Written by: Prof. Muhammad Farooq (Punjab College, Sargodha)

Calculus (B.Sc)

Chapter No.1

Real Number, Limit And Continuity

Integers: The numbers $0, \pm 1, \pm 2, \pm 3$,-----are called integers and the set $\{---3,-2,-1,0,1,2,3---\}$ is called the set of integers and it is denoted by Z (Z is for zahlen the German word for "number")or I.

The number 1,2,3,------ are called +ve integers IS denoted by Z^+ or I+ or natural numbers , whereas the -1,-2,-3,----are called negative integers IS denoted by Z- OR I- Note: 0 is neither positive nor negative, 0 is called non- negative integer.

Rational Numbers: The number of the form P/q where $q \neq 0$ and both p and q are integers called rational numbers . Rational numbers is denoted by Q e.g. 1/3,5/7,9/3,7/1,-3/7 etc

OR

The numbers whose decimal representations are terminating (اختتام پذیری) or recurring (occur again and again) بار بار آنا

NOTE. Every integer n is also a rational since n= n/1 i.e. we can write it in p/q form, But the converse is not true.

Irrational Numbers: The numbers whose decimal representations are non- recurring are called Irrational Numbers. or the number not expressible in the form p/q ,where $p,q \in z$ e.g. $\sqrt{2}, \sqrt{3}, \sqrt{5}$ ----- Irrational Numbers is denoted by $\overline{Q}$ NOTE.1 (1) If an integer n is not perfect square ,then $\sqrt{n}$ is an example of an Irrational Numbers .

(2) It is necessary to represent Irrational Number by approximation. Using the symbol $\approx$ for example $\sqrt{2} \approx 1.4142$, and $\pi \approx 3.1416$

Real Numbers: It is the set of rational and irrational Numbers and is denoted by R ($R = Q \cup \overline{Q}$)

Or

The union of rational and irrational number is called the set of real number

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Complex Number

The number of the form a+ib where $a,b \in R$ $i=\sqrt{-1}$ are called complex numbers ,for example 3+7i ,2-6i,-3+5i ,6i etc The set of such numbers is called the set of complex numbers and is denoted by C

Properties of Real numbers

1. If $a,b \in R$, then $a+b \in R$ (Closure Law of addition)
2. If $a,b \in R$, then $a+b=b+a$(Commutative law of add)
3. If $a,b,c \in R$, then $a+(b+c)=(a+b)+c$
   (Associative law of add)
4. If $a,b,c \in R$, then $a(b+c)= ab+ac$&$(a+b)c=ac+bc$
   (left and right distributive law' $\times$, over' +,)
5. There exist $0 \in R$ such that $0+a=a+0 =a$ $\forall a \in R$
   ('0, IS Called additive identity)
6. For each $a \in R$, there is an element $-a \in R$
   S.t $a+ (-a) =0$ (Existence of additive inverse)
7. If $a \in R$ then $1/a \in R$ s.t $a.1/a=1/a.a \forall a \in R$
   (Existence of multiplicative inverse)
8. If there is an element $1 \in R$ s.t $a \in R$
   $1.a=a.1$, $a \in R$
   (Existence of multiplicative identity)
9. If $\forall a, b \in R$, $ab=ba$ (Commutative Law 'x,)
10. If $\forall a, b, c \in R$, $(ab) c=a (bc)$
    (Associative law of 'x,)

Theorem: Prove that $\sqrt{2}$ is irrational
OR
There exists no rational number x such that $x^2=2$

Proof: Suppose on the contrary that is a rational number p/q such that

$$\frac{p}{q} = \sqrt{2} \Rightarrow p = q\sqrt{2}$$

Squaring $p^2 = 2q^2$ — (1) ∵ Where p & q having no common factor.

(1) implies p^2 is an even integer and so p is also even.

Therefore let $p = 2r$ ∵ where r is an integer.

From (1) $(2r)^2 = 2q^2$ (∵ Even $=2r$)

$4r^2 = 2q^2 \Rightarrow 2r^2 = q^2$ ∵ it implies q is also even.

Thus p and q have 2 as a common factor which contradicts our assumption $x=\sqrt{2}$ is not rational.

Hence $\sqrt{2}$ is irrational

Alternative

Th: Prove that $\sqrt{2}$ is not a rational number
OR Prove that $x^2=2$ is not satisfied by rational x.
OR Prove that $\sqrt{2}$ is an irrational number.

Proof: We prove this theorem by contradiction. For this consider that $\sqrt{2}$ is a rational number $\Rightarrow \sqrt{2} = \frac{p}{q}$, $p, q \in Z$; Let p and q are in its lowest form

$\Rightarrow$ Squaring

$$2 = \frac{p^2}{q^2}$$

$$\Rightarrow 2q = \frac{p^2}{q} \rightarrow (i)$$

Which is not possible; because L.H.S is an integer, whereas the R.H.S is a fraction. Which is a contradiction

Hence $\sqrt{2}$ is not a rational number.

Th: Prove that $\sqrt{n}$, where n is a prime is not a rational number.

Order Properties of R

① **Law of Trichotomy**

If $a, b \in R$, Then exactly one of following holds.
i. $a > b$ ii. $a < b$ iii. $a = b$

② If $a, b, c \in R$ and if $a > b$ & $b > c$, Then $a > c$ (Transitivity Proper

Theorem: Let $a, b, c, d \in R$

i. If $a > b$, Then $a+c > b+c$ & $a-c > b-c$

ii. If $a > b$, $c > d$, Then $a+c > b+d$

iii. If $a > b$, $c > d$ $\because c > 0$, Then $ac > bd$ & $\frac{a}{c} > \frac{b}{c}$

iv. If $a > b$ $c < 0$, Then $ac < bc$ & $\frac{a}{c} < \frac{b}{c}$

(v) If $a > 0$, Then $\frac{1}{a} > 0$ & if $a < 0$, Then $\frac{1}{a} < 0$

(vi) If a and b have Same Sign and $a > b$, Then $\frac{1}{a} < \frac{1}{b}$

vii) If $a > b$; Then $a > \frac{a+b}{2} > b \Rightarrow b < \frac{a+b}{2} < a$

viii If a, b have Same Sign, Then $ab > 0$ and if $ab < 0$ Then a and b have opposite Signs.

Absolute value OR Modulus of a R.

Definition Let x be a Real number, Then Absolute Value of x mean modulus of x, denoted by $|x|$ and is defined as

$$|x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$$

$\because$ **Theorem** If $x, y \in R$ Then

① $|x| = 0 \Leftrightarrow x = 0$

Let $|x| = 0$, Then by the definition of Absolute Value, $x = 0$.
Conversely, Let $x = 0$, Then by definition. $|x| = |0| = 0$
Hence $|x| = 0 \Leftrightarrow x = 0$

② $|-x| = |x| \quad \forall \; x \in R$.

If $x = 0$, Then $|-x| = |-0| = 0 = |0| = |x|$
$\Rightarrow |-x| = |x| \longrightarrow$ i.

If $x < 0$, Then $-x > 0$, So $|-x| = -x = |x|$
$\Rightarrow |-x| = |x| \longrightarrow$ ii.

If $x > 0$, Then $-x < 0$, So $|-x| = -(-x) = x = |x|$
$\Rightarrow |-x| = |x|$

From i. ii. & iii. $|-x| = |x|$.

③ $|xy| = |x||y| \quad \forall \; x, y \in R$.

If both x and y are zero, Then $xy = 0$, So
$|xy| = |0| = 0 = |0||0| = |x||y| \Rightarrow |xy| = |x||y|$

If $x > 0$ and $y = 0$, Then $xy = 0$, So
$|xy| = |0| = 0 = |x||0| = |x||y| \Rightarrow |xy| = |x||y|$

If $x < 0$ and $y = 0$, Then $xy = 0$, So
$|xy| = |0| = 0 = |x||0| = |x||y| \Rightarrow |xy| = |x||y|$

If $x = 0$ and $y > 0$, Then $xy = 0$, So
$|xy| = |0| = 0 = |0||y| = |x||y| \Rightarrow |xy| = |x||y|$

If $x = 0$ and $y < 0$, Then $xy = 0$, So
$|xy| = |0| = 0 = |0||y| = |x||y| \Rightarrow |xy| = |x||y|$

If $x > 0$ and $y < 0$, Then $xy < 0$ So
$|xy| = -(xy) = (x)(-y) = |x||y| \Rightarrow |xy| = |x||y|$

If $x<0$ and $y>0$, Then $xy<0$, So

$$|xy| = -(xy) = (-x)y = |x|\,|y| \Rightarrow |xy| = |x|\,|y|$$

Hence $|xy| = |x|\,|y| \quad \forall\ x,y \in R$

4 If $a \geq 0$, Then $|x| \leq a$ if and only if $-a \leq x \leq a$

$$x \leq a \quad \text{if} \quad x \geq 0 \longrightarrow (1)$$

$$-x \leq a \quad \text{if} \quad x < 0 \longrightarrow (2)$$

The inequality (2) Can be rewritten as

$$-a \leq x \Rightarrow x \geq -a \longrightarrow (3)$$

Combining (1) & (3), we have.

$$-a \leq x \leq a$$

Conversely Let $-a \leq x \leq a$, Then we Can Split it into following two inequalities

$$x \leq a \longrightarrow (4)$$

$$-a \leq x \longrightarrow (5)$$

The inequality (5) Can be rewritten as

$$-x \leq a \longrightarrow (6)$$

Thus from (4) & (6), we Obtain

$$|x| \leq a.$$

5 $|x+y| \leq |x| + |y|$ for all $x,y \in R$.

Since $|x|^2 = x^2 \quad \forall\ x \in R$.

$$\begin{aligned}
|x+y|^2 &= (x+y)^2 \\
&= x^2 + 2xy + y^2 && \because xy \leq |xy| \\
&\leq x^2 + 2|x||y| + y^2 && \because |xy| = |x||y| \\
&= |x|^2 + 2|x||y| + |y|^2 && \because |x|^2 = x^2 \\
&= \left[\,|x| + |y|\,\right]^2 && |y|^2 = y^2
\end{aligned}$$

$$\Rightarrow |x+y| \leq |x| + |y| \quad \forall\ x,y \in R$$

Note

If $x = 2$, $y = -3$

$|x+y|^2 < |x|^2 + 2|x||y| + |y|$

If $x = 2$, $y = 3$

$|x+y|^2 = |x|^2 + 2|x||y| + |$

Proper inequality hold only x & y having opposite Sign.

Deductions

D-(i) Since $|x+y| \leq |x| + |y| \quad \forall x, y \in R$

So Replacing y by $-y$, we have

$$|x-y| \leq |x| + |-y| \qquad \because |-y| = |y|$$

$$\Rightarrow |x-y| \leq |x| + |y|$$

D-(ii) $\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2} \qquad y \neq 0$

$$\Rightarrow \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad \forall x, y \in R, \ y \neq 0$$

D-(iii) $\big||x| - |y|\big| \leq |x-y| \qquad \forall x, y \in R.$

Consider $|x| = |x-y+y| \leq |x-y| + |y|$

$$\Rightarrow |x| - |y| \leq |x-y| \longrightarrow i.$$

Similarly $|y| = |y-x+x| \leq |y-x| + |x|$

$$\Rightarrow |y| - |x| \leq |y-x|$$

$$\Rightarrow |y| - |x| \leq |x-y|$$

$$\Rightarrow -|x-y| \leq |x| - |y| \longrightarrow ii$$

Combining (1) & ii, we have

$$-|x-y| \leq |x| - |y| \leq |x-y|$$

$$\Rightarrow \big||x| - |y|\big| \leq |x-y| \qquad \forall x, y \in R.$$

D-(iv) $\big||x| - |y|\big| \leq |x-y| \leq |x| + |y|$

By D(iii), we have $\big||x| - |y|\big| \leq |x-y| \qquad \forall x, y \in R$

by D(i) we have $|x-y| \leq |x| + |y|$

Combining these two Results, we have $\quad \forall x, y \in R$

$$\big||x| - |y|\big| \leq |x-y| \leq |x| + |y|$$

The Completeness Property of R

Upper Bound :- Let S be a non-empty Sub Set of Real numbers. An element $M \in R$ is Called an <u>upper bound</u> of S if $x \leq M$ for all $x \in S$

∵ If S is bounded above, Then an upper bound M of S is Called Least upper bound (l.u.b) or Supremum (Sup) of S if it is less than any Other Lower bound of S. We write $M = \text{Sup}\, S$ or $M = \text{l.u.b.}\, S$

Supremum Property ⇒ Every non-empty Set of Real number which has an upper bound has the Supremum

Lower Bound

An element $m \in R$ is Called a <u>Lower bound</u> of S if $m \leq x$ for all $x \in S$

If S is bounded below, Then a Lower bound m of S is Called greatest Lower bound (g.l.b) or infimum (inf) of S if m is larger than any Other lower bound. In this Case we write $m = \inf S$ or $m = \text{g.l.b.}\, S$

infimum Property Every non-empty of Real numbers which has a Lower bound has the Infimum.

<u>Note</u>: ① If S has an upper bound, Then S is Said to be <u>bounded above</u> and if S has a lower bound, Then S is Said to be <u>bounded below</u>

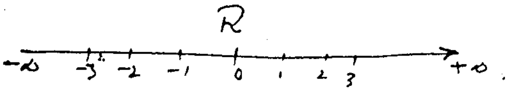
② A SubSet of R is Said to be <u>Bounded</u> if it is bounded above as well as bounded below

⇒ $m \leq x \leq M$

③ If Some Subset of R Lacks of upper bound or Lower bound Then it is Called an Unbounded Set

④ Completeness Property does not hold for the Set of Rational numbers. See Q 6 Ex 1.1

Example ① ⇒ $S = \{1, 2, 3 \cdots 20\}$ be a finite Set



Then Every Real number $M \geq 20$ is an upper bound of S and Every Real number $m \leq 1$ is lower bound of S. $20 = \text{l.u.b}$ & $1 = \text{g.l.b}$ ∵ S is Bounded Set

Example 1 $S = \{1, 2, 3, \cdots\}$ **S is bounded below, But not Not bounded above**

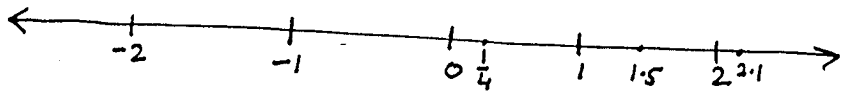
(iii) $S = \{ \ldots\ldots, -2, -1, 0, 1, 2 \}$ S is bounded above, but not bounded below.

(iv) $S = \{ \ldots -2, -1, 0, 1, 2, \ldots \}$ is neither bounded below nor bounded above.

(v) $S = \{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \}$, The Set S is bounded Set $\because$ S is bounded below and also bounded above

Note that $\quad Sup(S) = 1 \in S, \quad Inf(S) = 0 \notin S$

Real Line



The Set of Real number can be associated with points on a horizontal straight line. Identify every real number by a point of the line. This line is called the Real line:-

Interval:-

Any Section of the Real line is called an interval, There are the following three types of an Interval.

(1) Closed Interval

$$[a, b] = \{ x \in R, a \le x \le b \}$$

(ii) Open Interval

$$]a, b[= (a, b) = \{ x \in R \ a < x < b \}$$

(iii) Semi open or Semi closed.

$$[a, b[= [a, b) = \{ x \in R, a \le x < b \}$$

Similarly $]a, b] = (a, b] = \{ x \in R, a < x \le b \}$

Definition

i, If $a \in R$, The Set $]-\infty, a[= \{x \in R, x < a\}$

and $]a, \infty[= \{x \in R \; x > a\}$

are Called open rays or open half line,, determined by a.

ii If $a \in R$ The Set $]-\infty, a] = \{x \in R; x \leq a\}$

and $[a, \infty[= \{x \in R; x \geq a\}$

are Called ' Closed rays or Closed half lines" determined by a. The real number a is Called the end point of these rays.

Note $-\infty$ & ∞ are merely symbols and are not element of R

WORKing Rule For The Solution of Inequality:-

Step-1 Convert the inequality into an equation. Such equation is Called Associated Equation

Step-II Solve the Associated Equation these Solution is Called boundary number of Inequality.

Step-III Locate boundary numbers On the real line. And the real line divided into distinct Regions.

Step-IV Now Check These Region by Using arbitrary pt (Test pt) from the Region.

The Regions whose test points Satisfy the Inequality are in the Solution Set.

Step-v. Union of all those Regions which belong to Solution Set makes the Solution Set of inequality.

Note

(1) If a Rational expression occours in an inequality the ~~the~~ number where denominators Vanish are not points in the domain of Rational expression. Such number are Called Free boundary number

(2) Free boundary number are not the Part of Solution Set, Since the given expression is not defined at the point.

Exp(5) See Page - 9 (Book)

Binary Relation

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$

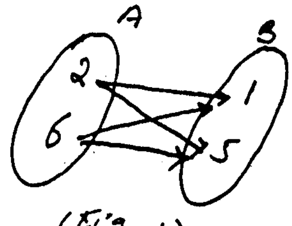
Then Cartesian product $A \times B$ of A and B is

$$A \times B = \{(2,1)\,(2,3)\,(2,5)\,(4,1)\,(4,3)\,(4,5)\,(6,1)\,(6,3)\,(6,5)\}$$

Then any Subset of $A \times B$ is Called B.R of $A \times B$

for example

$$R_1 = \{(2,1)\,(2,5), (6,1), (6,5)\}$$



(Fig-1)

$R_2 = \{(2,1)\ (4,3), (6,3)\}$

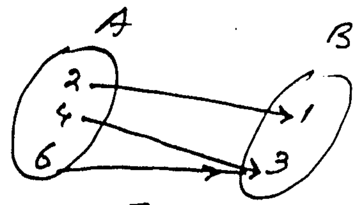
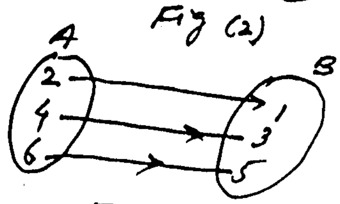


Fig (2)

and $R_3 = \{(2,1)\ (4,3), (6,5)\}$



Fig(3)

R_1, R_2, R_3 Sub Set of $A \times B$
or these B.R of $A \times B$

Domain of B. R.

Set of Ist element of all Order Pair of any Binary Relation R of a Set A is called domain of R and is written as Dom R.

Range of B. R.

Set of 2nd element of all Order Pair of any Binary Relation R of a Set A is Called Range of R denoted by Range R.

Function

Let A and B be any two non-empty Sets f is a Binary Relation from Set A to B Then f is Called function from A to B if

1. Dom f = A
2. In Binary Relation f Every element of Set A is attached only One element of Set B

It is defined by $f: A \longrightarrow B \Rightarrow f(A) = B$
Read as f is function A to B

Note Any B.R f will Not be a fn: which Consists of Such Ordered Pairs whose Ist elements are equal but Second element are different see fig-ii in B.R.

On to or SurJective Function

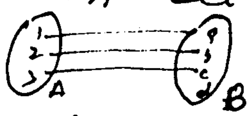
Let $f: A \longrightarrow B$ be a function from A to B. Then if Range $f = B$ then f is called Onto or Surjective function See Fig-3 Page 12

(1-1) Function

Let $f: A \longrightarrow B$ be a function then f is called (1-1) function or Injective function if distinct element of A have distinct images under f.

e.g $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

$R_1 = \{(1, a), (2, b), (3, c)\}$ is (1-1) Function because each element of A has distinct Image in Set B. But it is not onto Function



Range $R_1 \neq B$ (f is called Into function)

OR

A function f is (1-1) function from A to B if distinct elements of A have distinct Images in B i.e $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

or $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$

Bi-Jective Function

A Function which is both One-one and Onto at a time is Called bi-jective function.

Real Valued Function

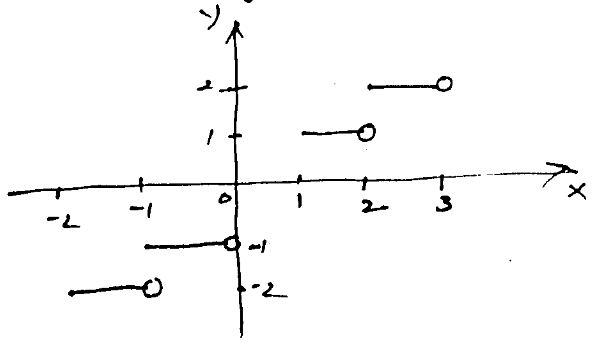
A function defined from R to R is Called Real Valued function of Real Variable.

Image of a Function:-

If (x, y) is an element of f, Then we Write $f(x) = y \Rightarrow f: x \to y$ or $y = f(x)$ Instead of $(x, y) \in f$
Then y is Called image of x under f
or y is also Called Value of f at the pt. x
The Set X is Called domain of f and the Set $\{y = f(x) \in R : x \in X\}$ of all Values of f is the Range of f
In Symbols, we Write $f: X \to R$

Bracket Function:-

A function $f: R \to R$ be defined as $f(x) = [x]$ is Called Bracket function or Greatest Integer The Value of $f(x)$ i.e y are Integer
If $n \leq x < n+1$ Then $[x] = n$ Where n is an integer. So function $f(x)$ has Constant Value on $[n, n+1]$. In graph the Circles on Right hand end pts of line Segment are not Part of graph:



$$
\begin{aligned}
y = f(x) &= 0 \quad \text{for } 0 \leq x < 1 \\
&= 1 \quad \text{if } 1 \leq x < 2 \\
&= 2 \quad \text{if } 2 \leq x < 3 \\
&\ldots \\
&= -1 \quad \text{if } -1 \leq x < 0 \\
&= -2 \quad -2 \leq x < -1 \\
&\ldots
\end{aligned}
$$

Exp $\Rightarrow$ $y = f(x) = 0$, $0 \leq x < 1 \Rightarrow (0,0)$ $(\cdot 1, 0)$, $(\cdot 2, 0)$ ...... $(\cdot 9, 0)$
$y = f(x) = 1$ $1 \leq x < 2 \Rightarrow (1,1)$ $(1\cdot 1, 1)$ $(1\cdot 3, 1)$ .... $(1\cdot 9, 1)$