h\to 0} h^2 \sin\frac{1}{h}$

$= -(0)^2 .$ Some no. in $[-1,1]$

$f(0-0) = 0$

& $f(0+0) = \lim_{x\to 0+0} f(x)$

$= \lim_{x\to 0+0} x^2 \sin\frac{1}{x}$

Put $x = 0+h$
where $h>0$ & $h\to 0$

$= \lim_{h\to 0} h^2 \sin\frac{1}{h}$

$= (0)^2 .$ (Some no. in $[-1,1]$)

$f(0+0) = 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$\Rightarrow f'(0) = \lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$

Now $Lf'(0) = \lim_{h\to 0-0} \dfrac{h^2 \sin\frac{1}{h} - 0}{h}$

$= \lim_{h\to 0-0} \left(h \sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1, 1]$

$= 0$

& $Rf'(0) = \lim_{h\to 0+0} \dfrac{f(h)-f(0)}{h}$

$= \lim_{h\to 0+0} \dfrac{h^2 \sin\frac{1}{h}}{h}$

$= \lim_{h\to 0+0} \left(h \sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1, 1]$

$= 0$

Since $Lf'(0) = Rf'(0)$

So f is derivable at $x = 0$

Q4 Is the function

$$f(x) = \begin{cases} (x-a)\cdot \sin\left(\frac{1}{x-a}\right) & x \neq a \\ 0 & x = a \end{cases}$$

Continuous & differentiable at $x = 0$

Sol. Given function is 10

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & x \neq a \\ 0 & x = a \end{cases}$$

Here $f(a) = 0$

& $f(a-0) = \lim\limits_{x \to a-0} f(x)$

$= \lim\limits_{x \to a-0} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= \lim\limits_{h \to 0} (a-h-a)\sin\left(\frac{1}{a-h-a}\right)$ Put $x = a-h$ where $h > 0$ & $h \to 0$

$= \lim\limits_{h \to 0} (-h)\sin\left(\frac{1}{-h}\right)$

$= \lim\limits_{h \to 0}\left(h\sin\frac{1}{h}\right)$

$= 0 \cdot$ Some no. in $[-1,1]$

$f(a-0) = 0$

& $f(a+0) = \lim\limits_{x \to a+0} f(x)$

$= \lim\limits_{x \to a+0} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= \lim\limits_{h \to 0} (a+h-a)\sin\left(\frac{1}{a+h-a}\right)$ Put $x = a+h$ where $h > 0$ & $h \to 0$

$= \lim\limits_{h \to 0} h\sin\frac{1}{h}$

$= 0 \cdot$ Some no. in $[-1,1]$

$f(a+0) = 0$

Since $f(a-0) = f(a+0) = f(a)$

So f is Continuous at $x = a$

As $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow \quad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$Lf'(a) = \lim\limits_{h\to 0-0} \dfrac{(\cancel{a}+h-\cancel{a})\sin\left(\frac{1}{a+h-a}\right)-0}{h}$

$= \lim\limits_{h\to 0-0} \dfrac{h\sin\frac{1}{h}}{h}$

$= \lim\limits_{h\to 0-0} \sin\dfrac{1}{h}$

$Lf'(a) = \text{Some no. in } [-1,1]$

So $Lf'(a)$ does not exist.

Hence f is not derivable at $x=a$

Q5 Let $f(x) = \begin{cases} x\tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Discuss the Continuity & differentiability of f at $x=0$

Sol:- Given function is

$f(x) = \begin{cases} x\tan^{-1}\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Here $f(0) = 0$

& $f(0-0) = \lim\limits_{x\to 0-0} f(x)$

$= \lim\limits_{x\to 0-0} x\tan^{-1}\dfrac{1}{x}$

$= 0 \cdot \text{Some real no.}$

So $f(0-0) = 0$

& $f(0+0) = \lim_{x\to 0+0} f(x)$

$= \lim_{x\to 0+0} x\tan^{-1}\frac{1}{x}$

$= 0 \cdot$ Some real no.

$= 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x=0$

As $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$\Rightarrow f'(0) = \lim_{h\to 0} \dfrac{f(h)-f(0)}{h}$

$Lf'(0) = \lim_{h\to 0-0} \dfrac{h\tan^{-1}\frac{1}{h} - 0}{h}$

$= \lim_{h\to 0-0} \tan^{-1}\frac{1}{h}$

which does not exist

So $Lf'(0)$ does not exist

Hence f is not derivable at $x=0$

Q6 Examine for Continuity & differentiability the function $x^{4/3}$ at $x=0$

Sol:- Given function is

$f(x) = x^{4/3}$

Here $f(0) = (0)^{4/3} = 0$

& $f(0-0) = \lim_{x\to 0-0} f(x)$

$= \lim_{x\to 0-0} x^{4/3}$

$$f(0-0) = 0$$

& $$f(0+0) = \lim_{x \to 0+0} x^{4/3}$$

$$= (0)^{4/3}$$

$$= 0$$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{f(h)-f(0)}{h}$$

$$Lf'(0) = \lim_{h \to 0-0} \frac{h^{4/3}-0}{h}$$

$$= \lim_{h \to 0-0} h^{4/3-1}$$

$$= \lim_{h \to 0-0} h^{1/3}$$

$$= 0$$

& $$Rf'(0) = \lim_{h \to 0+0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h \to 0+0} \frac{h^{4/3}-0}{h}$$

$$= \lim_{h \to 0+0} h^{1/3}$$

$$= (0)^{1/3}$$

$$= 0$$

Since $Lf'(0) = Rf'(0)$

So f is derivable at $x = 0$

Q7 Find the values of a & b so that f is Continuous & differentiable at $x=1$ where

$$f(x) = \begin{cases} x^3 & \text{if } x<1 \\ ax+b & \text{if } x \geq 1 \end{cases}$$

Sol:- Given function is

$$f(x) = \begin{cases} x^3 & \text{if } x<1 \\ ax+b & \text{if } x \geq 1 \end{cases}$$

Since f is Continuous at $x=1$

So $\quad f(1-0) = f(1+0)$

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1+0} f(x)$$

$$\lim_{x\to 1-0} x^3 = \lim_{x\to 1+0} (ax+b)$$

$$(1)^3 = a(1)+b$$

$$\Rightarrow a+b = 1 \quad \text{———— (1)}$$

Also as f is derivable at $x=1$

So $\quad Lf'(1) = Rf'(1)$

Now $\quad f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

So $\quad f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$

$$Lf'(1) = \lim_{h\to 0-0} \frac{(1+h)^3-1}{h}$$

$$= \lim_{h\to 0-0} \frac{\cancel{1}+3h+3h^2+h^3-\cancel{1}}{h}$$

$$= \lim_{h\to 0-0} (3+3h+h^2)$$

$$= 3+0+0$$

$$Lf'(1) = 3$$

$$\& \quad Rf'(1) = \lim_{h\to 0+0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0+0} \frac{[a(1+h)+b]-[a+b]}{h}$$

$$= \lim_{h\to 0+0} \frac{\cancel{a}+ah+\cancel{b}-\cancel{a}-\cancel{b}}{h}$$

$$= \lim_{h\to 0+0} \frac{ah}{h}$$

$$= \lim_{h\to 0+0} a$$

$$Rf'(1) = a$$

Since f is differentiable at $x = 1$

So $Lf'(1) = Rf'(1)$

$$\Rightarrow \quad 3 = a$$

or $\boxed{a = 3}$

Put in (1)

$3 + b = 1$

$\boxed{b = 2}$

<u>Q8</u> Let $f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \le \pi/6 \\ ax+b & \text{if } \frac{\pi}{6} < x \le 1 \end{cases}$

Derive the values of a & b so that f is Continuous & differentiable at $x = \pi/6$

<u>Sol</u>:- Given function is

$$f(x)=\begin{cases}\sin 2x & \text{if } 0<x\le \pi/6\\ ax+b & \text{if } \frac{\pi}{6}<x\le 1\end{cases}$$

Since f is Continuous at $x=\pi/6$

$$\Rightarrow f\left(\frac{\pi}{6}-0\right) = f\left(\frac{\pi}{6}+0\right)$$

or $\displaystyle\lim_{x\to\frac{\pi}{6}-0} f(x) = \lim_{x\to\frac{\pi}{6}+0} f(x)$

$$\lim_{x\to\frac{\pi}{6}-0}\sin 2x = \lim_{x\to\frac{\pi}{6}+0}(ax+b)$$

$$\sin 2\left(\frac{\pi}{6}\right) = a\left(\frac{\pi}{6}\right)+b$$

$$\sin \pi/3 = \frac{\pi a+6b}{6}$$

$$\frac{\sqrt{3}}{2} = \frac{\pi a+6b}{6}$$

$$2(\pi a+6b) = 6\sqrt{3}$$

$$\pi a+6b = 3\sqrt{3} \quad \text{———} \quad (1)$$

Also f is derivable at $x=\pi/6$

So $Lf'\left(\frac{\pi}{6}\right) = Rf'(\pi/6)$

Now $\displaystyle f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

So $\displaystyle f'(\pi/6) = \lim_{h\to 0}\frac{f\left(\frac{\pi}{6}+h\right)-f(\pi/6)}{h}$

$$Lf'(\pi/6) = \lim_{h\to 0-0}\frac{\sin 2\left(\frac{\pi}{6}+h\right)-\sin 2(\pi/6)}{h}$$

$$= \lim_{h\to 0-0}\frac{\sin\left(\frac{\pi}{3}+2h\right)-\sin \pi/3}{h}$$

$$Lf'\left(\frac{\pi}{6}\right) = \lim_{h\to 0-0} \frac{2\cos\left(\frac{\pi/3+2h+\pi/3}{2}\right).\sin\left(\frac{\frac{\pi}{3}+2h-\pi/3}{2}\right)}{h}$$

$$= \lim_{h\to 0-0} 2\cos(\pi/3+h).\frac{\sin h}{h}$$

$$= \left[\lim_{h\to 0-0} 2\cos(\pi/3+h)\right]\left[\lim_{h\to 0-0}\frac{\sin h}{h}\right]$$

$$= 2\cos(\pi/3+0).1$$

$$= 2\cos(\pi/3)$$

$$= 2\cdot\frac{1}{2}$$

$$Lf'(\pi/6) = 1$$

& $$Rf'(\pi/6) = \lim_{h\to 0+0}\frac{f(\pi/6+h)-f(\pi/6)}{h}$$

$$= \lim_{h\to 0+0}\frac{[a(\pi/6+h)+b]-[a(\pi/6)+b]}{h}$$

$$= \lim_{h\to 0+0}\frac{a\cancel{\pi/6}+ah\cancel{+b}-a\cancel{\pi/6}\cancel{-b}}{h}$$

$$= \lim_{h\to 0+0}\frac{a\cancel{h}}{\cancel{h}}$$

$$Rf'(\pi/6) = a$$

Since f is derivable at $x = \pi/6$

So $Lf'(\pi/6) = Rf'(\pi/6)$

or $1 = a$

$\boxed{a=1}$ Put in (1)

$\pi(1)+6b = 3\sqrt{3}$

$\Rightarrow 6b = 3\sqrt{3}-\pi$

$$\boxed{b = \frac{3\sqrt{3}-\pi}{6}}$$

Q9 If $f(x) = x\tanh\frac{1}{x}$, $x \neq 0$ & $f(0) = 0$.

Discuss the Continuity & differentiability of f at $x = 0$

Sol:- Given function is

$$f(x) = \begin{cases} x\tanh\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$\text{or } f(x) = \begin{cases} x\cdot\dfrac{e^{\frac{1}{x}} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Here $f(0) = 0$

$$\& \quad f(0-0) = \lim_{x\to 0-0} x\cdot\frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}$$

$$= \lim_{x\to 0-0} x\left[\frac{e^{\frac{1}{x}} - \frac{1}{e^{1/x}}}{e^{1/x} + \frac{1}{e^{1/x}}}\right]$$

$$= \lim_{x\to 0-0} x\left[\frac{e^{\frac{2}{x}} - 1}{e^{2/x} + 1}\right]$$

$$= \lim_{h\to 0} (-h)\left[\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right] \qquad \text{Put } x = 0-h \text{ where } h>0 \ \& \ h\to 0$$

$$= (0)\cdot\left[\frac{0-1}{0+1}\right]$$

$$= (0)(-1)$$

$$f(0-0) = 0$$

$$\& \quad f(0+0) = \lim_{x\to 0+0} f(x)$$

$$= \lim_{x\to 0+0} x\left[\frac{e^{\frac{2}{x}} - 1}{e^{2/x} + 1}\right]$$

$$= \lim_{h\to 0} h\left[\frac{e^{2/h}-1}{e^{2/h}+1}\right]$$ Put $x = 0+h$ where $h>0$ & $h\to 0$ 19

$$= \lim_{h\to 0} h\left[\frac{1-e^{-2/h}}{1+e^{-2/h}}\right]$$

$$= (0)\left[\frac{1-0}{1+0}\right]$$

$f(0+0) = 0$

Since $f(0-0) = f(0+0) = f(0)$

So f is Continuous at $x = 0$

As $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

Now $Lf'(0) = \lim_{h\to 0-0} \dfrac{h\left(\dfrac{e^{2/h}-1}{e^{2/h}+1}\right)}{h}$

$$= \lim_{h\to 0-0}\left(\frac{e^{2/h}-1}{e^{2/h}+1}\right)$$

$$= \frac{0-1}{0+1}$$

$Lf'(0) = -1$

& $Rf'(0) = \lim_{h\to 0+0} \dfrac{f(h)-f(0)}{h}$

$$= \lim_{h\to 0+0} \frac{h\cdot\left(\dfrac{e^{2/h}-1}{e^{2/h}+1}\right)-0}{h}$$

$$= \lim_{h\to 0+0}\left(\frac{e^{2/h}-1}{e^{2/h}+1}\right)$$

$$Rf'(0) = \lim_{h\to 0+0}\left(\frac{1-e^{-2/h}}{1+e^{-2/h}}\right) \qquad 2.$$

$$= \frac{1-0}{1+0}$$

$$Rf'(0) = 1$$

Since $Lf'(0) \neq Rf'(0)$

So f is not differentiable at $x=0$

Find the slope of the tangent line to the given curve at the indicated pt. (Problems 10–12).

Q10 $y = x^2$ at $(2,4)$

Sol:- Given eq. of curve is

$$y = x^2$$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x$$

or $\frac{dy}{dx} = 2(2)$ at $(2,4)$

$$\Rightarrow \frac{dy}{dx} = 4$$

which is req. slope of tangent line to given curve

Q11 $y = \frac{1}{x}$ at $(1,1)$

Sol:- Given eq. of curve is

$$y = \frac{1}{x}$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$\frac{dy}{dx} = \frac{-1}{(1)^2}$ at $(1,1)$

So $\frac{dy}{dx} = -1$, which is req. slope of tangent line to given curve.

Q12 $y = x^2 - 7x + 3$ at $(7,3)$

Sol:- Given eq. of Curve is 21

$$y = x^2 - 7x + 3$$

Diff. w.r.t. x

$$\frac{dy}{dx} = 2x - 7$$

or $\frac{dy}{dx} = 2(7) - 7$ at $(7,3)$

or $\frac{dy}{dx} = 14 - 7$

$$\frac{dy}{dx} = 7$$

which is the req. slope of tangent line to given Curve.

Q13 Let v be the velocity of a particle at any given time t. Deduce that the acceleration at this instant is $\frac{dv}{dt}$.

Sol:-

Suppose that a particle starts its motion along the line from fixed pt. O.

Let after time t, the particle reaches at the pt. P with vel. v at pt. P. Further suppose that after a small interval of time δt, it reaches at pt. Q with vel. δv. It means particle attains vel. δv in time δt in going from P to Q. Then avg acc. of particle $= \frac{\delta v}{\delta t}$

& acc. of particle $= \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$ — Ans.

Find the velocity & acc. at the end of 22. 0, 1, 2 seconds (Problems 14—16).

Q14 $\quad S = \dfrac{1}{t+1}$

Sol:- Given

$$S = \frac{1}{t+1}$$

Diff. w.r.t. t

$$\frac{ds}{dt} = \frac{-1}{(t+1)^2}$$

or $$v = \frac{-1}{(t+1)^2}$$

Put $t = 0, 1, 2$

$$v_0 = \frac{-1}{(0+1)^2} = -1$$

$$v_1 = \frac{-1}{(1+1)^2} = -\frac{1}{4}$$

$$v_2 = \frac{-1}{(2+1)^2} = \frac{-1}{9}$$

As $$v = \frac{-1}{(t+1)^2}$$

$$v = -(t+1)^{-2}$$

Diff. w.r.t. t

$$\frac{dv}{dt} = 2(t+1)^{-3}$$

$$a = \frac{2}{(t+1)^3}$$

Put $t = 0, 1, 2$

$$a_0 = \frac{2}{(0+1)^3} = \frac{2}{1} = 2$$

$$a_1 = \frac{2}{(1+1)^3} = \frac{2}{(2)^3} = \frac{2}{8} = \frac{1}{4}$$

$$a_2 = \frac{2}{(2+1)^3} = \frac{2}{(3)^3} = \frac{2}{27} \text{ —— Ans.}$$

Q15 $S = t^2 + 2t + 5$

Sol. Given

$S = t^2 + 2t + 5$

Diff. w.r.t. t

$\frac{ds}{dt} = 2t + 2$

$v = 2(t+1)$

Put $t = 0, 1, 2$

$v_0 = 2(0+1) = 2$

$v_1 = 2(1+1) = 4$

$v_2 = 2(2+1) = 6$

As $v = 2(t+1)$

Diff. w.r.t. t

$\frac{dv}{dt} = 2(1+0)$

$a = 2$

Put $t = 0, 1, 2$

$a_0 = 2$

$a_1 = 2$

$a_2 = 2$

Q16 $S = t^2(t-1)$

Sol. Given

$S = t^2(t-1)$

$S = t^3 - t^2$

Diff. w.r.t. t

$\frac{ds}{dt} = 3t^2 - 2t$

$v = 3t^2 - 2t$

Put $t = 0, 1, 2$

$v_0 = 3(0)^2 - 2(0) = 0$

$v_1 = 3(1)^2 - 2(1) = 3 - 2 = 1$

$v_2 = 3(2)^2 - 2(2) = 12 - 4 = 8$

As $v = 3t^2 - 2t$

Diff. w.r.t. t

$$\frac{dv}{dt} = 6t - 2$$

$a = 6t - 2$

Put $t = 0, 1, 2$

$a_0 = 6(0) - 2 = -2$

$a_1 = 6(1) - 2 = 4$

$a_2 = 6(2) - 2 = 10$

Q17 A pt. moves in a st. line so that its distance S (in meters) after time t (in seconds) is $S = 4t^2 - 16t + 12$

Find

(i) The average vel. in the interval $[1, 1+\Delta t]$

(ii) The velocity at $t = 1$

Sol:- Given

$$S = 4t^2 - 16t + 12$$

we know that average velocity of particle is

$$\frac{\Delta S}{\Delta t} = \frac{\text{total change in } S}{\text{total change in } t}$$

$$\frac{\Delta S}{\Delta t} = \frac{S(1+\Delta t) - S(1)}{1+\Delta t - 1}$$

$$= \frac{[4(1+\Delta t)^2 - 16(1+\Delta t) + 12] - [4(1)^2 - 16(1) + 12]}{\Delta t}$$

$$= \frac{4(1+2\Delta t+\Delta t^2) - 16 - 16\Delta t + 12 - 4 + 16 - 12}{\Delta t}$$

$$= \frac{\cancel{4} + 8\Delta t + 4\Delta t^2 - \cancel{16} - 16\Delta t + \cancel{12} - \cancel{4} + \cancel{16} - \cancel{12}}{\Delta t}$$

$$= \frac{4\Delta t^2 - 8\Delta t}{\Delta t}$$

$$= \frac{4\cancel{\Delta t}(\Delta t - 2)}{\cancel{\Delta t}}$$

$$= 4(\Delta t - 2)$$

$\frac{\Delta S}{\Delta t} = 4\Delta t - 8$ is the avg. vel. in interval $[1, 1+\Delta t]$

Now velocity of particle at $t = 1$ is

$$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \lim_{\Delta t \to 0} (4\Delta t - 8) \qquad \text{at } t = 1$$

$$\frac{ds}{dt} = 4(0) - 8$$

So $v = -8 \text{ m/sec}$ at $t = 1$

Q18 The position of a body (in feet) at time t seconds is

$$S = t^3 - 5t^2 + 9t$$

Find the body's acc. each time its velocity is zero.

Sol:

Sol. Given that

$$S = t^3 - 6t^2 + 9t$$

Diff. w.r.t. t

$$\frac{ds}{dt} = 3t^2 - 12t + 9$$

$$v = 3t^2 - 12t + 9$$

Diff. w.r.t. t

$$\frac{dv}{dt} = 6t - 12$$

or $a = 6t - 12$

If vel. of body is zero

then $3t^2 - 12t + 9 = 0$

$$t^2 - 4t + 3 = 0$$

$$t^2 - 3t - t + 3 = 0$$

$$t(t-3) - 1(t-3) = 0$$

$$(t-3)(t-1) = 0$$

$\boxed{t = 1, 3}$ is the time at which vel. of particle is zero

Now acc. of body at $t = 1$ is

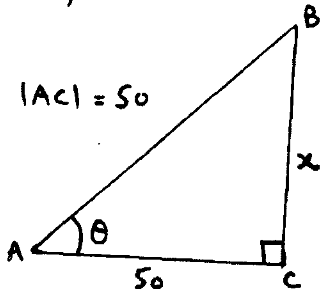
$$[a]_{t=1} = 6(1) - 12 = -6$$

& acc. of body at $t = 3$ is

$$[a]_{t=3} = 6(3) - 12 = 18 - 12 = 6$$

Q19 A ladder is placed 50 meters from a wall at an angle θ with the horisontal. The top of the ladder is x meters above the ground. If the bottom of the ladder is pushed towards the wall. Find the rate of change of x with respect to θ when $\theta = 45^\circ$. B27

Sol. Let AB be the ladder where $|BC| = x$ & $|AC| = 50$



Let $\angle CAB = \theta$

From right angled $\triangle ABC$

$$\tan\theta = \frac{x}{50}$$

$$\Rightarrow x = 50\tan\theta$$

Diff. w.r.t. θ

$$\frac{dx}{d\theta} = 50\sec^2\theta$$

$$\Rightarrow \left[\frac{dx}{d\theta}\right]_{\theta=\pi/4} = 50\sec^2\pi/4 = 50(\sqrt{2})^2 = 50\times 2$$

So $\left[\frac{dx}{d\theta}\right]_{\theta=\pi/4} = 100$ m/rad. is the rate of change of x w.r.t. θ

$$= \frac{100}{\frac{180}{\pi}}$$

$$= \frac{100\pi}{180}$$

$$= \frac{100\times 3.1416}{180}$$

$= 1.75$ m/degree is the rate of change of x w.r.t. θ at $\theta = 45^\circ$

Q20 The no. of litres of water in a tank t minutes after the water starts draining out of the tank is given by

$$f(t) = 200(30-t)^2$$

(i) What is the average rate at which the water flows out during the first 5 minutes?

(ii) How fast is the water running out at the end of 5 minutes?

Sol. Given that

$$f(t) = 200(30-t)^2$$

Then the average rate at which the water flows out during the first 5 minutes is

$$\frac{\Delta f}{\Delta t} = \frac{\text{total change in } f(t)}{\text{total change in } t}$$

$$= \frac{f(5)-f(1)}{5-1}$$

$$= \frac{200(30-5)^2 - 200(30-1)^2}{4}$$

$$= \frac{200(25)^2 - 200(29)^2}{4}$$

$$= \frac{200\left[(25)^2-(29)^2\right]}{4}$$

$$= 50\left[625-841\right]$$

$$= 50(-216)$$

$$= -10800$$

i.e., the average rate at which the water flows out during first 5 minutes is 10800 litres/minute.

Now as $f(t) = 200(30-t)^2$

Diff. w.r.t. t

$f'(t) = -400(30-t)$

Hence the rate at which the water runs out after 5 minutes is

$= f'(5)$

$= -400(30-5)$

$= -400 \times 25$

$= -10000$

$= 10000$ litres/minute

Q21 The height S (in feet) of a rocket t seconds after its launching is given by

$S = -t^3 + 96t^2 + 195t + 10, \quad t \geq 0$

(i) Find the velocity of rocket at any time t.

(ii) The velocity of the rocket when t = 0, 30, 50, 70 seconds. Interpret the results.

(iii) The max. height attained by the rocket.

Sol:- The height of a rocket t seconds after its launching is

$S = -t^3 + 96t^2 + 195t + 10$

Diff. w.r.t. t

$\frac{ds}{dt} = -3t^2 + 192t + 195$

or $v = -3t^2 + 192t + 195$ is the velocity of the rocket at any time t.

(ii) As the vel. v of rocket at any time t is

$v = -3t^2 + 192t + 195$

When $t = 0$, $v = -3(0)^2 + 192(0) + 195 = 195$ ft/sec.

When $t = 30$, $v = -3(30)^2 + 192(30) + 195 = -2700 + 5760 + 195$

$= 3255$ ft/sec.

When $t = 50$, $v = -3(50)^2 + 192(50) + 195 = -7500 + 9600 + 195$

$= 2295$ ft/sec.

When $t = 70$, $v = -3(70)^2 + 192(70) + 195 = -14700 + 13440 + 195$

$= -1065$ ft/sec.

Now we will interpret the above results.

When the rocket is launched, its vel. is $= 195$ ft/s

At $t = 30$ sec. its velocity becomes 3255 ft/sec. It means the vel. of rocket increases after some time b/w $t = 0$ & $t = 30$

At $t = 50$ sec. its vel. becomes 2295 ft/sec. which means vel. of rocket after some time b/w $t = 30$ & $t = 50$ sec.

Now if the vel. of rocket

Q22 The rupee cost $C(x)$ of producing x washing machines is 318

$$C(x) = 2000 + 100x - 0.1x^2$$

(i) Find the marginal cost at $x = 100$.

(ii) Show that the marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine

Sol.- Given cost function is

$$C(x) = 2000 + 100x - 0.1x^2$$

Diff. w.r.t. x

$$C'(x) = 100 - (0.1).2x$$

or $C'(x) = 100 - 0.2x$

So marginal cost at $x = 100$ is

$$C'(100) = 100 - 0.2(100)$$
$$= 100 - 20$$
$$= 80$$

So the marginal cost at $x = 100$ is 80Rs.

(ii) It is obvious that the cost of producing 101st washing machine is

$$C(101) - C(100)$$
$$= \left[2000 + 100(101) - 0.1(101)^2\right] - \left[2000 + 100(100) - 0.1(100)^2\right]$$
$$= (2000 + 10100 - 1020.10) - (2000 + 10000 - 1000)$$
$$= \cancel{2000} + 10100 - 1020.10 - \cancel{2000} - 10000 + 1000 = 80$$

Thus marginal cost at $x = 100$ is approximately the cost of producing 101st washing machine.

Q23 The revenue $R(x)$ (in rupees) of selling x units of desks is

$$R(x) = 2000\left(1 - \frac{1}{x+2}\right)$$

(i) Find the marginal revenue when x no. of desks are sold.

(ii) Use $R'(x)$ to estimate the increase that will result by selling the 9th desk.

Sol: The revenue $R(x)$ of selling x units of desks is

$$R(x) = 2000\left(1 - \frac{1}{x+2}\right)$$

Diff. w.r.t. x

$$R'(x) = 2000\left(0 + \frac{1}{(x+2)^2}\right)$$

$$\boxed{R'(x) = \frac{2000}{(x+2)^2}}$$

is the marginal revenue when x no. of desks are sold.

(ii) As $R'(x) = \dfrac{2000}{(x+2)^2}$

So the approximate increase in revenue that will result by selling the 9th desk is

$$= R'(8)$$

$$= \frac{2000}{(8+2)^2}$$

$$= \frac{2000}{100}$$

$$= 20 \text{ Rs.}$$

Q24 The Cost $C(x)$ (in rupees) of producing x units of fans is 33

$$C(x) = 100x + 200000$$

& the revenue $R(x)$ (in rupees) of selling these x no. of fans is

$$R(x) = -0.02x^2 + 400x$$

Find the profit function $P(x)$ & the marginal profit at $x = 2000$. Calculate the actual profit realized from the sale of 2001st fan.

Sol. The cost function & the revenue function are given as

$$C(x) = 100x + 200000$$

& $$R(x) = -0.02x^2 + 400x$$

We know that the profit function will be the difference of $R(x)$ & $C(x)$.

i.e., $P(x) = R(x) - C(x)$

$$= (-0.02x^2 + 400x) - (100x + 200000)$$

$$= -0.02x^2 + 400x - 100x - 200000$$

$$P(x) = -0.02x^2 + 300x - 200000$$

So marginal profit is

$$P'(x) = -0.04x + 300$$

Marginal profit at $x = 2000$ is

$$P'(2000) = (-0.04)(2000) + 300$$

So $P'(2000) = -80 + 300$

$= 220$ Rs.

Now actual profit from the sale of 2001st fan

$= P(2001) - P(2000)$

$= \left[-0.02(2001)^2 + 300(2001) - 200000\right]$

$- \left[-0.02(2000)^2 + 300(2000) - 200000\right]$

$= (-0.02)\left[(2001)^2 - (2000)^2\right] + 300$

$= (-0.02)(4001) + 300$

$= -80.02 + 300$

$= 219.98$ Rs.

Hence the marginal profit at $x = 2000$ is nearly equal to the profit from the sale of 2001st fan.