BSc (Part 1)
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LEIBNIZ RULE

Theorem: Let $y = uv$, where u, v are functions of 'x' and both possess derivatives of nth order. Then

$$y^{(n)} = \frac{d^n}{dx^n}(y) = \frac{d^n}{dx^n}(uv)$$

$$y^{(n)} = \binom{n}{0}u^{(n)}v + \binom{n}{1}u^{(n-1)}v' + \binom{n}{2}u^{(n-2)}v'' + \binom{n}{3}u^{(n-3)}v^{(3)} + \dots\dots + \binom{n}{r}u^{(n-r)}v^{(r)} + \dots\dots$$
$$+ \dots\dots + \binom{n}{n-1}u'v^{(n-1)} + \binom{n}{n}uv^{(n)} \longrightarrow ①$$

Proof: We'll prove this by Mathematical Induction.

For $n = 1$

$$y' = \binom{1}{0}u'v + \binom{1}{1}uv' = u'v + uv' \quad \text{true}$$

For $n = 2$

$$y'' = \binom{2}{0}u''v + \binom{2}{1}u'v' + \binom{2}{2}uv''$$
$$= u''v + 2u'v' + uv'' \quad \text{true.}$$

condition satisfied

$(uv)' = uv' + vu'$

$(uv)'' = D(uv' + vu')$

$= D(uv') + D(u'v)$

$= uv'' + u'v' + u'v' + u''v$

$(uv)'' = uv'' + 2u'v' + u''v$

Condition 2 :

Suppose this (1) is true for $n = k$ where $k \in Z^+$

$$y^{(k)} = \binom{k}{0}u^{(k)}v + \binom{k}{1}u^{(k-1)}v' + \binom{k}{2}u^{(k-2)}v'' + \dots\dots\dots + \binom{k}{k-1}u'v^{(k-1)} + \binom{k}{k}uv^{(k)} \longrightarrow ②$$

$\boxed{\text{C-3}}$ Now we prove ① true for $n = k+1$, where $k \in Z^+$

Differentiating ② with respect to 'x'

$$y^{(k+1)} = (uv)^{k+1} = D\left[\binom{k}{0}u^{(k)}v + \binom{k}{1}u^{(k-1)}v' + \binom{k}{2}u^{(k-2)}v'' + \cdots\cdots + \binom{k}{k-1}u'v^{(k-1)} + \binom{k}{k}uv^{(k)}\right]$$

$$y^{(k+1)} = \binom{k}{0}D\left(u^{(k)}v\right) + \binom{k}{1}D\left(u^{(k-1)}v'\right) + \binom{k}{2}D\left(u^{(k-2)}v''\right) + \cdots\cdots + \binom{k}{k-1}D\left(u'v^{(k-1)}\right) + \binom{k}{k}D\left(uv^{(k)}\right)$$

$$y^{(k+1)} = \binom{k}{0}\left[u^{(k)}v' + u^{(k+1)}v\right] + \binom{k}{1}\left[u^{(k-1)}v'' + u^{(k)}v'\right] + \binom{k}{2}\left[u^{(k-2)}v''' + u^{(k-1)}v''\right] + \cdots\cdots$$
$$+ \cdots\cdots + \binom{k}{k-1}\left[u'v^{(k)} + u''v^{(k-1)}\right] + \binom{k}{k}\left[uv^{(k+1)} + u'v^{(k)}\right]$$

$$y^{(k+1)} = \binom{k}{0}u^{(k)}v' + \binom{k}{0}u^{(k+1)}v + \binom{k}{1}u^{(k-1)}v'' + \binom{k}{1}u^{(k)}v' + \binom{k}{2}u^{(k-2)}v''' + \binom{k}{2}v^{(k-1)}v'' + \cdots\cdots$$
$$+ \cdots\cdots + \binom{k}{k-1}u'v^{(k)} + \binom{k}{k-1}u''v^{(k-1)} + \binom{k}{k}uv^{(k+1)} + \binom{k}{k}u'v^{(k)}$$

$$y^{(k+1)} = \binom{k}{0}u^{(k+1)}v + \left[\binom{k}{0} + \binom{k}{1}\right]u^{(k)}v' + \left[\binom{k}{1} + \binom{k}{2}\right]u^{(k-1)}v'' + \left[\binom{k}{2} + \binom{k}{3}\right]u^{(k-2)}v''' + \cdots\cdots$$
$$+ \cdots\cdots + \left[\binom{k}{k-2} + \binom{k}{k-1}\right]u''v^{(k-1)} + \left[\binom{k}{k-1} + \binom{k}{k}\right]u'v^{(k)} + \binom{k}{k}\cancel{uv}^{(k+1)}$$

$$\because \quad \because \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1} \quad \forall \quad n$$

$$y^{(k+1)} = \binom{k}{0}u^{(k+1)}v + \binom{k+1}{1}u^{(k)}v' + \binom{k+1}{2}u^{(k-1)}v'' + \binom{k+1}{3}u^{(k-2)}v''' + \cdots\cdots$$
$$+ \cdots\cdots + \binom{k+1}{k-1}u''v^{(k-1)} + \binom{k+1}{k}u'v^{(k)} + \binom{k}{k}uv^{(k+1)}$$

$\because \binom{k}{0} = 1$ and $\binom{k+1}{0} = 1$ So $\binom{k}{0} = \binom{k+1}{0}$

$\because \binom{k}{k} = 1$ and $\binom{k+1}{k+1} = 1$ So $\binom{k+1}{k+1} = \binom{k}{k}$

$$y^{(k+1)} = \binom{k+1}{0}u^{(k+1)}v + \binom{k+1}{1}u^{(k)}v' + \binom{k+1}{2}u^{(k-1)}v'' + \binom{k+1}{3}u^{(k-2)}v''' + \cdots\cdots + \cdots\cdots + \binom{k+1}{k-1}u''v^{(k-1)} + \binom{k+1}{k}u'v^{(k)} + \binom{k+1}{k+1}uv^{(k+1)}$$

So this theorem is true for $n = k+1$, where $k \in \mathbb{Z}^+$

So Statement 1 is proved by Mathematical Induction.

$2(1)-1$

$2(1)+1$ $\quad y^{(3)}(0) = (-1)(1^2)$

$2(2)-1$

$2(2)+1$ $\quad y^{(5)}(0) = (-1)^2(1^2)(3^2)$

$2(3)-1$

$2(3)+1$ $\quad y^{7}(0) = (-1)^3(1^2)(3^2)(5^2)$

$2(4)+1$ $\quad y^{9}(0) = (-1)^4(1^2)(3^2)(5^2)(7^2)$

$2(4)-1$

$$y^{(2n+1)}(0) = (-1)^n(1^2)(2^2)(3^2)\cdots\cdots(2n-1)^2$$

$2(1)$

$2(2)$

$2(3)$

$y^{(2n)}(0)$

Find first order partial derivatives of given functions.

15. $f(x,y) = x^{y^2}$

$f_x = y^2 x^{y^2-1}$

$f_y = x^{y^2}\ln x . \frac{d}{d}(y^2)$

$f_y = 2y^2\, x^{y^2}\ln x$

or

$f = x^{y^2}$

$\ln f = \ln x^{y^2}$

$\ln f = y^2 \ln x$

$\frac{1}{f}\frac{\partial f}{\partial y} = 2y\ln x$

$\frac{\partial f}{\partial y} = f . 2y\ln x$

$= x^{y^2} . 2y\ln x$

$f_y = \frac{\partial f}{\partial y} = 2y\, x^{y^2}\ln x$

16. $f(x,y) = e^{x^2+y^2}$

$f_x = e^{x^2+y^2} . \frac{\partial}{\partial x}(x^2+y^2)$

$f_x = e^{x^2+y^2}(2x)$

$f_y = e^{x^2+y^2}\frac{\partial}{\partial y}(x^2+y^2)$

$f_y = e^{x^2+y^2}(0+2y)$

$f_y = 2y\, e^{x^2+y^2}$

17. $f = \tan^{-1}(y/x)$

$f_x = \frac{1}{1+(\frac{y}{x})^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

$f_x = \frac{1}{\frac{x^2+y^2}{x^2}} . y\frac{\partial}{\partial x}\left(\frac{1}{x}\right)$

$f_x = \frac{\cancel{x^2}}{x^2+y^2} . y . \left(-\frac{1}{\cancel{x^2}}\right)$

$f_x = \frac{-y}{x^2+y^2}$

$f = \tan^{-1}(y/x)$

$f_y = \frac{1}{1+(y/x)^2} . \frac{\partial}{\partial y} . \left(\frac{y}{x}\right)$

$= \frac{1}{\frac{x^2+y^2}{x^2}} . \frac{1}{x} . \frac{\partial}{\partial y}(y)$

$= \frac{x^{\cancel{2}}}{x^2+y^2} . \frac{1}{\cancel{x}}(1)$

$f_y = \frac{x}{x^2+y^2}$

18. $f(x,y) = \tan^{-1}(x+y)$

$f_x = \frac{\partial}{\partial x}\left(\tan^{-1}(x+y)\right)$

$= \frac{1}{1+(x+y)^2}\frac{\partial}{\partial x}(x+y)$

$= \frac{1}{1+(x+y)^2}(1+0)$

$f_x = \frac{1}{1+(x+y)^2}$

$f = \tan^{-1}(x+y)$

$f_y = \frac{1}{1+(x+y)^2}\frac{\partial}{\partial y}(x+y)$

$= \frac{1}{1+(x+y)^2} . (0+1)$

$f_y = \frac{1}{1+(x+y)^2}$

$$f(x,y) = e^{ax}\sin by$$

$$f_x = \frac{\partial}{\partial x}(e^{ax}\sin by)$$

$$= \sin by \cdot \frac{\partial}{\partial x} e^{ax}.$$

$$= \sin by \cdot e^{ax}\frac{\partial}{\partial x}(ax)$$

$$= \sin by \; e^{ax} \cdot (a)$$

$$f_x = ae^{ax}\sin by$$

$$f = e^{ax}\sin by$$

$$f_y = \frac{\partial}{\partial y} e^{ax}\sin by$$

$$f_y = e^{ax}\frac{\partial}{\partial y}\sin by$$

$$f_y = e^{ax}\cos by \frac{\partial}{\partial y}(by)$$

$$f_y = e^{ax}\cos by\,(b)$$

$$f_y = be^{ax}\cos by$$

20. $$f(x,y) = \ln(x^2+y^2)$$

$$f_x = \frac{\partial}{\partial x}(\ln(x^2+y^2))$$

$$= \frac{1}{\ln(x^2+y^2)}\frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{1}{\ln(x^2+y^2)}(2x+0)$$

$$f_x = \frac{2x}{\ln(x^2+y^2)}$$

$$f = \ln(x^2+y^2)$$

$$f_y = \frac{\partial}{\partial y}\ln(x^2+y^2)$$

$$= \frac{1}{\ln(x^2+y^2)}\frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{\ln(x^2+y^2)}\cdot(0+2y)$$

$$f_y = \frac{2y}{\ln(x^2+y^2)}$$

f

21. $$f(x,y) = \ln\left[\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$$ (2)

$$f = \ln(\sqrt{x^2+y^2}-x) - \ln(\sqrt{x^2+y^2}+x)$$

$$f_x = \frac{\partial}{\partial x}\left[\ln(\sqrt{x^2+y^2}-x)\right] - \frac{\partial}{\partial x}\left[\ln(\sqrt{x^2+y^2}+x)\right]$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x}\frac{\partial}{\partial x}(\sqrt{x^2+y^2}-x) - \frac{1}{\sqrt{x^2+y^2}+x}\frac{\partial}{\partial x}(\sqrt{x^2+y^2}+x)$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x}\left[\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial x}(x^2+y^2)-1\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial x}(x^2+y^2)+1\right]$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x}\left[\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}-1\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{2x}{2\sqrt{x^2+y^2}}+1\right]$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x}(-)\left[\frac{-x}{\sqrt{x^2+y^2}}+1\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{x}{\sqrt{x^2+y^2}}+1\right]$$

$$f_x = \frac{-1}{\cancel{\sqrt{x^2+y^2}-x}}\left[\cancel{\frac{-x+\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}}\right] - \frac{1}{\cancel{\sqrt{x^2+y^2}+x}}\left[\cancel{\frac{x+\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}}\right]$$

$$f_x = \frac{-1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}}$$

$$f_x = \frac{-2}{\sqrt{x^2+y^2}}$$

$$f = \ln\left[\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$$

$$f = \ln\left[\sqrt{x^2+y^2}-x\right] - \ln\left[\sqrt{x^2+y^2}+x\right]$$

$$f_y = \frac{\partial}{\partial x}\left[\ln\left[\sqrt{x^2+y^2}-x\right]\right] - \frac{\partial}{\partial x}\left[\ln(\sqrt{x^2+y^2}+x)\right]$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\frac{\partial}{\partial y}\left[\sqrt{x^2+y^2}-x\right] - \frac{1}{\sqrt{x^2+y^2}+x}\frac{\partial}{\partial y}\left[\sqrt{x^2+y^2}+x\right] \quad ③$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\left[\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial y}(x^2+y^2) - 0\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{1}{2\sqrt{x^2+y^2}}\frac{\partial}{\partial y}(x^2+y^2)+0\right]$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\left[\frac{(2y+0)}{2\sqrt{x^2+y^2}}\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{1}{2\sqrt{x^2+y^2}}(0+2y)\right]$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\left[\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right] - \frac{1}{\sqrt{x^2+y^2}+x}\left[\frac{2y}{2\sqrt{x^2+y^2}}\right]$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\cdot\frac{y}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\frac{y}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}\left[\frac{1}{\sqrt{x^2+y^2}-x} - \frac{1}{\sqrt{x^2+y^2}+x}\right]$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}\left[\frac{\sqrt{x^2+y^2}+x-(\sqrt{x^2+y^2}-x)}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{\cancel{\sqrt{x^2+y^2}}+x-\cancel{\sqrt{x^2+y^2}}+x}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\cdot\frac{2x}{\left[(\sqrt{x^2+y^2})^2-(x)^2\right]}$$

$$= \frac{2xy}{\sqrt{x^2+y^2}\left[\cancel{x^2}+y^2-\cancel{x^2}\right]} = \frac{2x\cancel{y}}{y^{\cancel{2}}\sqrt{x^2+y^2}}$$

$$f_y = \frac{2x}{y\sqrt{x^2+y^2}}$$

22. $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

$$f_x = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1}\frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}\cdot(x^2+y^2+z^2)^{-3/2}(2x+0+0)$$

$$= -\frac{2x}{2(x^2+y^2+z^2)^{3/2}}$$

$$f_x = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$f_y = \frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{-1}{2(x^2+y^2+z^2)^{3/2}}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{-1}{2(x^2+y^2+z^2)^{3/2}}(0+2y+0)$$

$$f_y = \frac{-\cancel{2}y}{\cancel{2}(x^2+y^2+z^2)^{3/2}} = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$f_z = \frac{-1}{2(x^2+y^2+z^2)^{3/2}}\cdot\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$f_z = \frac{-1}{2(x^2+y^2+z^2)^{3/2}}(0+0+2z)$$

$$f_z = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Find all the four second order derivatives. (4)

23. $f = e^{x-y}$

$f_x = \frac{\partial f}{\partial x} = e^{x-y}\frac{\partial}{\partial x}(x-y)$

$f_x = e^{x-y}(1-0)$

$f_x = e^{x-y}$

$f_{xy} = \frac{\partial}{\partial y}(f_x)$

$f_{xy} = \frac{\partial}{\partial y}(e^{x-y})$

$= e^{x-y}\frac{\partial}{\partial y}(x-y)$

$= e^{x-y}(0-1)$

$\boxed{f_{xy} = -e^{x-y}}$

$f_x = e^{x-y}$

$f_{xx} = \frac{\partial}{\partial x}(e^{x-y})$

$= e^{x-y}(1-0)$

$\boxed{f_{xx} = e^{x-y}}$

$f_y = e^{x-y}\frac{\partial}{\partial y}(x-y)$

$= e^{x-y}(0-1)$

$f_y = -e^{x-y}$

$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(-e^{x-y})$

$f_{yy} = -e^{x-y}\frac{\partial}{\partial y}(x-y)$

$= -e^{x-y}(0-1)$

$\boxed{f_{yy} = +e^{x-y}}$

$f_y = -e^{x-y}$

$f_{yx} = \frac{\partial}{\partial x}(-e^{x-y})$

$= -e^{x-y}\frac{\partial}{\partial x}(x-y)$

$= -e^{x-y}(1-0)$

$\boxed{f_{yx} = -e^{x-y}}$

24. $f = \frac{x+y}{x-y}$

$f_x = \frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right)$

$f_x = \frac{(x-y)\,\partial/\partial x(x+y) - (x+y)\frac{\partial}{\partial x}(x-y)}{(x-y)^2}$

$f_x = \frac{(x-y)(1+0) - (x+y)(1-0)}{(x-y)^2}$

$f_x = \frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2}$

$f_x = \frac{-2y}{(x-y)^2}$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-2y}{(x-y)^2}\right)$

$= -2y\frac{\partial}{\partial x}(x-y)^{-2}$

$= -2y\cdot(-2)(x-y)^{-3}\frac{\partial}{\partial x}(x-y)$

$= 4y(x-y)^{-3}\cdot(1-0)$

$= \frac{4y}{(x-y)^3}$

$f_y = \frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right)$

$f_y = \frac{(x-y)\frac{\partial}{\partial y}(x+y) - (x+y)\frac{\partial}{\partial y}(x-}{(x-y)^2}$

$f_y = \frac{(x-y)(0+1) - (x+y)(0-1)}{(x-y)^2}$

$f_y = \frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2}$

$f_y = \frac{2x}{(x-y)^2}$

$f_{yy} = \frac{\partial}{\partial y}\left(\frac{2x}{(x-y)^2}\right) = 2x\frac{\partial}{\partial y}(x-$

$f_{yy} = 2x.\left[-2(x-y)^{-3}\frac{\partial}{\partial y}(x-y\right.$

$= -\frac{4x}{(x-y)^3}(0-1)$

$f_{yy} = \frac{4x}{(x-y)^3}$

$$f_x = \frac{-2y}{(x-y)^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left[\frac{-2y}{(x-y)^2}\right]$$

$$= -2\,\frac{\partial}{\partial y}\left[\frac{y}{(x-y)^2}\right]$$

$$= -2\left[\frac{(x-y)^2(1) - y.2(x-y)(0-1)}{(x-y)^4}\right]$$

$$f_{xy} = -2\left[\frac{(x-y)^2 + 2y(x-y)}{(x-y)^4}\right]$$

$$= -2\cancel{(x-y)}\left[\frac{x-y+2y}{(x-y)^{\cancel{4}\,3}}\right]$$

$$= -2\,\frac{(x+y)}{(x-y)^3}$$

$$f_y = \frac{2x}{(x-y)^2}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)$$

$$f_{yx} = \frac{(x-y)^2(2) - (2x)2(x-y)(1)}{(x-y)^4}$$

$$f_{yx} = \frac{2\cancel{(x-y)}}{(x-y)^3}\left[\frac{x-y-2x}{\cancel{x-y}}\right]$$

$$f_{yx} = 2\left[\frac{-x-y}{(x-y)^3}\right]$$

$$f_{yx} = \frac{-2(x+y)}{(x-y)^3}$$

25. $$f = e^{x^y}$$

$$f_x = e^{x^y}\frac{\partial}{\partial x}(x^y)$$

$$f_x = e^{x^y}.\,y x^{y-1}$$

$$f_{xx} = y\,\frac{\partial}{\partial x}\left(e^{x^y}x^{y-1}\right)$$

$$f_{xx} = y\left[e^{x^y}\frac{\partial}{\partial x}(x^{y-1}) + x^{y-1}\frac{\partial}{\partial x}e^{x^y}\right]$$

$$f_{xx} = y\left[e^{x^y}.(y-1)x^{y-2} + x^{y-1}e^{x^y}.\frac{\partial}{\partial x}x^y\right]$$

$$f_{xx} = y\left[(y-1)e^{x^y}x^{y-2} + x^{y-1}e^{x^y}y x^{y-1}\right]$$

$$f_{xx} = \left[y(y-1)x^{y-2} + y^2 x^{y-1+y-1}\right]e^{x^y}$$

$$= \left[y(y-1)x^{y-2} + y^2x^{2y-2}\right]e^{x^y}$$

$$f_x = y e^{x^y}.\,x^{y-1}$$

$$f_{xy} = y e^{x^y}\frac{\partial}{\partial y}x^{y-1} + y x^{y-1}\frac{\partial}{\partial x}e^{x^y} + e^{x^y}x^{y-1}\frac{\partial}{\partial y}(y)$$

$$f_{xy} = y e^{x^y}.\,x^{y-1}\ln x + y x^{y-1}.\,e^{x^y}\frac{\partial}{\partial x}x^y + e^{x^y}x^{y-1}(1)$$

$$= y e^{x^y}x^{y-1}\ln x + y x^{y-1}e^{x^y}x^y\ln x + e^{x^y}x^{y-1}$$

$$= e^{x^y}\left[y x^{y-1}\ln x + y x^{2y-1}\ln x + x^{y-1}\right]$$

$$f = e^{x^y}$$

$$f_y = e^{x^y}\frac{\partial}{\partial x}(x^y) \qquad \because \ln a . a^x = \frac{d}{dx}a^x$$

$$= e^{x^y}.\,x^y\ln x$$

$$f_{yy} = \ln x\left[\frac{\partial}{\partial y}(x^y.\,e^{x^y})\right]$$

$$= \ln x\left[e^{x^y}\frac{\partial}{\partial y}x^y + x^y\frac{\partial}{\partial y}e^{x^y}\right]$$

$$= \ln x\left[e^{x^y}.\,x^y\ln x + x^y.\,e^{x^y}\frac{\partial}{\partial y}x^y\right]$$

$$= \ln x\left[e^{x^y}.\,x^y\ln x + x^y e^{x^y}.\,x^y\ln x\right]$$

$$= e^{x^y}\left[x^y(\ln x)^2 + x^{y+y}(\ln x)^2\right]$$

$$= e^{x^y}\left[x^y(\ln x)^2 + x^{2y}(\ln x)^2\right]$$

$$f_y = e^{x^y}x^y\ln x$$

$$f_{yx} = \frac{\partial}{\partial x}\left[e^{x^y}x^y\ln x\right]$$

$$f_{yx} = e^{x^y}x^y\frac{\partial}{\partial x}\ln x + e^{x^y}\ln x\frac{\partial}{\partial x}x^y + x^y\ln x\frac{\partial}{\partial x}e^{x^y}$$

$$= e^{x^y}x^y.\frac{1}{x} + e^{x^y}\ln x\, y x^{y-1} + x^y\ln x.\,e^{x^y}\frac{\partial}{\partial x}x^y$$

$$= e^{x^y}x^{y-1} + e^{x^y}\ln x\, y x^{y-1} + x^y\ln x.\,e^{x^y}.\,y x^{y-1}$$

$$f_{yx} = e^{x^y}\left[x^{y-1} + y x^{y-1}\ln x + y x^{y+y-1}\ln x\right]$$

$$f_{yx} = e^{x^y}\left[x^{y-1} + y x^{y-1}\ln x + y x^{2x-1}\ln x\right]$$

$$f = \tan(\tan^{-1}x + \tan^{-1}y)$$

$$f = \frac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\tan(\tan^{-1}y)}$$

$$\because \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

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$$f = \frac{x+y}{1-xy}$$

$$f_x = \frac{(1-xy)(1+0) - (x+y)(0-y)}{(1-xy)^2}$$

$$f_x = \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2}$$

$$f_x = \frac{1+y^2}{(1-xy)^2}$$

$$f_{xx} = [1+y^2]\frac{\partial}{\partial x}(1-xy)^{-2}$$

$$f_{xx} = (1+y^2)(-2)(1-xy)^{-3}\frac{\partial}{\partial x}(1-xy)$$

$$f_{xx} = (1+y^2)\frac{(-2)}{(1-xy)^3}\cdot(0-y)$$

$$f_{xx} = \frac{-2(-y)(1+y^2)}{(1-xy)^3}$$

$$f_{xy} = \frac{2y(1+y^2)}{(1-xy)^3}$$

$$f_x = \frac{1+y^2}{(1-xy)^2}$$

$$f_{xy} = \frac{(1-xy)^2(0+2y) - (1+y^2)(\frac{\partial}{\partial y}(1-xy)^2}{(1-xy)^4}$$

$$f_{xy} = \frac{2y(1-xy)^2 - (1+y^2).2(1-xy)\frac{\partial}{\partial y}(1-xy)}{(1-xy)^4}$$

$$f_{xy} = \frac{2y(1-xy)^2 - (1+y^2).2(1-xy)(-x)}{(1-xy)^4}$$

$$f_{xy} = \frac{2y(1-xy)^2 + 2x(1-xy)(1+y^2)}{(1-xy)^4}$$

$$f_{xy} = \frac{\cancel{(1-xy)}(2y(1-xy) + 2x(1+y^2))}{(1-xy)^{\cancel{4}3}}$$

$$f_{xy} = \frac{2y - \cancel{2xy^2} + 2x + \cancel{2xy^2}}{(1-xy)^2}$$

$$f_{xy} = \frac{2(x+y)}{(1-xy)^2}$$

$$f = \frac{x+y}{1-xy}$$

$$f_y = \frac{(1-xy)(0+1) - (x+y)(0-x)}{(1-xy)^2}$$

$$f_y = \frac{1 - \cancel{xy} + x^2 + \cancel{xy}}{(1-xy)^2}$$

$$f_y = \frac{1+x^2}{(1-xy)^2}$$

$$f_{yy} = (1+x^2)\frac{\partial}{\partial y}(1-xy)^{-2}$$

$$f_{yy} = (1+x^2).(-2)(1-xy)^{-3}\frac{\partial}{\partial y}(1-xy)$$

$$f_{yy} = \frac{-2(1+x^2)}{(1-xy)^3}\cdot(0-x)$$

$$f_{yy} = \frac{-2(1+x^2)(-x)}{(1-xy)^3}$$

$$f_{yy} = \frac{2x(1+x^2)}{(1-xy)^3}$$

$$f_y = \frac{1+x^2}{(1-xy)^2}$$

$$f_{yx} = \frac{(1-xy)^2(2x) - (1+x^2)2(1-xy)(-y)}{(1-xy)^4}$$

$$f_{yx} = \frac{\cancel{(1-xy)}[2x(1-xy) + 2y(1+x^2)]}{(1-xy)^{\cancel{4}3}}$$

$$f_{yx} = \frac{2x - \cancel{2x^2y} + 2y + \cancel{2x^2y}}{(1-xy)^3}$$

$$f_{yx} = \frac{2(x+y)}{(1-xy)^3}$$

In Problem 27-32. verify $f_{xy} = f_{yx}$. 7

27. $f(x,y) = e^{xy}\cos(bx+c)$

$f = e^{xy}\cos(bx+c)$

$f_x = e^{xy}\frac{\partial}{\partial x}\cos(bx+c) + \cos(bx+c)\frac{\partial}{\partial x}e^{xy}$

$f_x = e^{xy}(-\sin(bx+c))\frac{\partial}{\partial x}(bx+c) + \cos(bx+c)e^{xy}\frac{\partial}{\partial x}(xy)$

$f_x = -e^{xy}\sin(bx+c)(b) + \cos(bx+c)e^{xy}(y)$

$f_x = -b\,e^{xy}\sin(bx+c) + y\cos(bx+c)e^{xy}$

Partial derivatiue w.r.t "y",

$f_{xy} = -b\frac{\partial}{\partial y}e^{xy}\sin(bx+c) + \frac{\partial}{\partial y}ye^{xy}\cos(bx+c)$

$= -b\sin(bx+c)\left[\frac{\partial}{\partial y}e^{xy}\right] + \cos(bx+c)\left[\frac{\partial}{\partial y}e^{xy}.y\right]$

$= -b\sin(bx+c)\left[e^{xy}\frac{\partial}{\partial y}(xy)\right] + \cos(bx+c)\left[e^{xy}\frac{\partial}{\partial y}(y) + y\frac{\partial}{\partial y}e^{xy}\right]$

$= -b\sin(bx+c)\left[e^{xy}.x\right] + \cos(bx+c)\left[e^{xy} + ye^{xy}\frac{\partial}{\partial y}(xy)\right]$

$= -bx\,e^{xy}\sin(bx+c) + \cos(bx+c)(e^{xy} + ye^{xy}.x)$

$= -bxe^{xy}\sin(bx+c) + \cos(bx+c)e^{xy} + xye^{xy}\cos(bx+c)$

$f_{xy} = e^{xy}\left[-bx\sin(bx+c) + \cos(bx+c) + xy\cos(bx+c)\right] \longrightarrow 1$

$f = e^{xy}\cos(bx+c)$

$f_y = \cos(bx+c)\frac{\partial}{\partial y}e^{xy}$

$f_y = \cos(bx+c).e^{xy}\frac{\partial}{\partial y}(xy)$

$f_y = \cos(bx+c)e^{xy}(x)$

$f_y = x\,e^{xy}\cos(bx+c)$

Product rule;

$f_{yx} = xe^{xy}\frac{\partial}{\partial x}\cos(bx+c) + x\cos(bx+c)\frac{\partial}{\partial x}e^{xy} + e^{xy}\cos(bx+c)\frac{\partial}{\partial x}(x)$

$f_{yx} = xe^{xy}(-\sin(bx+c))\frac{\partial}{\partial x}(bx+c) + x\cos(bx+c)e^{xy}\frac{\partial}{\partial x}(xy) + e^{xy}\cos(bx+c)(1)$

$= -xe^{xy}\sin(bx+c)(b) + x\cos(bx+c)e^{xy}(y) + e^{xy}\cos(bx+c)$

$f_{yx} = e^{xy}\left[-xb\sin(bx+c) + xy\cos(bx+c) + \cos(bx+c)\right. \longrightarrow 2$

from 1 and 2

$f_{xy} = f_{yx}$

28. $f(x,y) = \ln(e^x + e^y)$ (8)

$f = \ln(e^x+e^y)$

$f_x = \frac{1}{e^x+e^y}\frac{\partial}{\partial x}(e^x+e^y)$

$f_x = \frac{1}{e^x+e^y}(e^x+0)$

$f_x = \frac{e^x}{e^x+e^y}$

$f_{xy} = \frac{\partial}{\partial y}\frac{e^x}{(e^x+e^y)} = \frac{\partial}{\partial y} e^x(e^x+e^y)^{-1}$

$f_{xy} = e^x \cdot \frac{\partial}{\partial y}(e^x+e^y)^{-1}$

$f_{xy} = e^x \cdot (-1)(e^x+e^y)^{-2} \cdot \frac{\partial}{\partial y}(e^x+e^y)$

$f_{xy} = -\frac{e^x}{(e^x+e^y)^2} \cdot (0+e^y)$

$f_{xy} = -\frac{e^x \cdot e^y}{(e^x+ y)^2}$

$f_{xy} = -\frac{e^{x+y}}{(e^x+e^y)^2} \longrightarrow (1)$

$f = \ln(e^x+e^y)$

$f_y = \frac{1}{e^x+e^y}\frac{\partial}{\partial y}(e^x+e^y)$

$= \frac{1}{e^x+e^y}(0+e^y)$

$f_y = \frac{e^y}{e^x+e^y} = e^y(e^x+e^y)^{-1}$

$f_{yx} = e^y \frac{\partial}{\partial x}(e^x+e^y)^{-1}$

$f_{yx} = e^y \cdot (-1)(e^x+e^y)^{-2}\frac{\partial}{\partial x}(e^x+e^y)$

$f_{yx} = -e^y \cdot \frac{1}{(e^x+e^y)^2} \cdot (e^x+0)$

$f_{yx} = -\frac{e^x \cdot e^y}{(e^x+e^y)^2}$

$f_{yx} = -\frac{e^{x+y}}{(e^x+e^y)^2} \longrightarrow (2)$

from (1) and (2)

$f_{xy} = f_{yx}$.

29. $f(x,y) = \ln\left(\frac{x^2+y^2}{xy}\right) = \ln(x^2+y^2) - \ln(xy)$

$f = \ln(x^2+y^2) - \ln(xy)$

$f_x = \frac{1}{x^2+y^2}\cdot\frac{\partial}{\partial x}(x^2+y^2) - \frac{1}{xy}\frac{\partial}{\partial x}(xy)$

$f_x = \frac{1}{x^2+y^2}(2x+0) - \frac{1}{xy}(y)$

$f_x = \frac{2x}{x^2+y^2} - \frac{1}{x} = 2x(x^2+y^2)^{-1} - \frac{1}{x}$

$f_{xy} = 2x\frac{\partial}{\partial y}(x^2+y^2)^{-1} - \frac{\partial}{\partial y}\left(\frac{1}{x}\right)$

$= 2x.(-1)(x^2+y^2)^{-2}\frac{\partial}{\partial y}(x^2+y^2) - 0$

$= \frac{-2x}{(x^2+y^2)^2}(0+2y) = \frac{-4xy}{(x^2+y^2)^2}$

$f_{xy} = \frac{-4xy}{(x^2+y^2)^2} \longrightarrow (1)$

$f = \ln(x^2+y^2) - \ln xy$

$f_y = \frac{1}{x^2+y^2}\cdot(0+2y) - \frac{1}{xy}(x)$

$f_y = \frac{2y}{x^2+y^2} - \frac{1}{y}$

$f_{yx} = 2y\frac{\partial}{\partial x}(x^2+y^2)^{-1} - \frac{\partial}{\partial x}(1/y)$

$f_{yx} = 2y(-1)(x^2+y^2)^{-2}\frac{\partial}{\partial x}(x^2+y^2) - 0$

$f_{yx} = -\frac{2y}{(x^2+y^2)^2}\cdot(2x+0)$

$f_{yx} = \frac{-4xy}{(x^2+y^2)^2}$

from 1 & 2 $f_{xy} = f_{yx}$

30. $f(x,y) = x^y + y^x$

$$f = x^y + y^x$$
$$f_x = \frac{\partial}{\partial x} x^y + \frac{\partial}{\partial x} y^x$$
$$f_x = y x^{y-1} + y^x \ln y$$
$$f_{xy} = y\frac{\partial}{\partial y} x^{y-1} + x^{y-1}\frac{\partial}{\partial y}(y) + y^x \frac{\partial}{\partial y}\ln y + \ln y \frac{\partial}{\partial y} y^x$$
$$f_{xy} = y.\, x^{y-1}\ln(x)\frac{\partial}{\partial y}(y-1) + x^{y-1} + y^x.\frac{1}{y} + \ln y.\, x y^{x-1}$$
$$= y x^{y-1}\ln x\,(1) + x^{y-1} + y^{x-1} + y^{x-1} x \ln y$$
$$f_{xy} = x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \longrightarrow 1$$

$\because \frac{d}{dx} a^x = a^x \ln a$

$$f = x^y + y^x$$
$$f_y = \frac{\partial}{\partial y} x^y + \frac{\partial}{\partial y} y^x$$
$$f_y = x^y \ln x + x y^{x-1}$$
$$f_{yx} = x^y \frac{\partial}{\partial x}\ln x + \ln x \frac{\partial}{\partial x} x^y + x\frac{\partial}{\partial x} y^{x-1} + y^{x-1}\frac{\partial}{\partial x}(x)$$
$$f_{yx} = x^y.\frac{1}{x} + \ln x.\, y.\, x^{y-1} + x.\, y^{x-1}\ln y \frac{\partial}{\partial x}(x-1) + y^{x-1}(1)$$
$$f_{yx} = x^{y-1} + y x^{y-1}\ln x + x y^{x-1}\ln y + y^{x-1}$$
$$f_{yx} = x^{y-1}(1 + y\ln x) + y^{x-1}(x\ln y + 1) \quad 2$$

from 1 and 2 $f_{xy} = f_{yx}$

31. $f(x,y) = x\sin xy + y\cos xy$

$$f_x = x\frac{\partial}{\partial x}\sin(xy) + \sin(xy)\frac{\partial}{\partial x}(x) + y\frac{\partial}{\partial x}\cos(xy)$$
$$= x\cos(xy)\frac{\partial}{\partial x}(xy) + \sin xy + y(-\sin(xy))\frac{\partial}{\partial x}(xy)$$
$$= x\cos(xy).\,(y) + \sin xy + y(-\sin xy)(y)$$
$$= xy\cos(xy) + \sin xy - y^2 \sin xy$$
$$f_{xy} = x\frac{\partial}{\partial y}[y\cos xy] + \frac{\partial}{\partial y}[\sin xy] - \frac{\partial}{\partial y}[y^2 \sin xy]$$
$$f_{xy} = x\left[y\frac{\partial}{\partial y}\cos xy + \cos xy\frac{\partial}{\partial y}y\right] + \cos xy.\frac{\partial}{\partial y}(xy) - \left[\sin xy\frac{\partial}{\partial y}y^2 + y^2\frac{\partial}{\partial y}\sin xy\right]$$
$$f_{xy} = x\left[-y\sin xy.\frac{\partial}{\partial y}(xy) + \cos xy\right] + \cos xy.\, x - \left[\sin xy.\,2y + y^2\cos xy.\frac{\partial}{\partial y}xy\right]$$
$$f_{xy} = -xy\sin xy.\,(x) + x\cos xy + x\cos xy - 2y\sin xy - y^2\cos xy.\,x$$
$$f_{xy} = \sin xy(-x^2y - 2y) + 2x\cos xy - xy^2\cos xy$$
$$f_{xy} = -\sin xy\,(x^2y + 2y) + \cos xy\,(2x - xy^2) \longrightarrow 1$$

$$f = x\sin xy + y\cos xy$$

$$f_y = x\frac{\partial}{\partial y}\sin xy + \cos xy\frac{\partial}{\partial y}(y) + y\frac{\partial}{\partial y}\cos xy$$

$$= x\cos xy\frac{\partial}{\partial y}(xy) + \cos xy + y(-\sin xy)\frac{\partial}{\partial y}(xy)$$

$$= x\cos xy.(x) + \cos xy - y\sin xy.(x)$$

$$= x^2\cos xy + \cos xy - xy\sin xy$$

$$f_y = (x^2+1)\cos xy - xy\sin xy$$

$$f_{yx} = (x^2+1)\frac{\partial}{\partial x}\cos xy + \cos xy\frac{\partial}{\partial x}(x^2+1) - y\frac{\partial}{\partial x}(x\sin xy)$$ product rule

$$= (x^2+1)(-\sin xy)\frac{\partial}{\partial x}(xy) + \cos(xy)(2x+0) - y\left[x.\cos xy\frac{\partial}{\partial x}(xy) + \sin xy(1)\right]$$

$$= (x^2+1)(-\sin xy)(y) + 2x\cos xy - y(x\cos xy.y + \sin xy]$$

$$= (x^2+1)(-y\sin xy) + 2x\cos xy - xy^2\cos xy - y\sin xy$$

$$= -x^2y\sin xy - y\sin xy + 2x\cos xy - xy^2\cos xy - y\sin xy$$

$$= -x^2y\sin xy - 2y\sin xy + (2x - xy^2)\cos xy$$

$$= -\sin xy(x^2y + 2y) + (2x - xy^2)\cos xy \longrightarrow (2)$$

From (1) & (2)

$$f_{xy} = f_{yx}$$

32. $$f(x,y) = \frac{xy}{\sqrt{1+x^2+y^2}} = xy(1+x^2+y^2)^{-1/2}$$

$$f_x = y.\frac{\partial}{\partial x}\ x(1+x^2+y^2)^{-1/2}$$

$$= y\left[x\frac{\partial}{\partial x}(1+x^2+y^2)^{-1/2} + (1+x^2+y^2)^{-1/2}\frac{\partial}{\partial x}(x)\right]$$

$$= y\left[x.\left(-\frac{1}{2}\right)(1+x^2+y^2)^{-3/2}\underbrace{\frac{\partial}{\partial x}(1+x^2+y^2)}_{(2x)} + \frac{1}{(1+x^2+y^2)^{1/2}}\right]$$

$$= y\left[-\frac{x}{2}.\frac{1}{(1+x^2+y^2)^{3/2}}.(0+2x+0) + \frac{1}{(1+x^2+y^2)^{1/2}}\right]$$

$$= y\left[\frac{-\cancel{2}x^2}{\cancel{2}(1+x^2+y^2)^{3/2}} + \frac{1}{(1+x^2+y^2)^{1/2}}\right]$$

$$= y\left[\frac{-\cancel{x^2} + 1 + \cancel{x^2} + y^2}{(1+x^2+y^2)^{3/2}}\right] = \frac{y(1+y^2)}{(1+x^2+y^2)^{3/2}}$$

$$f_x = \frac{y(1+y^2)}{(1+x^2+y^2)^{3/2}} = \frac{y+y^3}{(1+x^2+y^2)^{3/2}}$$ (11)

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{y+y^3}{(1+x^2+y^2)^{3/2}}\right)$$ Quotient rule

$$= \frac{(1+x^2+y^2)^{3/2}\frac{\partial}{\partial y}(y+y^3) - (y+y^3)\frac{\partial}{\partial y}(1+x^2+y^2)^{3/2}}{\left[(1+x^2+y^2)^{3/\cancel{2}}\right]^{\cancel{2}}}$$

power rule

$$= \frac{(1+x^2+y^2)^{3/2}(1+3y^2) - (y+y^3)\left(\frac{3}{2}\right)(1+x^2+y^2)^{1/2}\frac{\partial}{\partial y}\underset{(0+0+2y)}{(1+x^2+y^2)}}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{3/2}(1+3y^2) - (y+y^3)\cdot\left(\frac{3}{\cancel{2}}\right)(1+x^2+y^2)^{1/2}(\cancel{2}y)}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{1/2}}{(1+x^2+y^2)^3}\left[(1+x^2+y^2)(1+3y^2) - 3(y+y^3)y\right]$$

$$= \frac{1}{(1+x^2+y^2)^{3-\frac{1}{2}}}\left[1+x^2+y^2+\cancel{3y^2}+3x^2y^2+\cancel{3y^4}-\cancel{3y^2}-\cancel{3y^4}\right]$$

$$= \frac{1}{(1+x^2+y^2)^{5/2}}\left[1+x^2+y^2+3x^2y^2\right]$$

$$f_{xy} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^{5/2}} \longrightarrow (1)$$

$$f = \frac{xy}{\sqrt{1+x^2+y^2}}$$

$$f_y = \frac{\partial}{\partial y}\cdot\frac{xy}{\sqrt{1+x^2+y^2}} = x\frac{\partial}{\partial y}\frac{y}{\sqrt{1+x^2+y^2}}$$

$$f_y = x.\ \frac{\sqrt{1+x^2+y^2}\,\frac{\partial}{\partial y}(y) - y\cdot\frac{\partial}{\partial y}\sqrt{1+x^2+y^2}}{\left(\sqrt{1+x^2+y^2}\right)^2}$$

$$f_y = x\left[\frac{\sqrt{1+x^2+y^2} - y\cdot\frac{1}{2\sqrt{x^2+y^2+1}}\cdot\frac{\partial}{\partial y}(1+x^2+y^2)}{\left(\sqrt{1+x^2+y^2}\right)^{\cancel{2}}}\right]$$

$$f_y = x\left[\frac{\sqrt{1+x^2+y^2} - \frac{y\,(\cancel{2}y)}{\cancel{2}\sqrt{x^2+y^2+1}}}{1+x^2+y^2}\right]$$

$$f_y = \frac{x}{1+x^2+y^2}\left[\frac{1+x^2+\cancel{y^2}-\cancel{y^2}}{\sqrt{x^2+y^2+1}}\right]$$

$$f_y = \frac{x(1+x^2)}{(1+x^2+y^2)^{3/2}}$$

$$f_y = \frac{x+x^3}{(1+x^2+y^2)^{3/2}}$$

$$f_{yx} = \frac{(1+x^2+y^2)^{3/2}\frac{\partial}{\partial x}(x+x^3) - (x+x^3)\frac{\partial}{\partial x}(1+x^2+y^2)^{3/2}}{\left[(1+x^2+y^2)^{\frac{3}{2}}\right]^2} \quad \text{Quotient rule}$$

$$f_{yx} = \frac{(1+x^2+y^2)^{3/2}(1+x^2) - (x+x^3)\cdot\frac{3}{2}(x^2+1+y^2)^{1/2}\frac{\partial}{\partial x}(1+x^2+y^2)}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{3/2}(1+3x^2) - \frac{3}{\cancel{2}}(x+x^3)(1+x^2+y^2)^{1/2}(\cancel{2}x)}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{1/2}}{(1+x^2+y^2)^3}\left[(1+x^2+y^2)(1+3x^2) - 3(x+x^3)(x)\right]$$

$$= \frac{1}{(1+x^2+y^2)^{5/2}}\left[1+x^2+y^2+3\cancel{x^2}+3\cancel{x^4}+3x^2y^2-3\cancel{x^2}-3\cancel{x^4}\right]$$

$$f_{yx} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^{5/2}} \longrightarrow (2)$$

from (1) and (2)

$$f_{xy} = f_{yx}$$

Show that each of the following function satisfies Laplace's eq.

(13)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

33. $f(x,y) = \sin x \sinh y$

$$\frac{\partial f}{\partial x} = f_x = \frac{\partial}{\partial x}(\sin x \sinh y)$$
$$\frac{\partial f}{\partial x} = \sinh y \frac{\partial}{\partial x}\sin x$$
$$\frac{\partial f}{\partial x} = \sinh y \cos x$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \sinh y \frac{\partial}{\partial x}(\cos x)$$
$$\frac{\partial^2 f}{\partial x^2} = \sinh y(-\sin x)$$
$$= -\sin x \sinh y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\sin x \sinh y)$$
$$= \sin x \frac{\partial}{\partial y}(\sinh y)$$
$$\frac{\partial f}{\partial y} = \sin x \cosh y$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \sin x \frac{\partial}{\partial y}(\cosh y)$$
$$\frac{\partial^2 f}{\partial y^2} = \sin x \sinh y$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\sin x \cancel{\sinh y} + \sin x \sinh y$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

34. $f(x,y) = e^{-x}\cos y$

$$\frac{\partial f}{\partial x} = \cos y \frac{\partial}{\partial x} e^{-x}$$
$$= \cos y\, e^{-x}\frac{\partial}{\partial x}(-x)$$
$$\frac{\partial f}{\partial x} = -e^{-x}\cos y$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -\cos y \frac{\partial}{\partial x} e^{-x}$$
$$\frac{\partial^2 f}{\partial x^2} = -\cos y\, e^{-x}\frac{\partial}{\partial x}(-x)$$
$$= -\cos y\, e^{-x}(-1)$$
$$\frac{\partial^2 f}{\partial x^2} = e^{-x}\cos y$$

$$\frac{\partial f}{\partial y} = e^{-x}\frac{\partial}{\partial y}\cos y$$
$$= e^{-x}(-\sin y)$$
$$\frac{\partial f}{\partial y} = -e^{-x}\sin y$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -e^{-x}\frac{\partial}{\partial y}\sin y$$
$$= -e^{-x}\cos y$$
$$\frac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-x}\cancel{\cos y} - e^{x}\cancel{\cos y} = 0$$

35. $f(x,y) = \ln\sqrt{x^2+y^2}$

$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \ln\sqrt{x^2+y^2}$

$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x}(x^2+y^2)^{1/2}$

$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \frac{\partial}{\partial x}(x^2+y^2)$

$\frac{\partial f}{\partial x} = \frac{1}{2(\sqrt{x^2+y^2})^2}(2x+0)$

$= \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)}$

$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{(x^2+y^2)(1) - x(2x+0)}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \ln\sqrt{x^2+y^2}$

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y}(x^2+y^2)^{1/2}$

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \frac{\partial}{\partial y}(x^2+y^2)$

$\frac{\partial f}{\partial y} = \frac{1}{2(x^2+y^2)} \cdot (0+2y)$

$\frac{\partial f}{\partial y} = \frac{2y}{2(x^2+y^2)}$

$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{(x^2+y^2)(1) - y(0+2y)}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial y^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$

$= \frac{\cancel{-x^2}+\cancel{y^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

36. $f(x,y) = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \frac{\partial}{\partial x}\left(\frac{2xy}{x^2-y^2}\right)$

$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2y \frac{\partial}{\partial x}\left(\frac{x}{x^2-y^2}\right)$

$= \frac{1}{\frac{(x^2-y^2)^2+4x^2y^2}{(x^2-y^2)^2}} \cdot 2y \cdot \frac{(x^2-y^2)(1) - x(2x-0)}{(x^2-y^2)^2}$

$= \frac{2y\cancel{(x^2-y^2)^2}}{x^4+y^4-2x^2y^2+4x^2y^2} \cdot \frac{x^2-y^2-2x^2}{\cancel{(x^2-y^2)^2}} = \frac{2y}{x^4+y^4+2x^2y^2} \cdot \frac{-x^2-y^2}{1}$

$$\therefore \frac{\partial f}{\partial x} = \frac{2y}{(x^2+y^2)\cancel{^2}} \cdot \left[-\cancel{(x^2+y^2)}\right]$$

$$= \frac{-2y}{x^2+y^2} = -2y(x^2+y^2)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = -2y \frac{\partial}{\partial x}(x^2+y^2)^{-1} = -2y \cdot (-1)(x^2+y^2)^{-2} \frac{\partial}{\partial x}(x^2+y^2)$$

$$= \frac{2y}{(x^2+y^2)^2} \cdot (2x+0)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2}$$

$$f = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$$

$$f_y = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2x \frac{\partial}{\partial y}\left(\frac{y}{x^2-y^2}\right)$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} \cdot 2x \cdot \frac{(x^2-y^2)(1)-y(-2y)}{\cancel{(x^2-y^2)^2}}$$

$$= \frac{2x}{x^4+y^4-2x^2y^2+4x^2y^2} \cdot \frac{x^2-y^2+2y^2}{1}$$

$$= \frac{2x}{x^4+y^4+2x^2y^2} \cdot (x^2+y^2)$$

$$= \frac{2x}{(x^2+y^2)^{\cancel{2}}} \cdot \cancel{(x^2+y^2)} = 2x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \frac{\partial}{\partial y}(x^2+y^2)^{-1}$$

$$= 2x \cdot (-1)(x^2+y^2)^{-2} \frac{\partial}{\partial y}(x^2+y^2)$$

$$= -\frac{2x}{(x^2+y^2)^2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \cancel{\frac{4xy}{(x^2+y^2)^2}} - \cancel{\frac{4xy}{(x^2+y^2)^2}} = 0$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$