Exercise 2.3

Find \( \Delta y, dy, \Delta y - dy \).

Method:

\[
y = f(x) \\
y + \Delta y = f(x + \Delta x) \\
y + \Delta y - y = f(x + \Delta x) - f(x) \\
\Delta y = f(x + \Delta x) - f(x) \\
y = f(x) \\
dy = f'(x) dx \\
\frac{dy}{dx} = f'(x) \\
\boxed{dy = f'(x) dx}
\]

(i) \( y = x^3 - 1 \) \( x = 1 \) \( \Delta x = -0.5 \)

\[
\Delta y = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x)^3 - 1 - x^3 + 1} \\
\Delta y = (x + \Delta x)^3 - x^3 \\
\Delta y = (1 - 0.5)^3 - (1)^3 \\
\Delta y = 0.125 - 1 \\
\Delta y = -0.875 \\
y = x^3 - 1 \\
dy = 3x^2 \\
dy = 3x^2 dx \\
dy = 3(1)(0.5) \\
dy = -1.5 \\
\Delta y - dy = -0.875 - (-1.5) \\
\Delta y - dy = -0.875 + 1.5 \\
\Delta y - dy = 0.625
\]

(ii) \( y = \sqrt{3x - 2} \) \( x = 2 \) \( \Delta x = 0.3 \)

\[
\Delta y = \frac{f(x + \Delta x) - f(x)}{(x + \Delta x)^2 - 2 - 3x - 2} \\
\Delta y = \sqrt{3(2 + 0.3)^2 - 2} - \sqrt{3(2)^2 - 2} \\
\Delta y = 2.2136 - 2 \\
\Delta y = 0.2136 \\
y = \sqrt{3x - 2} \\
dy = \frac{3}{2\sqrt{3x - 2}} dx \\
\frac{dy}{dx} = \frac{3}{2\sqrt{3(2) - 2}} \\
\frac{dy}{dx} = \frac{3}{2\sqrt{6 - 2}} \\
\frac{dy}{dx} = \frac{3}{2(2)} \\
dy = 0.225 \\
\Delta y - dy = 0.2136 - 0.225 \\
\Delta y - dy = -0.0114
\]

2. Using differentials to approximate.

(iii) \( \sqrt{26.2} \) \( \Delta x = 1.2 \)

\[
y = f(x) \text{ with } x=25, \quad \Delta x = 1.2 \\
y = \frac{1}{25} \\
dy = \frac{1}{2\sqrt{25}} dx \\
dy = \frac{1}{2\sqrt{25}} (1.2) = \frac{1}{2}(1.2) \\
dy = 0.12 \]
(3.02) 4 = (3+0.02) 4

\[ f(x+\Delta x) \approx y + dy \]

\[ dy = f'(x) \Delta x \]

\[ f(x) = x^4 \]

\[ dy = 4x^3 \Delta x \]

\[ y = x^4 \]

\[ \Delta x = 0.02 \]

\[ dy = 4(0.02)^3 \]

\[ dy = 0.01504 \]

\[ y = 0.5 \]

\[ \Delta y = 0.01504 \]
Let \( x \) be the side of a cube.

The side of a cube is measured \( \pm \frac{2}{100} \) of its length.

\[
\frac{dx}{x} = \pm \frac{2}{100} = \pm 0.02
\]

The surface area of one face of the cube is \( A = x^2 \),

\[
dA = 2x \, dx
\]

Percentage error in surface area is \( \frac{\%}{} \text{age error in area} = \frac{dA}{A} \times 100 \)

\[
= \frac{2x \, dx}{x^2} \times 100
\]

\[
= \frac{2 \, dx}{x} \times 100
\]

\[
= 2 (\pm 0.02) \times 100
\]

\[
= \pm 4\%
\]

4. A box with square base .......... Volume of Box.

Sol. Let \( x \) be width of box.

Then height of box = \( 2(\text{width}) = 2x \)

\[ x = 8.5 \text{ inches} \]

Possible error in width = \( dx = \pm 0.3 \text{ inches} \)

Volume of box = \( \text{Length} \times \text{Width} \times \text{Height} \)

\[ V = x \times (x) \times (2x) \]

\[ V = 2x^3 \]

We find error in volume i.e. \( dV \).

\[
dV = 2x^2 \, dx = 6x^2 \, dx
\]

\[
dV = 6(8.5)^2 (\pm 0.3)
\]

\[
= \pm 130.05 \text{ (inches)}^3
\]

So error in volume of box is \( \pm 130.05 \text{ (inches)}^3 \).

5. Radius \( x \) of the circle increases .......... \( \frac{\%}{} \text{age change in area} \).

Sol. Let \( x \) be the radius of circle.

\[ x = 10 \]

\[ x + \Delta x = 10.1 \]

\[ \Delta x = 10.1 - x \]

\[ \Delta x = 10.1 - 10 \]

\[ dx = \Delta x = 0.1 \]
Area of circle = \( A = \pi x^2 \)

we find percentage change in area

\[
\frac{\% \text{ age change}}{A} = \frac{\frac{dA}{A} \times 100}{\frac{2\pi x dx}{\pi x^2} \times 100}
= 2 \frac{d}{dx} \times 100
= 2 \frac{0.1}{10} \times 100 = 2\%
\]

\% age change in area = 2%

6. The diameter of plant ..................... plant change.

Sol.

Let \( r \) be radius of plant

(i) Let \( C \) be circumference of plant

then \( C = 2\pi r \)

Increase in Circumference \( = dC \)

\[
C = 2\pi r
\]

\[
dC = 2\pi dr
\]

\[2 = 2\pi dr \Rightarrow dr = \frac{1}{\pi}
\]

⇒ increase in radius \( = \frac{1}{\pi} \)

⇒ increase in diameter \( = 2dr = \frac{2}{\pi} \)

(ii) Area of cross-section of plant \( = A = \pi r^2 \)

\[
A = \pi r^2
\]

d\( A = 2\pi r dr \)

\[
dA = 2\pi (4)(\frac{1}{\pi})
\]

\[dA = 8\]

⇒ change in area \( = 8 \) inches.

7. Sand pouring from a chute ................. increase by \( 2 \text{ cm}^3 \).

Sol.

Let \( r \) be radius ; \( h \) be

heighth of conical pile.

\( r = 10 \text{ cm} \)

given condition

altitude = radius

\( h = r \)

⇒ \( V \) be volume then \( dV = 2\text{ cm}^3 \)

Volume of cone \( = \frac{1}{3} \pi r^2 h \)

Volume of conical pile \( = V = \frac{1}{3} \pi r^2 (r) \)

\[V = \frac{1}{3} \pi r^3 \]
\[ dV = \frac{1}{3} \pi (3r^2 \, dr) \]
\[ dV = \pi r^2 \, dr \]

Put values:
\[ 2 = \pi (10)^2 \, dr = 100 \pi \, dr \]
\[ dr = \frac{2}{100 \pi} \]
\[ dr = \frac{1}{50 \pi} \]

Change in radius = \( \frac{1}{50 \pi} \) cm

8. A dome is in the shape .................. paint required.

Sol. Let radius of dome = \( r = 60 \) feet

Let \( V \) be volume of dome.

Volume of hemisphere = \( \frac{2}{3} \pi r^3 \)

So Volume of dome = \( \frac{2}{3} \pi r^3 \)

we find \( dV \), where \( dr = 0.01 \) inch
\[ dr = 0.01 \text{ feet} \]
\[ \Rightarrow dr = \frac{12}{1200} \text{ feet} \]

\[ dV = \frac{2}{3} \pi (60)^2 \times \frac{1}{1200} \times \frac{1200}{60^2} \]
\[ dV = 6 \pi \text{ ft}^3 \]


Sol:

Let \( x \) be the length of side of base.

Area of \( OABCD = \text{Area of } \Delta OAD + \text{Area of Square ABCD} \)

Area of Square \( ABCD = \text{length} \times \text{width} \)

Area of \( \Delta OAD = \frac{1}{2}(\text{length} \times \text{width}) \)
\[ = \frac{1}{2}(xW) \]
\[ = \frac{1}{2} \frac{\sqrt{3}}{2} x \times x \]

Area of \( \Delta OAD = \frac{\sqrt{3}}{4} x^2 \)

\[ \sin 60^\circ = \frac{l}{x} \]
\[ l = x \sin 60^\circ \]
\[ l = \frac{\sqrt{3}}{2} x \]
a of \( OABCD = x^2 + \frac{\sqrt{3}}{4} x^2 \)

\[
A = x^2 \left[ 1 + \frac{\sqrt{3}}{4} \right]
\]

\[
dA = 2x \cdot dx \left[ 1 + \frac{\sqrt{3}}{4} \right]
\]

\[
= 2 \cdot \frac{1}{100} x \left[ 1 + \frac{\sqrt{3}}{4} \right]
\]

\[
= \frac{1}{50} x^2 \left[ 1 + \frac{\sqrt{3}}{4} \right]
\]

Percentage error in area = \( \frac{dA}{A} \times 100 \)

\[
= \frac{\frac{1}{50} x^2 \left[ 1 + \frac{\sqrt{3}}{4} \right]}{x^2 \left[ 1 + \frac{\sqrt{3}}{4} \right]} \times 100
\]

\[
= \frac{100}{50}
\]

Percentage error in area = 2%

10. A boy makes paper .................. capacity of cup.

Sol.

Let \( r \) be radius of base

" \( h \) be height of cup.

\( h = 4r \)

\( r = 2 \text{ cm} \)

\( r + \Delta r = 1.5 \)

\( \Delta r = 1.5 - r \)

\( = 1.5 - 2 \)

\( dr = \Delta r = 0.5 \text{ cm} \)

Volume of cone = \( \frac{1}{3} \pi r^2 h \)

\[
V = \frac{1}{3} \pi r^2 (4r) = \frac{4}{3} \pi r^3
\]

\[
dV = \frac{4}{3} \pi \cdot 3r^2 \, dr
\]

\[
dV = 4\pi r^2 \, dr
\]

\[
dV = 4\pi (2)^2 (0.5)
\]

\[
dV = -8\pi
\]

Decrease in capacity of cup = \( dV = -8\pi \text{ cm}^3 \).

11. To estimate the height ................... height so found.

Sol.

Let \( x \) be the height of Minar-e-Pakistan.
then from figure,
\[ \frac{OM}{OE} = \frac{AC}{AB} \]
\[ \Rightarrow \frac{x}{25} = \frac{3}{4} \]
\[ \Rightarrow x = 75 \]
height of minar-e-Pakistan = 75 m.

(ii) if 'y' is actual length of shadow.

from figure,
\[ \frac{OM}{OB} = \frac{AC}{AB} \]
\[ \frac{x}{y+24} = \frac{3}{y} \]
\[ xy = 3(y+24) \]
\[ xy = 3y + 72 \]

differentiate,
\[ xdy + ydx = 3dy + 0 \]
\[ ydx = 3dy - 3dy = (3-x)dy \]
\[ \frac{dy}{y} = \frac{dx}{3-x} \rightarrow (1) \]

percentage error in length of shadow = 1% 
\[ \frac{dy}{y} \times 100 = \pm 1 \% \]
\[ \frac{dy}{y} = \frac{1}{100} = \pm 0.01 \]

\( (1) \Rightarrow 0.01 = \frac{dx}{3-x} \Rightarrow dx = \pm 0.01 (3-y) \)

we find percentage error in height of minar.
\[ \% \text{age error} = \frac{dx}{y} \times 100 \]
\[ = \pm 0.01 (\frac{3-y}{y}) \times 100 \]
\[ = \pm 0.01 (\frac{3-75}{75}) \times 100 = \pm \frac{1 \times 72}{75} = \pm 0.96 \% \]

\[ \% \text{age error in height of minar-e-Pakistan} = \pm 0.96 \% \]
Oil Spilled from tanker \[ \text{10 ft} \]

1. Let ‘\(r\)’ be radius of circle. \(r = 40 \text{ ft}\)
Then rate of change of radius \(\frac{dr}{dt} = 2 \text{ ft/sec}\)
we find rate of change of area, \(\frac{dA}{dt}\)
\[
A = \pi r^2
\]
\[
\frac{dA}{dt} = \pi (2r\frac{dr}{dt})
\]
\[
\frac{dA}{dt} = 2\pi (40)(2)
\]
\[
= 160\pi \text{ ft}^2/\text{sec}
\]
area of circle increases at rate \(160\pi \text{ ft}^2/\text{sec}\).

13. From a point ‘\(O\)’ after 5 sec.

Sol.

Let \(A\) be position of car \(1\) after ‘\(t\)’
seconds \(B\) ‘\(t\)’
\(\text{car2 after ‘\(t\)’}\)

\[
OA = x = t^2 + t
\]
\[
OB = y = t^2 + 3t
\]

By Pathagorus theorem (from fig)
\[
S^2 = x^2 + y^2
\]
\[
S^2 = (t^2 + t)^2 + (t^2 + 3t)^2
\]
at \(t = 5\) sec
\[
S^2 = (5^2 + 5)^2 + (5^2 + 3(5)) = (30)^2 + (25 + 15)^2
\]
\[
S^2 = 30^2 + 40^2 = 2500
\]
\[
S = 50
\]

\[
S^2 = x^2 + y^2 \rightarrow \text{(1)}
\]

we find rate of change of distance at 5 sec. \(\text{ie } \frac{ds}{dt} \Bigg|_{t=5}\)
diff \((1)\) w.r.t ‘\(t\’
\[
2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}
\]
\[
2(5) \frac{ds}{dt} = 2 \left[ (t^2 + t) \frac{dx}{dt} + (t^2 + 3t) \frac{dy}{dt} \right]
\]
\[
\frac{\Delta S}{\Delta t} = \left[ (t^2 + t)(2t + 1) + (t^2 + 3t)(2t + 3) \right]
\]
at \(t = 5\)
\[
(50) \frac{ds}{dt} \Bigg|_{t=5} = \left[ (5^2 + 5)(2\cdot 5 + 1) + (5^2 + 3\cdot 5)(2\cdot 5 + 3) \right]
\]
50. \( \frac{ds}{dt} \bigg|_{t=5} = \frac{1}{50} \left[ (30)(11) + (40)(13) \right] \)
\[
\frac{ds}{dt} \bigg|_{t=5} = \frac{1}{50} \left[ 330 + 520 \right] = \frac{850}{50} = 17
\]
Distance between cars changes at the rate of 17 ft/sec.

14. Sand falls from ....................... pile is 5 ft high.

Sol.
Let ‘r’ be radius of pile.
" h " height " " = h = 5 ft

\[ h = 2r \]
\[ \Rightarrow r = \frac{h}{2} \]

Volume of cone = \( \frac{1}{3} \pi r^2 h \)
Volume of pile: \( V = \frac{1}{3} \pi \left( \frac{h}{2} \right)^2 h \)

\[ V = \frac{1}{12} \pi h^3 \rightarrow (1) \]

Sand falls from container at rate = \( \frac{dV}{dt} = 10 \text{ ft}^3/\text{min} \)
We find rate of increase in height = \( \frac{dh}{dt} = ? \)

Diff. \( (1) \) w.r.t \( t \).

\[ \frac{dV}{dt} = \frac{1}{12} \pi \cdot 3h^2 \frac{dh}{dt} = \frac{1}{4} \pi h^2 \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{4 \frac{dV}{dt}}{\pi h^2} \]
\[ \frac{dh}{dt} = \frac{4}{\pi \left( \frac{5}{2} \right)^2} \]
\[ \frac{dh}{dt} = \frac{4 \cdot 2^2}{\frac{25}{4} \pi} = \frac{8}{\frac{25}{4} \pi} = 0.509 \text{ ft/min} \]

Height of pile is changing at rate of 0.51 ft/min

15. A 6 ft tall man ....................... shadow changing.

Sol.
Let distance of man from lamp post = 0M = x
Let distance of tip of shadow from 0 = z
from triangles,

\[ \frac{OP}{OA} = \frac{BM}{AM} \]

\[ \frac{16}{z} = \frac{6}{z-x} \]

\[ 16z - 16x = 6z \]

\[ 16z - 6z - 16x = 0 \Rightarrow 10z - 16x = 0 \]

\[ 2(5z - 8x) = 0 \Rightarrow 5z - 8x = 0 \]

\[ 5z = 8x \]

Differentiating w.r.t. \( t \)

\[ \frac{d}{dt} \left( 5z \right) = 8 \frac{dx}{dt} = 5 \text{ (Speed)} \]

\[ \frac{d}{dt} \frac{dz}{dt} = \frac{40}{5} \Rightarrow \frac{dz}{dt} = 8 \]

Tip of man's shadow moves at the rate of 8 ft/sec.

If \( y \) is length of shadow

\[ AB = y \]

from fig:

\[ \frac{OP}{OA} = \frac{BM}{AM} \]

\[ \frac{16}{x+y} = \frac{6}{y} \]

\[ 16y = 16x + 16y \]

\[ \Rightarrow 16y - 6x - 6y = 0 \]

\[ \Rightarrow 10y - 6x = 0 \Rightarrow 2(5y - 3x) = 0 \]

\[ \Rightarrow 5y - 3x = 0 \]

differentiating w.r.t. \( x \)

\[ 5 \frac{dy}{dx} - 3 \frac{dx}{dx} = 0 \]

\[ 5 \frac{dy}{dt} - 3 \frac{dx}{dt} = 0 \Rightarrow 5 \frac{dy}{dt} = 15 \]

\[ \frac{dy}{dt} = \frac{15}{5} \]

\[ \frac{dy}{dt} = 3 \]

Shadow is changing at the rate of 3 ft/sec.
Sol: Let $y$ be the altitude of rocket = 3000 ft
Let Distance b/w man and rocket = $x$

By Pythagoras theorem,
\[ x^2 = y^2 + 4000^2 \rightarrow (1) \]
\[ x^2 = 3000^2 + 4000^2 \]
\[ x^2 = 9,000,000 + 16,000,000 \]
\[ x^2 = 25,000,000 \]
\[ x = \sqrt{25,000,000} \]
\[ x = 5000 \]

**Diff. (1) w.r.t ‘t’**
\[ 2x \frac{dx}{dt} = 2y \frac{dy}{dt} + 100 \]
\[ \frac{x}{dt} = \frac{y}{dt} \]
\[ 5000 \frac{dx}{dt} = (600)(3000) \]
\[ \frac{dx}{dt} = \frac{(600)(3000)}{5000} \]
\[ \frac{dx}{dt} = 360 \]

Distance b/w rocket and man is changing at rate of 360 ft/sec.

---

17. A aeroplane flying horizontally at an altitude $y$ is increasing after 30 seconds.

**Sol:**
Let $O$ be the observer on ground.
Let $P$ be the plane.
Let $OP = x$
$AP = y$

Altitude = $OA = 3$ miles.

By Pythagoras theorem
\[ x^2 = 3^2 + y^2 \]
\[ x^2 = 9 + 4^2 = 9 + 16 = 25 \]
\[ x = 5 \]

**Diff. w.r.t ‘t’**
\[ 2x \frac{dx}{dt} = 0 + 2y \frac{dy}{dt} \]
\[ x \frac{dx}{dt} = y \frac{dy}{dt} \]

\[ y = \frac{y}{t} \]
\[ \frac{y}{t} = \frac{400 \times 30}{3600} \]
\[ \frac{y}{t} = 4 \]

\[ y = 4 \]
rate of change of distance from observer to plane $\frac{dx}{dt} = ?$

\[
\frac{dy}{dt} = 480
\]

\[x \frac{dx}{dt} = y \frac{dy}{dt}\]

(5) \[
\frac{dx}{dt} = (4)(480) = 1920 \delta \text{ ft/s}
\]

\[
\frac{dx}{dt} = \frac{384}{5} \text{ miles/hr}
\]

18. A boy flies kites released in 70m.

Let 'x' be the length of string.

Altitude $OB = 30m$

\[
OA = y
\]

\[
\frac{dy}{dt} = 2 \text{ m/sec}
\]

\[
\frac{dx}{dt} = ?
\]

From figure

\[
x^2 = y^2 + 30^2
\]

\[
2x \frac{dx}{dt} = 2y \frac{dy}{dt} = 0
\]

\[
x \frac{dx}{dt} = y \frac{dy}{dt}
\]

\[
\frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt}
\]

\[
= \frac{2 \sqrt{110}}{70} \times (2)
\]

\[
\frac{dx}{dt} = \frac{4 \sqrt{110}}{7}
\]

Thus string is being let out at the rate of $\frac{4 \sqrt{110}}{7} \text{ m/sec}$.

19. A water tank is halfway up?

Sol.

Let $BO = x$
height of frustum of cone = 6

So height of cone = AO = AB + OB = x + 6

Let CP be the water level

and CP = r

Let BC = y

from \( \triangle AOB \) and \( \triangle BOR \)

\[
\frac{AO}{BO} = \frac{BR}{OB}
\]

\[
\frac{4}{x+6} = \frac{2}{y+6}
\]

\[
4x = 2y + 12
\]

\[
\Rightarrow 4x - 2y = 12 \Rightarrow 2x = 6y 
\]

\[
\Rightarrow x = 3
\]

from \( \triangle BOR \) and \( \triangle COP \)

\[
\frac{BR}{OB} = \frac{CP}{CO} \]

\[
\frac{2}{x} = \frac{r}{y+6}
\]

\[
\frac{2}{3} = \frac{r}{y+6}
\]

\[
r = \frac{y + 6}{3}
\]

Volume of frustum:

\[
V = \frac{1}{3} \pi r^2 (y+6) - \frac{1}{3} \pi (2)^2 (6)
\]

\[
= \frac{1}{3} \pi \left( \frac{y+6}{3} \right)^2 (y+6) - \frac{1}{3} \pi 4 \times 6^2
\]

\[
= \frac{1}{3 \times 9} \pi (y+6)^3 - 8 \pi
\]

\[
V = \frac{4}{27} \pi (y+6)^3 - 8 \pi
\]

\[
\frac{dV}{dt} = 20 \text{ m}^3/\text{min}
\]

\[
\frac{dV}{dt} = \frac{1}{27} \pi 3(y+6)^2 \frac{dy}{dt} = 0
\]

\[
20 = \frac{1}{9} \pi (y+6)^2 \frac{dy}{dt} \quad \rightarrow \text{(1)}
\]

\[
\text{water is half way up} \Rightarrow y = \frac{1}{2} (\text{height of frustum})
\]

\[
y = \frac{1}{2} (6) = 3
\]

\[
y = 3 \text{ m}
\]

\[
\text{(1)} \Rightarrow 20 = \frac{1}{9} \pi (3+6)^2 \frac{dy}{dt}
\]

\[
20 = \frac{1}{9} \pi 9^2 \frac{dy}{dt}
\]

\[
\Rightarrow \frac{dy}{dt} = \frac{20}{9 \pi} \text{ m/min}
\]
A 20 m long water trough contains water. How much water is there in the trough?

**Solution:**

**Volume of water:** Length × Area of cross-sectional area

**Cross-sectional area:**

Area of triangle = \( \frac{1}{2} \) width × height

\[
\text{Area} = \frac{1}{2} \times \frac{2x}{\sqrt{3}} \times x
\]

\[
= \frac{x^2}{\sqrt{3}}
\]

**Volume:**

\[
V = 12 \times \frac{x^2}{\sqrt{3}}
\]

\[
\frac{dV}{dt} = \frac{12x}{\sqrt{3}} \cdot \frac{dx}{dt}
\]

\[
\frac{dV}{dt} = \frac{24}{\sqrt{3}} \cdot \frac{dx}{dt}
\]

**Water level:**

\[
= x = \frac{3}{2}
\]

\[
\frac{dx}{dt} = 4
\]

\[
4 = \frac{24}{\sqrt{3}} \left( \frac{3}{2} \right) \frac{dx}{dt}
\]

\[
\frac{\sqrt{3}}{3} = \frac{dx}{dt}
\]

\[
\frac{\sqrt{3}}{3 \times 3} = \frac{dx}{dt}
\]

\[
\Rightarrow \frac{dx}{dt} = \frac{1}{3\sqrt{3}}
\]

**Water level is rising at the rate of** \( \frac{1}{3\sqrt{3}} \) m/min.