nth derivatives:

1. Let $y = (ax+b)^m$ <u>Exercise 2.5</u>

$$y' = m(ax+b)^{m-1}\frac{d}{dx}(ax+b) = m(ax+b)^{m-1}(a+0)$$

$$= am\,(ax+b)^{m-1}$$

$$y'' = ma\,\frac{d}{dx}(ax+b)^{m-1} = ma\left[(m-1)(ax+b)^{m-2}\frac{d}{dx}(ax+b)\right];$$

$$y'' = ma\left[(m-1)(ax+b)^{m-2}(a+0)\right]$$

$$y'' = m(m-1)a^2\,(ax+b)^{m-2}$$

$$y''' = m(m-1)a^2\,\frac{d}{dx}(ax+b)^{m-2} = m(m-1)a^2\left[(m-2)(ax+b)^{(m-3)}\frac{d}{dx}(ax+b)\right]$$

$$y''' = m(m-1)\,a^2\left[(m-2)(ax+b)^{m-3}\,(a+0)\right]$$

$$y''' = m(m-1)(m-2)\,a^3\,(ax+b)^{m-3}$$

$$\vdots$$

$$y^{(n)} = m(m-1)(m-2)\ldots(m-(n-1))a^n(ax+b)^{m-n}$$

$$y^{(n)} = \frac{m(m-1)(m-2)\ldots(m-(n-1))(m-n)!}{(m-n)!}\,a^n\,(ax+b)^{m-n}$$

$$y^{(n)} = \frac{m!}{(m-n)!}\,a^n\,(ax+b)^{m-n}$$

<u>**Corollary 1**</u>: if $m = -1$

$$y = (ax+b)^{-1} = \frac{1}{ax+b}$$

$$y^{(n)} = (-1)(-1-1)(-1-2)\cdots(-1-(n-1))a^n\,(ax+b)^{-1-n}$$

$$= (-1)(-2)(-3)\ldots\ldots(\cancel{-1}-n+\cancel{1})\,a^n\,(ax+b)^{-(1+n)}$$

$$= \frac{(-1)^n\,(1)(2)(3)\ldots(n)\,a^n}{(ax+b)^{n+1}}$$

$$= \frac{(-1)^n\,n!\,a^n}{(ax+b)^{n+1}}$$

<u>**Corollary 2**</u>: Let $y = \ln(ax+b)$

$$y' = \frac{1}{ax+b}\,\frac{d}{dx}(ax+b) = \frac{1}{ax+b}(a+0)$$

$$y' = \frac{a}{ax+b}$$

Taking its $(n-1)$th derivative.

$$y^{(n)} = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{a}{ax+b}\right] \quad ②$$

$$y^n = a\,\frac{d^{n-1}}{dx^{n-1}}\left[\frac{1}{ax+b}\right]$$

$$y^n = a\left(\frac{(-1)^{n-1}(n-1)!\,a^{n-1}}{(ax+b)^n}\right)$$

$$y^n = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$$

In each of Problem 1-4, find the nth derivative.

1. $\dfrac{x}{x^2-a^2} = \dfrac{x}{(x-a)(x+a)}$

partial fraction

$$\frac{x}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} \longrightarrow ①$$

multiplying both sides with $(x-a)(x+a)$

$$\cancel{(x-a)}\cancel{(x+a)}\cdot\frac{x}{\cancel{(x-a)}\cancel{(x+a)}} = \frac{A}{\cancel{x-a}}(x+a)\cancel{(x-a)} + \frac{B}{\cancel{(x+a)}}\cancel{(x+a)}(x-a)$$

$$x = A(x+a) + B(x-a) \rightarrow ②$$

put $x=-a$ in ②

$$-a = A(-a+a)+B(-a-a)$$
$$-a = 0 + B(-2a)$$
$$-a = -2aB$$
$$B = \frac{-a}{-2a} = \frac{1}{2}$$
$$\boxed{B=\frac{1}{2}}$$

put $x=a$ in ②

$$a = A(a+a)+B(a-a)$$
$$a = 2aA + 0$$
$$A = \frac{a}{2a}$$
$$\boxed{A=\frac{1}{2}}$$

putting values in ①

$$\frac{x}{x^2-a^2} = \frac{1}{2(x-a)} + \frac{1}{2(x+a)}$$

$$\frac{x}{x^2-a^2} = \frac{1}{2}\left[\frac{1}{x-a} + \frac{1}{x+a}\right]$$

Taking nth derivative

$$\frac{d^n}{dx^n}\left[\frac{x}{x^2-a^2}\right] = \frac{1}{2}\left[\frac{d^n}{dx^n}\left(\frac{1}{x-a}\right) + \frac{d^n}{dx^n}\left(\frac{1}{x+a}\right)\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^n n!\,(1)^n}{(x-a)^{n+1}} + \frac{(-1)^n n!\,(1)^n}{(x+a)^{n+1}}\right]$$

$$\frac{d^n}{dx^n}\left[\frac{x}{x^2-a^2}\right] = \frac{1}{2}\left[\frac{(-1)^n n!}{(x-a)^{n+1}} + \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$ ③

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x-a)^{n+1}} + \frac{1}{(x+a)^{n+1}}\right]$$

2. $\dfrac{x^4}{(x-1)(x-2)} = \dfrac{x^4}{x^2-3x+2}$ (improper fraction)

$$\begin{array}{r|l} & x^2+3x+7 \\ x^2-3x+2 & \cancel{x^4} \\ & \pm\cancel{x^4} \mp 3x^3 \pm 2x^2 \\ \hline & 3\cancel{x^3} - 2x^2 \\ & \pm\cancel{3x^3} \mp 9x^2 \pm 6x \\ \hline & +7\cancel{x^2} - 6x \\ & \pm\cancel{7x^2} \mp 21x \pm 14 \\ \hline & 15x-14 \end{array}$$

$$\frac{x^4}{(x-1)(x-2)} = x^2+3x+7+\frac{15x-14}{x^2-3x+2}$$

$$= x^2+3x+7+\frac{15x-14}{(x-1)(x-2)} \longrightarrow ①$$

Resolving into partial fraction

$$\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \longrightarrow ②$$

$$15x-14 = A(x-2) + B(x-1) \longrightarrow ③$$

put $x=2$ in ③

$$15(2)-14 = A(2-2) + B(2-1)$$

$$30-14 = A(0) + B(1)$$

$$\boxed{16 = B}$$

put $x=1$ in ③

$$15(1)-14 = A(1-2) + B(1-1)$$

$$1 = B(-1) + 0$$

$$\boxed{B = -1}$$

put values in ②

$$\frac{15x-14}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{16}{x-2} = -\frac{1}{x-1} + \frac{16}{x-2}$$

put this in ①

$$\frac{x^4}{(x-1)(x-2)} = x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}$$

Taking nth derivative.

$$\frac{d^n}{dx^n}\left[\frac{x^4}{(x-1)(x-2)}\right] = \frac{d^n}{dx^n}\left[x^2+3x+7-\frac{1}{x-1}+\frac{16}{x-2}\right]$$

$$= \frac{d^n}{dx^n}(x^2) + 3\frac{d^n}{dx^n}(x) + \frac{d^n}{dx^n}(7) - \frac{d^n}{dx^n}\left(\frac{1}{x-1}\right) + 16\frac{d^n}{dx^n}\left(\frac{1}{x-2}\right)$$

$$= 0+0+0-\left[\frac{(-1)^n n!(1)^n}{(x-1)^{n+1}}\right] + 16\left[\frac{(-1)^n n!(1)^n}{(x-2)^{n+1}}\right]$$

$$= (-1)^n n!\left[-\frac{(1)^n}{(x-1)^{n+1}} + \frac{16(1)^n}{(x-2)^{n+1}}\right]$$

$$= (-1)^n n!\left[-\frac{1}{(x-1)^{n+1}} + \frac{16}{(x-2)^{n+1}}\right]$$

$y = e^{ax}\sin(bx+c)$

differentiating w.r.t 'x' (4)

$$y' = e^{ax}\frac{d}{dx}\sin(bx+c) + \sin(bx+c)\frac{d}{dx}e^{ax}$$

$$= e^{ax}\cos(bx+c).b + \sin(bx+c).ae^{ax}$$

$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

put $a = r\cos\theta$, $b = r\sin\theta$, $\tan\theta = \frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$y' = e^{ax}[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)]$$

$a^2+b^2 = r^2(\sin^2\theta+\cos^2\theta)$

$$y' = re^{ax}[\sin(bx+c)\cos\theta + \sin\theta\cos(bx+c)]$$

$a^2+b^2 = r^2$

$\Rightarrow r = \sqrt{a^2+b^2}$

$$y' = re^{ax}\sin(bx+c+\theta)$$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Similarly

$$y'' = r\frac{d}{dx}[e^{ax}\sin(bx+c+\theta)]$$

$$= r[re^{ax}\sin(bx+c+\theta+\theta)]$$

$$y'' = [r^2e^{ax}\sin(bx+c+2\theta)]$$

$$y''' = r^3e^{ax}\sin(bx+c+3\theta)$$

$\because r = (a^2+b^2)^{1/2}$

$r^n = [(a^2+b^2)^{1/2}]^n$

$r^n = (a^2+b^2)^{n/2}$

$$y^{(n)} = r^n e^{ax}\sin(bx+c+n\theta)$$

$$y^{(n)} = (a^2+b^2)^{n/2} e^{ax}\sin(bx+c+n\tan^{-1}(\tfrac{b}{a})]$$

4. $$y = e^{ax}\cos^2x\sin x$$

$$y = e^{ax}\left[\frac{1+\cos2x}{2}\right]\sin x$$

$\because \cos\alpha = \sqrt{\frac{1+\cos2\alpha}{2}}$

$$y = \frac{e^{ax}}{2}(1+\cos2x)\sin x$$

$$y = \frac{1}{2}[e^{ax}\sin x + e^{ax}\cos2x\sin x]$$

$$y = \frac{1}{2}\left[e^{ax}\sin x + e^{ax}\frac{1}{2}[2\cos2x\sin x]\right]$$

$\because 2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$$= \frac{1}{2}\left[e^{ax}\sin x + \frac{e^{ax}}{2}[\sin(2x+x) - \sin(2x-x)]\right]$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{e^{ax}}{4}[\sin3x - \sin x]$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{e^{ax}}{4}\sin3x - \frac{e^{ax}}{4}\sin x$$

$$= \left(\frac{1}{2}-\frac{1}{4}\right)e^{ax}\sin x + \frac{1}{4}e^{ax}\sin3x$$

$$= \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin3x.$$

Taking nth derivative

$$\frac{d^n}{dx^n}(y) = \frac{1}{4}\frac{d^n}{dx^n}\left(e^{ax}\sin x\right) + \frac{1}{4}\frac{d^n}{dx^n}\left(e^{ax}\sin 3x\right) \qquad (5)$$

$a=a,\ b=1,\ c=0$ $\qquad$ $a=a,\ b=3,\ c=0$

$$y^{(n)} = \frac{1}{4}(a^2+1)^{n/2} e^{ax}\sin\left(x+0+n\tan^{-1}\left(\frac{1}{a}\right)\right) + \frac{1}{4}\left[(a^2+9)^{n/2} e^{ax}\sin\left(3x+0+n\tan^{-1}\left(\frac{3}{a}\right)\right)\right]$$

$$y^n = \frac{1}{4}(a^2+1)^{n/2} e^{ax}\sin\left(x+n\tan^{-1}\left(\frac{1}{a}\right)\right) + \frac{1}{4}(a^2+9)^{n/2} e^{ax}\sin\left(3x+n\tan^{-1}\left(\frac{3}{a}\right)\right)$$

5. if $x^y = e^{x-y}$, find $\frac{d^ny}{dx^n}$.

$$\ln x^y = \ln e^{x-y}$$
$$y\ln x = (x-y)\ln e \qquad \because \ln e = 1$$
$$y\ln x = x-y$$
$$y\ln x + y = x$$
$$y(\ln x+1) = x \longrightarrow (1)$$

Let

$u = y$ $\qquad$ $v = 1+\ln x$

$$v^{(n)} = 0 + \frac{(-1)^{n-1}(n-1)!\,(1)^n}{x^n}$$
$$v^n = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Differentiating (1) by Leibniz' Theorem,

$$y^n(1+\ln x) + n y^{(n-1)}\times\left(\frac{1}{x}\right) + \frac{n(n-1)}{2!}y^{(n-2)}\left(-\frac{1}{x^2}\right) + \cdots\cdots + \cdots\cdots + n y' \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + y\frac{(-1)^{n-1}(n-1)!}{x^n} = 0$$

6. if $f(x) = \overline{\ln(1+\sqrt{1-x})}$, prove that

$$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$$

Sol:

$$f(x) = \ln(1+\sqrt{1-x})$$
$$f'(x) = \frac{1}{1+\sqrt{1-x}}\frac{d}{dx}(1+\sqrt{1-x})$$

$$f'(x) = \frac{1}{1+\sqrt{1-x}}\left(0+\frac{1}{2\sqrt{1-x}}(-1)\right)$$

$$= \frac{1}{1+\sqrt{1-x}}\left(\frac{-1}{2\sqrt{1-x}}\right)$$

$$2\sqrt{1-x}\, f'(x) = \frac{-1}{1+\sqrt{1-x}} \times \frac{1-\sqrt{1-x}}{1-\sqrt{1-x}}$$

$$2\sqrt{1-x}\, f'(x) = \frac{-1+\sqrt{1-x}}{1-(\sqrt{1-x})^2} = \frac{-1+\sqrt{1-x}}{1-(1-x)} = \frac{-1+\sqrt{1-x}}{\cancel{1}-\cancel{1}+x}$$

$$2\sqrt{1-x}\, f'(x) = \frac{-1+\sqrt{1-x}}{x}$$

$$2x\sqrt{1-x}\, f'(x) = -1+\sqrt{1-x}$$

differentiating w.r.t 'x'

$$2\left[x\sqrt{1-x}\frac{d}{dx}f'(x) + f'(x)\frac{d}{dx}\left[x\sqrt{1-x}\right]\right] = -0+\frac{1}{2\sqrt{1-x}}(-1)$$

$$2\left[x\sqrt{1-x}\, f''(x) + f'(x)\left[x\cdot\frac{d}{dx}\sqrt{1-x}+\sqrt{1-x}\frac{d}{dx}(x)\right]\right] = -\frac{1}{2\sqrt{1-x}}$$

$$2\left[x\sqrt{1-x}\, f''(x) + f'(x)\left[x\cdot\frac{1}{2\sqrt{1-x}}(-1)+\sqrt{1-x}\right]\right] = -\frac{1}{2\sqrt{1-x}}$$

$$2x\sqrt{1-x}\, f''(x) + 2f'(x)\left[\frac{-x+2(1-x)}{2\sqrt{1-x}}\right] = -\frac{1}{2\sqrt{1-x}}$$

$$2x\sqrt{1-x}\, f''(x) + 2f'(x)\left[\frac{-x+2-2x}{2\sqrt{1-x}}\right] = -\frac{1}{2\sqrt{1-x}}$$

multiplying both sides with $2\sqrt{1-x}$

$$2x\sqrt{1-x}\, f''(x).\,2\sqrt{1-x} + 2f'(x).\,\cancel{2\sqrt{1-x}}\left[\frac{2-3x}{\cancel{2\sqrt{1-x}}}\right] = -\frac{1}{\cancel{2\sqrt{1-x}}} \times \cancel{2\sqrt{1-x}}$$

$$4x(\sqrt{1-x})^2 f''(x) + 2f'(x)\,(2-3x) = -1$$

$$4x(1-x)f''(x) + 2(2-3)f'(x) + 1 = 0$$

---

7. if $y = \tan^{-1}x$

Show $(1+x^2)y'' + 2xy' = 0$

Hence find value of y^n when $x=0$.

Sol. $y = \tan^{-1}x$ $\qquad y(0) = \tan^{-1}(0) = 0$

$y' = \frac{1}{1+x^2}$ $\qquad y'(0) = \frac{1}{1+0} = 1$

(7)

$+x^2)y' = 1$

diff again by product rule.

$(1+x^2)y'' + y'(0+2x) = 0$

$(1+x^2)y'' + 2xy' = 0$

$(1+0)y''(0) + 2(0)y'(0) = 0$

$y''(0) + 2(0)(1) = 0$

$y''(0) + 0 = 0$

$y''(0) = 0$

Differentiating by Leibnitz theorem

$$(1+x^2)y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{2!}(2)y^{(n)} + 2xy^{n+1} + n(2)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + 2nx\,y^{(n+1)} + \frac{n^2-n}{\cancel{2}} \times \cancel{2}y^{(n)} + 2xy^{n+1} + 2ny^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + 2xy^{(n+1)}(n+1) + y^{(n)}(n^2-n+2n) = 0$$

$$(1+x^2)^{y+2} + 2(n+1)xy^{(n+1)} + (n^2+n)y^{(n)} = 0$$

putting $x=0$, $(1+0)y^{(n+2)}(0) + 2(n+1)(0)y^{(n+1)}(0) + (n^2+n)y^{(n)}(0) = 0$

$$y^{(n+2)}(0) + 0 + n(n+1)y^{(n)} = 0$$

$$y^{(n+2)}(0) = -n(n+1)y^{(n)}(0) \rightarrow Ⓐ$$

For even values of n;

putting $n=2$ in Ⓐ $y^{(2+2)}(0) = -2(2+1)y^{(2)}(0)$

$y^{(4)}(0) = -2(3)\ y''(0) = -6(0) \quad \because y''(0)=0$

$y^{(4)}(0) = 0$

putting $n=4$ in Ⓐ $y^{(4+2)}(0) = -4(4-1)y^{(4)}(0)$

$y^{(6)}(0) = -4(3)(0) = 0 \quad \because y^{(4)}(0) = 0$

$y^{(6)}(0) = 0$

Generalizing, we get $y^{(2n)}(0) = 0$ ω

for odd values of 'n'.

putting $n=1$ in Ⓐ $y^{(1+2)}(0) = -1(1+1)y'(0) \quad \because y'(0)=1$

$y^{(3)}(0) = -2(1) = -2 \Rightarrow y^{(3)}(0) = (-1)^1 2!$

putting $n=3$ in (A) $y^{(3+2)}(0) = -3(3+1)y^{(3)}(0)$ $\because y^{(3)}(0) = -2$

$$= -3(4)(-2)$$
$$= (-1)(-1)\,4.3.2$$
$$y^{(5)}(0) = (-1)^2\,4!$$ $\boxed{y^{(2(2)+1)}(0)}$ (8)

putting $n=5$ in (A) $y^{(5+2)}(0) = -5(5+1)y^{(5)}(0)$

$$= -5(6)(-1)^2\,4!$$
$$= (-1)(-1)^2\,6.5.4!$$
$$y^{(7)}(0) = (-1)^3\,6!$$ $\because \boxed{y^{(2(3)+1)}(0)}$

$[2(2)]!$ $[2(3)]!$

Generalizing;

$$y^{(2n+1)}(0) = (-1)^{n}(2n)! \quad [2(n)]!$$

8. if $y = \sin(a\sin^{-1}x)$, prove that
$$(1-x^2)y^{(n+2)} = (2n+1)xy^{(n+1)} + (n^2-a^2)y^n$$

Sol:
$$y = \sin(a\sin^{-1}x)$$
$$y' = \cos(a\sin^{-1}x)\frac{d}{dx}(a\sin^{-1}x)$$
$$y' = \cos(a\sin^{-1}x).a.\frac{1}{\sqrt{1-x^2}}$$
$$\sqrt{1-x^2}\,y' = a\cos(a\sin^{-1}x)$$
Squaring both sides;
$$(1-x^2)(y')^2 = a^2\cos^2(a\sin^{-1}x)$$
$$(1-x^2)(y')^2 = a^2[1-\sin^2(a\sin^{-1}x)]$$
$$(1-x^2)(y')^2 = a^2[1-y^2]$$
Differentiating again.
$$(1-x^2)\frac{d}{dx}(y')^2 + (y')^2\frac{d}{dx}(1-x^2) = a^2\frac{d}{dy}(1-y^2)$$
$$(1-x^2)2y'y'' + (y')^2(-2x) = a^2(0-2yy')$$
$$2y'[(1-x^2)y'' - xy'] = -2a^2yy'$$
$$(1-x^2)y'' - xy' = -a^2y$$
$$\Rightarrow (1-x^2)y'' - xy' + a^2y = 0$$
Differentiating 'n' times by Leibniz's Theorem;
$$(1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \frac{n(n-1)}{2!}(-2)y^{(n)} - [xy^{(n+1)} + n(1)y^{(n)}] + a^2y^{(n)} = 0$$
$$(1-x^2)y^{(n+2)} - 2nxy^{n+1} - n(n-1)y^{(n)} - xy^{(n+1)} - ny^{(n)} + a^2y^{(n)} = 0$$
$$(1-x^2)y^{(n+2)} - (2n+1)xy^{n+1} + y^n(-n^2 + \cancel{n} - \cancel{n} + a^2) = 0$$
$$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} + (a^2-n^2)y^{(n)} = 0$$ **Proved** ■

g $n=3$ in (A) $y^{(3+2)}(0) = -3(3+1)y^{(3)}(0)$ $\because y^{(3)}(0)=-2$

$= -3(4)(-2)$

$= (-1)(-1)\,4.3.2$

$y^{(5)}(0) = (-1)^2\,4!$ $\boxed{y^{(2(2)+1)}(0)}$ (8)

putting $n=5$ in (A) $y^{(5+2)}(0) = -5(5+1)y^{(5)}(0)$

$= -5(6)\,(-1)^2\,4!$

$= (-1)(-1)^2\,6.5.4!$

$y^{(7)}(0) = (-1)^3\,6!$ $\because \boxed{y^{(2(3)+1)}(0)}$

⋮

Generalizing;

$$y^{(2n+1)}(0) = (-1)^2\,(2n)!$$

8. if $y = \sin(a\sin^{-1}x)$, prove that
$(1-x^2)y^{(n+2)} = (2n+1)\,xy^{(n+1)} + (n^2-a^2)y^n$

Sol:

$y = \sin(a\sin^{-1}x)$

$y' = \cos(a\sin^{-1}x)\frac{d}{dx}(a\sin^{-1}x)$

$y' = \cos(a\sin^{-1}x).\,a.\frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\,y' = a\cos(a\sin^{-1}x)$

Squaring both sides;

$(1-x^2)(y')^2 = a^2\cos^2(a\sin^{-1}x)$

$(1-x^2)(y')^2 = a^2[1-\sin^2(a\sin^{-1}x)]$

$(1-x^2)(y')^2 = a^2[1-y^2]$

Differentiating again.

$(1-x^2)\frac{d}{dx}(y')^2 + (y')^2\frac{d}{dx}(1-x^2) = a^2\frac{d}{dy}(1-y^2)$

$(1-x^2)\,2y'y'' + (y')^2(-2x) = a^2(0-2yy')$

$2y'[(1-x^2)y'' - xy'] = -2a^2yy'$

$(1-x^2)y'' - xy' = -a^2y$

$\Rightarrow (1-x^2)y'' - xy' + a^2y = 0$

Differentiating 'n' times by Leibniz's Theorem;

$(1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \frac{n(n-1)}{2!}(-2)y^{(n)} - [xy^{(n+1)} + n(1)y^{(n)}] + a^2y^{(n)} = 0$

$(1-x^2)y^{(n+2)} - 2nx\,y^{n+1} - n(n-1)y^{(n)} - xy^{(n+1)} - ny^{(n)} + a^2y^{(n)} = 0$

$(1-x^2)y^{(n+2)} - (2n+1)xy^{n+1} + y^n(-n^2+n-n+a^2) = 0$

$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} + (a^2-n^2)y^{(n)} = 0$ **Proved** ■

if $y = e^{m\sin^{-1}x}$, show that

$(1-x^2)y^{(n+2)} - (2n+1)xy^{(n+1)} - (n^2+m^2)y^{(n)} = 0$

Find the value of y^n at $x=0$.

Sol:

$$y = e^{m\sin^{-1}x} \qquad y = e^{m\sin^{-1}(0)} = e^0 = 1 \quad \boxed{y=1}$$

$$\frac{dy}{dx} = y' = e^{m\sin^{-1}x}\frac{d}{dx}(m\sin^{-1}x)$$

$$y' = e^{m\sin^{-1}x}\cdot\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y' = m\,e^{m\sin^{-1}x} = my \qquad \sqrt{1-0}\,y'(0) = m(1) \Rightarrow \boxed{y'(0)=m}$$

$$\sqrt{1-x^2}\,y' = my$$

$$(1-x^2)(y')^2 = m^2y^2 \qquad \text{Squaring both sides}$$

$$(1-x^2)2y'y'' + (y')^2(-2x) = m^2\cdot 2yy' \qquad \text{(diff. w.r.t 'x')}$$

$$\cancel{2y'}((1-x^2)y'' + (-xy')) = \cancel{2y'}\cdot m^2y$$

$$(1-x^2)y'' - xy' = m^2y$$

$$(1-x^2)y'' - xy' - m^2y = 0 \qquad (1-0)y'' - 0(m) - m^2(1) = 0 \quad \boxed{y''=m^2}$$

Differentiating 'n' times by Leibniz' Theorem.

$$(1-x^2)y^{(n+2)} + n(-2x)y^{(n+1)} + \frac{n(n-1)}{\cancel{2!}}(-\cancel{2})y^{(n)} - \left[xy^{(n+1)} + n(1)y^{(n)}\right] - m^2y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - 2nxy^{(n+1)} - n(n-1)y^{(n)} - xy^{(n+1)} - ny^{(n)} - m^2y^{(n)} = 0$$

$$(1-x^2)y^{(n+2)} - xy^{(n+1)}[2n+1] - y^{(n)}(n^2 - \cancel{n} + \cancel{n} + m^2) = 0$$

$$(1-x^2)y^{(n+2)} - xy^{(n+1)}(2n+1) - y^{(n)}(n^2+m^2) = 0$$

putting $x = 0$

$$(1-0)y^{(n+2)} - 0 - y^{(n)}(n^2+m^2) = 0$$

$$y^{(n+2)}(0) = y^{(n)}\cdot(n^2+m^2) \longrightarrow (A)$$

for even values of n,

put $n=2$ in (A) $\quad y^{(2+2)}(0) = y^2(0)\,.\,(2^2+m^2) \quad y^{2(1)}$

$(y^{2(2)}) \leftarrow y^{(4)}(0) = m^2(m^2+2^2) \qquad \because y^{(2)}(0) = m^2$

put $n=4$ in (A) $\quad y^{(4+2)}(0) = y^{(4)}(0)\,(4^2+m^2)$

$y^{2(3)} \leftarrow y^{(6)}(0) = m^2(m^2+2^2)(m^2+4^2)$

put $n=6$ in (A) $\quad y^{(6+2)}(0) = y^{(6)}(0)\,.\,(6^2+m^2)$

$y^{2(4)} \leftarrow y^{(8)}(0) = m^2(m^2+2^2)(m^2+4^2)(m^2+6^2)$

eralizing;

$$y^{2n}(0) = m^2(m^2+2^2)(m^2+4^2)(m^2+6^2)\ldots\ldots(m^2+(2n-2)^2)$$

for odd values of n.

putting $n=1$ in (A) $\quad y^{(3)}(0) = y^{(1)}(0)(m^2+1^2) \qquad \because y'(0)=m$

$2(1)+1 \qquad y^{(3)}(0) = m(m^2+1^2)$

putting $n=3$ in (A) $\quad y^{(3+2)}(0) = y^{(3)}(0).(m^2+3^2)$

$2(2)+1 \qquad y^{(5)}(0) = m(m^2+1^2)(m^2+3^2)$

putting $n=5$ in (A) $\quad y^{(5+2)}(0) = y^{(5)}(0)(m^2+5^2)$

$2(3)+1 \qquad y^{(7)}(0) = m(m^2+1^2)(m^2+3^2)(m^2+5^2)$

generalized form

$$y^{(2n+1)}(0) = m(m^2+1^2)(m^2+3^2)(m^2+3^2)\ldots\ldots(m^2+(2n-1)^2)$$

10. Find $y^{(n)}(0)$ if

(i) $\quad y = \ln(x+\sqrt{1+x^2})$

$$y' = \frac{1}{x+\sqrt{1+x^2}}\frac{d}{dx}\left(x+\sqrt{1+x^2}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}}(\cancel{2}x)\right) = \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{\cancel{x+\sqrt{1+x^2}}}\left(\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}\right)$$

$$y' = \frac{1}{\sqrt{1+x^2}} \qquad\qquad y'(0) = \frac{1}{\sqrt{1+0}} = 1$$

$\sqrt{1+x^2}\,y' = 1 \qquad$ Squaring both sides

$(1+x^2)(y')^2 = 1. \qquad$ differentiating w.r.t 'x'

$(1+x^2)2y'y'' + (2x)(y')^2 = 0$

$2y'\left[(1+x^2)y'' + xy'\right] = 0$

$(1+x^2)y'' + xy' = 0 \qquad (1+0)y''(0) + 2(0)(1) = 0$

differentiating 'n' times by Leibniz theorem; $\boxed{y''(0)=0}$

$$(1+x^2)y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{\cancel{2!}}(\cancel{2})y^{(n)} + xy^{(n+1)} + n(1)y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2-\cancel{n}+\cancel{n}) = 0$$

$$(1+x^2)y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2) = 0$$

putting '$x=0$' $\quad (1+0)y^{(n+2)}(0) + 0 + y^{n}(0)\,n^2 = 0$

$$y^{(n+2)}(0) = -y^{(n)}(0)n^2 \longrightarrow (A)$$ (11)

for even values of 'n'

put $n=2$ in (A); $y^{(2+2)}(0) = -y^{(2)}(0)\cdot(2)^2$

$y^{(4)}(0) = -(0)(2) = 0$ $\quad \because y''(0)=0$

put $n=4$ in (A) $\quad y^{(4+2)}(0) = -y^{(4)}(0).\ (4)^2$

$y^{(6)}(0) = -(0)(4^2) = 0$

generalizing;

$$y^{(2n)}(0) = 0$$

for odd values of 'n' $\quad 2(1)+1 \quad (-1)^1(1^2) \quad (-1)^n$

put $n=1$ in (A); $\quad y^{(1+2)}(0) = -y^{(1)}(0)(1)^2$

$y^{(3)}(0) = (-1)(1^2)$

put $n=3$ in (A); $\quad y^{(3+2)}(0) = -y^{(3)}(0)\ (3^2)$

$= (-1)(-1)(1^2)(3^2)$

$2(2)+1 \quad y^{(5)}(0) = (-1)^2(1^2)(3^2)$

put $n=5$ in (A); $\quad y^{(5+2)}(0) = -y^{(5)}(0)\ (5^2)$

$= (-1)(-1)^2(1^2)(3^2)(5^2)$

$2(3)+1 \quad y^{(7)}(0) = (-1)^3(1^2)(3^2)(5^2)$

generalizing;

$n+1 \quad y^{(2n+1)}(0) = (-1)^n(1^2)(3^2)(5^2)(7^2)\ldots\ldots[(2n-1)^2]$

(ii) $$y = (x+\sqrt{1+x^2})^m$$

$y(0) = (0+\sqrt{1+0})^m = (1)^m$
$\Rightarrow y(0)=1$

$$y' = m(x+\sqrt{1+x^2})^{m-1}\frac{d}{dx}(x+\sqrt{1+x^2})$$

$$= m(x+\sqrt{1+x^2})^{m-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{1+x^2})^{m-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$= \frac{m(x+\sqrt{1+x^2})^{m-1+1}}{\sqrt{1+x^2}}$$

$$y' = \frac{m(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}} = \frac{my}{\sqrt{1+x^2}}$$

$\sqrt{1+x^2}\,y' = my \qquad \sqrt{1+0}\,y'(0) = m(1)$

$\boxed{y'(0)=m}$

Squaring both sides.

$$(1+x^2)y'^2 = m^2y^2$$

differentiating w.r.t 'x'.

$$(1+x^2).\,2y'y'' + y'^2(2x) = m^2.\,2yy'$$

$$2y'[(1+x^2)y'' + xy'] = 2y'.\,m^2y$$

$$(1+x^2)y'' + xy' = m^2y \longrightarrow (1+0)y'' + 0 - m^2(1) = 0 \quad \boxed{y'' = m^2}$$ (12)

$$(1+x^2)y'' + xy' - m^2y = 0$$

differentiating 'n' times by Leibniz' theorem.

$$(1+x^2)y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{\cancel{2!}}(\cancel{2})y^{(n)} + xy^{(n+1)} + n(1)y^{(n)} - m^2y^{(n)} = 0$$

$$(1+x^2)y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2 - \cancel{n} + \cancel{n} - m^2) = 0$$

$$(1+x^2)y^{(n+2)} + xy^{(n+1)}(2n+1) + y^{(n)}(n^2 - m^2) = 0$$

putting $x = 0$

$$(1+0)y^{(n+2)}(0) + 0 + y^{n}(0).(n^2 - m^2) = 0$$

$$y^{(n+2)}(0) = -y^{n}(0)(n^2 - m^2)$$

$$y^{(n+2)}(0) = y^{(n)}(0)(m^2 - n^2) \longrightarrow (A)$$

for even values of n.

putting $n=2$ in (A) $\quad y^{(2+2)}(0) = y^{(2)}(0)(m^2 - 2^2)$

$y^{(4)}(0) = m^2(m^2 - 2^2) \quad \because y''(0) = m^2$

putting $n=4$ in (A) $\quad y^{(4+2)}(0) = y^{(4)}(0).(m^2 - 4^2)$

$y^{(6)}(0) = m^2(m^2 - 2^2)(m^2 - 4^2)$

putting $n=6$ in (A) $\quad y^{(6+2)}(0) = y^{(4)}(0).(m^2 - 6^2)$

$y^{(8)}(0) = m^2(m^2 - 2^2)(m^2 - 4^2)(m^2 - 6^2)$

$2n$

generallizing,

$$y^{(2n)}(0) = m^2(m - 2^2)(m^2 - 4^2)(m^2 - (2n-2)^2)$$

for odd values of n.

putting $n=1$ in (A) $\quad y^{(1+2)}(0) = y^{(1)}(0)(m^2 - 1^2) \quad \because y'(0) = m$

$y^{(3)}(0) = m(m^2 - 1^2)$

putting $n=3$ in (A) $\quad y^{(3+2)}(0) = y^{(3)}(0).(m^2 - 3^2)$

$y^{(5)}(0) = m(m^2 - 1^2)(m^2 - 3^2)$

putting $n=5$ in (A) $\quad y^{(5+2)}(0) = y^{(5)}(0).(m^2 - 5^2)$

$y^{(7)}(0) = m(m^2 - 1^2)(m^2 - 3^2)(m^2 - 5^2)$

generalizing;

$$y^{(2n+1)}(0) = m(m^2 - 1^2)(m^2 - 3^2) \ldots\ldots (m^2 - (2n-1)^2)$$

11. if $y = \overline{a\cos(\ln x) + b\sin(\ln x)}$, prove that

$$x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$$

$$y' = a(-\sin(\ln x)).\frac{1}{x} + b\cos(\ln x).\frac{1}{x}$$

$$y' = \frac{1}{x}\left(-a\sin(\ln x) + b\cos(\ln x)\right)$$

$$xy' = -a\sin(\ln x) + b\cos(\ln x)$$

$$xy'' + y'(1) = -a\cos(\ln x).\frac{1}{x} + b(-\sin(\ln x))\frac{1}{x}$$

$$xy'' + y' = \frac{-1}{x}\left(\underbrace{a\cos(\ln x) + b\sin(\ln x)}_{y}\right)$$

$$x^2y'' + xy' = -y$$

$$x^2y'' + xy' + y = 0$$

Differentiating n times by Leibniz theorem,

$$x^2y^{(n+2)} + n(2x)y^{(n+1)} + \frac{n(n-1)}{\cancel{2!}}(\cancel{2})y^{(n)} + xy^{(n+1)} + n(1)y^{(n)} + y^{(n)} = 0$$

$$x^2y^{(n+2)} + xy^{(n+1)}[2n+1] + y^{(n)}[n^2 - \cancel{n} + \cancel{n} + 1] = 0$$

$$\underline{x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0.}$$

12. **Show that**

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} \cdots\cdots - \frac{1}{n}\right]$$

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) \qquad \text{here} \qquad u = \frac{1}{x}, \; v = \ln x$$

$$u' = \frac{-1}{x^2}, \; u'' = \frac{2}{x^3}, \; u^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}, \; u^{(n)} = \frac{(-1)^n n!}{x^{n+1}}$$

$$v^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

By Leibniz Theorem;

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \binom{n}{0}u^{(n)}v + \binom{n}{1}u^{(n-1)}v' + \binom{n}{2}u^{(n-2)}v'' + \cdots$$
$$+ \cdots\cdots\cdots + \binom{n}{n-1}u'v^{(n-1)} + \binom{n}{n}uv^{(n)}.$$

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = (1)\left[\frac{(-1)^n n!}{x^{n+1}}\right]\ln x + (n)\left[\frac{(-1)^{n-1}(n-1)!}{x^n}\right]\left(\frac{+1}{x}\right) + \frac{n(n-1)}{2!}\left[\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\right]\left(\frac{-1}{x^2}\right)$$
$$+ \cdots\cdots\cdots + (1)\left(\frac{1}{x}\right)\left(\frac{(-1)^{n-1}(n-1)!}{x^n}\right)$$

$$= \frac{(-1)^n n!}{x^{n+1}} \ln x + \frac{(-1)^{n-1}(n-1)!(n)}{x^n \cdot x} + \frac{[-n(n-1)(-1)^{n-2}(n-2)!]}{x^{n-1} \cdot x^2 \cdot 2!} \quad (14)$$

$$+ \cdots\cdots + \frac{(-1)^{n-1}(n-1)!}{x^n \cdot x}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \ln x + \frac{(-1)^{n-1}(-1)(-1)\, n(n-1)!}{x^{n+1}} + \left[\frac{-n(n-1)(n-2)!\,(-1)^{n-2}(-1)}{x^{n-1+2} \cdot 2!}\right.$$

$$+ \cdots\cdots + \frac{(-1)^{n-1}(-1)(-1)\, n(n-1)!}{n \cdot x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \ln x + \frac{(-1)(-1)^{n-1+1}\, n!}{x^{n+1}} + \frac{(-1)\, n!\, (-1)^{2+n-2}}{x^{n+1}(2!)} + \cdots$$

$$+ \cdots\cdots + \frac{(-1)(-1)^{n-1+1}}{n \cdot x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \ln x - \frac{(-1)^n n!}{x^{n+1}} - \frac{n!(-1)^n}{x^{n+1}(2)} - \cdots$$

$$- \cdots\cdots - \frac{(-1)^n}{n\, x^{n+1}}$$

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \cdots\cdots - \frac{1}{n}\right]$$

Proved.

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BSc Part I