EXERCISE 1.3

Discuss the continuity of the following functions at the indicated points.

1. \( f(x) = |x - 3| \) at \( x = 3 \)
2. \( f(x) = \begin{cases} \frac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases} \) at \( x = 3 \)
3. \( f(x) = \begin{cases} x - 4 & \text{if } -1 < x \leq 2 \\ x^2 - 6 & \text{if } 2 < x < 5 \end{cases} \) at \( x = 2 \)
4. \( f(x) = \begin{cases} \frac{x^3 - 27}{x^2 - 9} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \) at \( x = 3 \)
5. \( f(x) = \begin{cases} \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \) at \( x = 0 \)
6. \( f(x) = \sin x \) for all \( x \in \mathbb{R} \).
7. \( f(x) = \begin{cases} \frac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - \frac{a^2}{x} & \text{if } x > a \end{cases} \) at \( x = a \)
8. \( \) Determine the points of continuity of the function \( f(x) = x - \lfloor x \rfloor \) for all \( x \in \mathbb{R} \).
9. Discuss the continuity of \( x - |x| \) at \( x = 1 \).
10. Show that the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by
    \[
    f(x) = \begin{cases} 
    x & \text{if } x \text{ is irrational} \\
    1 - x & \text{if } x \text{ is rational}
    \end{cases}
    \]
    is continuous at \( x = \frac{1}{2} \).
11. Show that the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) defined by
    \[
    f(x) = \begin{cases} 
    \frac{1}{x} & \text{if } x \text{ is irrational} \\
    1 - x & \text{if } x \text{ is rational}
    \end{cases}
    \]
    is continuous on \( [0, 1] \). Is \( f(x) \) bounded on this interval? Explain.
12. Let \( f(x) = \begin{cases} \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \)

Is \( f \) continuous at \( x = 0 \)?

13. Let \( f(x) = \begin{cases} (x - a) \sin \left( \frac{1}{x - a} \right) & \text{if } x \neq a \\ 0 & \text{if } x = 0 \end{cases} \)

Discuss the continuity of \( f \) at \( x = a \).

14. Let \( f(x) = \begin{cases} x \cos \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Show that \( f \) is continuous at \( x = 0 \).

15. Let \( f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Discuss the continuity of \( f \) at \( x = 0 \).

16. Let \( f(x) = \begin{cases} x \sin \left( \frac{|x|}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \)

Discuss the continuity of \( f \) at \( x = 0 \).

17. Find \( c \) such that the function

\[
f(x) = \begin{cases} \frac{1 - \sqrt{x}}{x - 1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases}
\]

is continuous for all \( x \in [0, 1] \).

In Problems 18 – 20, find the points of discontinuity of the given function

18. \( f(x) = \begin{cases} x + 4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x - 4 & \text{if } 2 \leq x \leq 6 \end{cases} \)

19. \( g(x) = \begin{cases} -4 - x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2 + 46 & \text{if } x > 10 \end{cases} \)

20. \( f(x) = \begin{cases} x + 2 & \text{if } 0 \leq x < 1 \\ x & \text{if } 1 \leq x < 2 \\ x + 5 & \text{if } 2 \leq x < 3 \end{cases} \)

21. Find constants \( a \) and \( b \) such that the function \( f \) defined by

\[
f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ ax + b & \text{if } -1 \leq x < 1 \\ x^2 + 2 & \text{if } x \geq 1 \end{cases}
\]

is continuous for all \( x \).
Find the interval on which the given function is continuous. Also find points where it is discontinuous. (Problems 22-26):

22. \( f(x) = \frac{x^2 - 5}{x - 1} \)

23. \( f(x) = \frac{x}{|x|} \)

24. \( f(x) = \frac{\sin x}{x} \)

25. \( f(x) = \tan x \)

26. \( f(x) = \begin{cases} \sin x & \text{if } x \leq \pi / 4 \\ \cos x & \text{if } x > \pi / 4 \end{cases} \)

In Problems 27 – 34, examine whether the given function is continuous at \( x = 0 \)

27. \( f(x) = \begin{cases} (1 + 3x)^{1/2} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases} \)

28. \( f(x) = \begin{cases} (1 + x)^{1/2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

29. \( f(x) = \begin{cases} (1 + 2x)^{1/2} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases} \)

30. \( f(x) = \begin{cases} e^{-1/2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

31. \( f(x) = \begin{cases} \frac{e^{1/2}}{1 + e^{1/2}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

32. \( f(x) = \begin{cases} \frac{e^{1/2}}{e^{1/2} - 1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

33. \( f(x) = \begin{cases} \frac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \)

34. \( f(x) = \begin{cases} \frac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ 2/3 & \text{if } x = 0 \end{cases} \)

35. Let \( f(x) = x^2 \) and

\( g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ |x - 4| & \text{if } x > 0 \end{cases} \)

Determine whether \( f \circ g \) and \( g \circ f \) are continuous at \( x = 0 \).
Continuity

A function \( y = f(x) \) is said to be continuous at a point \( x = a \) \( \in \mathbb{D}_f \) if:

(i) \( f(x) \) is defined at \( x = a \)

(ii) \( \lim_{{x \to a}} f(x) = \lim_{{x \to a}} f(x) \)

(iii) \( \lim_{{x \to a}} f(x) = f(a) \)

Exercise No. 1.3

1. \( f(x) = \left| x - 3 \right| - 4 \)
   \( \text{at } x = 3 \)
   
   **Value**
   
   \( f(3) = 13 - 3 = 10 \)
   
   \( \Rightarrow x = 3 + h \)
   
   \( \Rightarrow \lim_{{h \to 0}} \left| 3 + h - 3 \right| = 0 \longrightarrow \text{(i)} \)

2. \( f(x) = \frac{x^2 - 9}{x - 3} \) \( \forall x \neq 3 \)
   
   **Value**
   
   \( f(3) = \lim_{{x \to 3}} \frac{x^2 - 9}{x - 3} = \lim_{{x \to 3}} \frac{(x-3)(x+3)}{x-3} = x+3 = 6 \) \( \text{at } x = 3 \)

3. \( f(x) = \begin{cases} \frac{1}{x} & 0 < x \leq 2 \\ \frac{1}{x^2} & 2 < x < 5 \end{cases} \) \( \forall x = 3 \)
   
   **Value**
   
   \( f(2) = \frac{1}{2} = \frac{1}{2} \rightarrow \text{iv} \)

\( \text{Written by: Prof. Muhammad Farooq} \)
\[ f(x) = \begin{cases} \frac{2x^2 - 9}{x^2 - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \]

At \( x = 3 \), the limit is given and is 6:

\[ \lim_{{x \to 3}} f(x) = \lim_{{x \to 3}} \frac{2x^2 - 9}{x^2 - 3} = \frac{2 \cdot 3^2 - 9}{3^2 - 3} = \frac{9}{6} = \frac{3}{2} \]

\( i \), \( ii \), and \( iii \):

\( f(3) \neq \frac{3}{2} \) for \( x \rightarrow 3 \)

\( \Rightarrow f(x) \) is discontinuous at \( x = 3 \)

As:

\[ f(x) = \begin{cases} \sin \left( \frac{x}{2} \right) & \text{if } x = 0 \\ 2 & \text{if } x = 0 \end{cases} \]

Ex.2 Page (4 Limit)

\( x \to 0 \) \( f(x) \) does not exist.

Hence the given function is not continuous at \( x = 0 \)

\( \sin x \neq x \) for \( x \in \mathbb{R} \)

\( f(x) \) is defined at \( x = 0 \)

\( \lim_{{x \to 0^+}} f(x) \) for \( L \)

\[ x \to a^- (\frac{x^2}{a^2}) \]

\[ \text{Written by: Prof. Muhammad Farooq} \]
\[
\lim_{{h \to 0}} \left( \frac{(a-h)^2}{a} - a \right) \\
= \frac{a^2}{a} - a = 0
\]

\(R.H.\ \lim_{x \to a} f(x) \downarrow\)
\(\lim_{x \to a} (a - \frac{a^2}{x}) = \lim_{h \to 0} (a - \frac{a^2}{a+h}) = a - \frac{a^2}{a} = 0\)
\(\Rightarrow \lim_{x \to a} f(x) = R.H.\ \lim_{x \to a} f(x) = 0\)
\(\Rightarrow x \to a \ f(x) = 0\)

Limit of \(f(x)\) exists at \(x = 0\)

\(i) \ \lim_{x \to 0} f(x) = f(0) = 0\)

All three conditions are satisfied:
- \(f(x)\) is Cont.: at \(x = 0\)

\(\text{Case I}\)

Determine the points of Continuity.

\(\text{if the function} \ f(x) = x - [x]\)

\(\text{for all} \ x \in \mathbb{R}\).

Note: \([x]\) is called Bracket \(f_n\)
(Greatest integral value of \(x\)
But not greater than \(x\)
(\(x\) is Decimal)

\(\text{Case II}\)

\(\text{let} \ x = 2.5\) (Take Fractional Value \(\in \mathbb{R}\))

\(\text{then} \ f(2.5) = 2.5 - [2.5] = 2.5 - 2 = 0.5\)

And \(\lim_{x \to 2.5} f(x) = L+ (x - [x]) = 2.5 - [2.5] = 2.5 - 2 = 0.5\)

\(\text{for} \ 1 \ \text{and} \ 2, \ \text{we get}\)

\(f(2.5) = x \to 2.5 \ f(x) = 0.5\)

\(f_n\) is Cont. at any Fractional value of \(x \in \mathbb{R}\).

\(\text{Case III}\)

When \(x\) is Integer either +ve or -ve

Suppose that \(x = c = 5\)

Then \(f(c) = x - [x]\) will

Then \(f(5) = 5 - [5] = 5 - 5 = 0\)

\(\text{And let}\)

\(\lim_{x \to 5-0} f(x) = L+ (x - [x]) = 5 - [5-0] = 5 - 4 = 1\)

\(\lim_{x \to 5+0} f(x) = L- (x - [x]) = 5 - [5+0] = 5 - 5 = 0\)

\(\lim \text{does not exist at} \ x = c = 5 \in \mathbb{R}.\)
For any two \( a \) and \( b \) in \( \mathbb{R} \), let \( f \) be a function defined on \( [a, b] \). Consider the integral of \( f \) from \( a \) to \( b \) defined by

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i) \Delta x
\]

where \( \Delta x = \frac{b-a}{n} \) and \( x_i = a + i\Delta x \) for \( i = 1, 2, \ldots, n \).

If \( f \) is continuous for all \( x \in [a, b] \), then the integral exists.

However, if \( f \) is continuous at every other real value of \( x \), \( f \) is continuous for all real values of \( x \).

Let \( f(x) = 1 - |x| \) at \( x = 1 \)

\[
f(1) = 1 - |1| = 1 - 1 = 0
\]

\[
x \to 1^{-} f(x) = x \to 1^{-} 0 = 1 - x
\]

\[
x \to 1^{+} f(x) = x \to 1^{+} 0 = 1 - x
\]

\[
\therefore \quad x \to 1^{-} f(x) = x \to 1^{+} 0 = f(1)
\]

\( f \) is continuous at \( x = 1 \).

Show that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by

\[
f(x) = \begin{cases} 
1 & \text{if } x \text{ is irrational} \\
1 - x & \text{if } x \text{ is rational}
\end{cases}
\]

is continuous at \( x = \frac{1}{2} \).

\( \therefore \quad \lim_{x \to \frac{1}{2}} f(x) = f \left( \frac{1}{2} \right) = \frac{1}{2} \)

\( \therefore \quad f \) is continuous at \( x = \frac{1}{2} \).

Note: The numbers \( a \) and \( b \) can be written to the form of \( \frac{p}{q} \) and \( q \) an integer. When \( q \neq 0 \) is called rational.
1) Show that the function $f: [0,1] \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is bounded on the interval. Explain.

**Solution:** Let $a$ be an arbitrary real number belonging to $[0,1].$

**Value:**

\[ f(a) = \frac{1}{a} \] \quad \text{if } a \neq 0

\[ f(a) = 0 \] \quad \text{if } a = 0

**Limit**

\[ \lim_{{x \to a^+}} f(x) = \lim_{{h \to 0^+}} \left( \frac{1}{a-h} \right) \]

\[ = \frac{1}{a} \]

\[ = \lim_{{h \to 0^+}} \left( \frac{1}{a+h} \right) \]

\[ \text{as } h \to 0 \]

\[ \text{Limit exists.} \]

**ii:**

\[ f(a) = \lim_{{x \to a}} f(x) = \frac{1}{a} \]

$a$ is arbitrary real number in $(0,1]$

**Concl:** on $(0,1]$

**Explain:** $x \to$ any value in $[0,1].$

**Let** see that $f(x) = 1$ when $x = 1$

$x = 1$ is its lower bound.

But value of $x$ becomes decreasing from 1. The value of $f(x) \to \infty \quad \text{as } x \to 0$

So fn: has not upper bound.

Thus fn: $f(x)$ is not bounded above.

Hence $f(x)$ is unbounded.

2) Let $f(x) = \begin{cases} 
\cos \left( \frac{x}{x} \right) & x \neq 0 \\
0 & x = 0 
\end{cases}$

\[ f \text{ is cont. at } x = 0 \]

**Value:**

\[ f(0) = 0 \quad \text{(given)} \]

**Limit**

\[ \lim_{{x \to 0^+}} f(x) = \lim_{{h \to 0^+}} f(h) \]

\[ \lim_{{x \to 0^+}} \cos \left( \frac{x}{x} \right) \]

\[ \cos \left( \frac{1}{h} \right) \]

\[ \text{does not exist.} \]

\[ \lim_{{x \to 0^-}} f(x) \]

\[ \lim_{{h \to 0^-}} f(h) \]

\[ \cos \left( \frac{1}{h} \right) \]

\[ \text{does not exist.} \]

\[ \text{fn is discontinuous at } x = 0 \]

**3**

\[ f(x) = \begin{cases} 
(x-a)^2 \sin \left( \frac{1}{x-a} \right) & x \neq a \\
0 & x = a 
\end{cases} \]

**Concl:** at $x = 0$

**Sol:**

\[ f(0) = 0 \quad \text{(given)} \]

**Limit**

\[ \lim_{{x \to a^+}} f(x) = \lim_{{h \to 0^+}} f(h) \]

\[ \lim_{{x \to a^-}} (x-a)^2 \sin \left( \frac{1}{x-a} \right) \]

\[ \sin \left( \frac{1}{x-a} \right) \]

\[ \text{as } h \to 0 \]

\[ \text{does not exist.} \]

\[ \lim_{{x \to a^-}} (x-a)^2 \sin \left( \frac{1}{x-a} \right) \]

\[ \sin \left( \frac{1}{x-a} \right) \]

\[ \text{as } h \to 0 \]

\[ \text{does not exist.} \]

\[ \text{fn is discontinuous at } x = 0 \]

\[ f(x) = \begin{cases} 
(x-a)^2 \sin \left( \frac{1}{x-a} \right) & x \neq a \\
0 & x = a 
\end{cases} \]

**R.H.L.**

\[ \lim_{{x \to a^+}} f(x) = \lim_{{h \to 0^+}} f(h) \]

\[ \lim_{{x \to a^-}} (x-a)^2 \sin \left( \frac{1}{x-a} \right) \]

\[ \sin \left( \frac{1}{x-a} \right) \]

\[ \text{as } h \to 0 \]

\[ \text{does not exist.} \]

\[ \lim_{{x \to a^-}} (x-a)^2 \sin \left( \frac{1}{x-a} \right) \]

\[ \sin \left( \frac{1}{x-a} \right) \]

\[ \text{as } h \to 0 \]

\[ \text{does not exist.} \]
\[ f(x) = \begin{cases} 1 - \frac{\sqrt{x}}{x^2 - 1} & \text{for } 0 < x < 1 \\ C & \text{for } x = 1 \end{cases} \]

\[ f(1) = C \rightarrow (i) \]

\[ \frac{f(x)}{x-1} = \frac{1 - \sqrt{x}}{x-1} - \frac{1}{x-1} = -\frac{1}{\sqrt{x+1}} \]

\[ f(x) \text{ is cont: } x < 0, x > 1 \]

\[ f(x) \text{ is cont: at } x = 0 \]

\[ f(x) \text{ is cont: at } x = 1 \]

\[ f(x) \text{ is cont: at } x = -2 \]

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\[ f(x) \text{ is cont: at } x = 2 \]

\[ f(x) \text{ is cont: at } x = -2 \]

\[ f(x) \text{ is cont: at } x = 2 \]
\[ g(x) = \begin{cases} 
 x^3 & \text{if } x < 1 \\
 -4 - x^2 & \text{if } 1 \leq x \leq 10 \\
 6x^2 + 46 & \text{if } x > 10 
\end{cases} \]

At \( x = 1 \):
\[ g(1) = -4 - (1)^2 = -4 - 1 = -5 \]

Left:
\[ \lim_{{x \to 1^-}} g(x) = \lim_{{x \to 1^-}} (x^3) = (1)^3 = 1 \]

Right:
\[ \lim_{{x \to 1^+}} g(x) = \lim_{{x \to 1^+}} (-4 - x^2) = -5 \]

\( g(x) \) is not continuous at \( x = 1 \).

At \( x = 10 \):
\[ g(10) = 600 + 46 = 646 \]

Left:
\[ \lim_{{x \to 10^-}} g(x) = \lim_{{x \to 10^-}} (6x^2 + 46) = 600 + 46 = 646 \]

Right:
\[ \lim_{{x \to 10^+}} g(x) = \lim_{{x \to 10^+}} (x^3) = (10)^3 = 1000 \]

\( g(x) \) is also discontinuous at \( x = 10 \).

\[ f(x) = \begin{cases} 
 x^2 & \text{if } x < -1 \\
 ax + b & \text{if } -1 \leq x < 1 \\
 x^2 + 2 & \text{if } x \geq 1 
\end{cases} \]

Continues for all \( x \).

At \( x = -1 \):
\[ f(-1) = (-1)^2 = 1 \]

Left:
\[ \lim_{{x \to -1^-}} f(x) = \lim_{{x \to -1^-}} (x^3) = (-1)^3 = -1 \]

Right:
\[ \lim_{{x \to -1^+}} f(x) = \lim_{{x \to -1^+}} (ax + b) = \lim_{{x \to -1^+}} (-1 + b) = -1 + b \]

By iii, \( b = 1 \).

By i, \( a + b = 3 \).

Add by ii:
\[ a + b = 1 \]
\[ a + b = 3 \]
\[ b = 2 \Rightarrow b = 1 \]

Using i:
\[ a + 1 = 3 \]
\[ a = 2 \]
Find the interval on which the given function is continuous. Also find points where it is discontinuous (22 - 26)

22 \[ f(x) = \frac{x^2 - 5}{x - 4} \]

Clearly at \( x = 4 \), the value of \( f(x) \) does not exist.
\( f(x) \) is not continuous at \( x = 4 \).

And continuous for all \( x \in \mathbb{R} - \{4\} \)

23 \[ f(x) = \frac{x}{|x|} \]

As the function is not defined at \( x = 0 \).
So it is discontinuous at \( x = 0 \).
The function is cont. at every value of \( x \) when \( x \in \mathbb{R} - \{0\} \).

24 \[ f(x) = \frac{\sin x}{x} \]

\( f(x) \) is not defined at \( x = 0 \) (Dis. Cont. Pt).
Every value of \( \sin x \) and \( x \) is continuous when \( x \neq 0 \)
i.e \( x \in \mathbb{R} - \{0\} \)

25 \[ f(x) = \tan x \]

Since \( \tan \left( \frac{\pi}{2} \right) \)

Value of \( \tan x \) at \( x = \frac{(2n+1)\frac{\pi}{2}}{2} \)
does not exist. Therefore \( f(x) \) is discontinuous at \( x = \frac{(2n+1)\frac{\pi}{2}}{2} \).

26 \[ f(x) = \sin x, x \leq \frac{\pi}{4} \]

\[ = \cos x, x > \frac{\pi}{4} \]

(i) \[ f\left( \frac{\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \]

(ii) \[ x \to \frac{\pi}{4}, f(x) = x \to \frac{\pi}{4} \]
\[ \cos x = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \]

\[ x \to \frac{\pi}{4}, f(x) = x \to \frac{\pi}{4} \]
\[ \sin x = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \]

\[ x \to \frac{\pi}{4}, f(x) = x \to \frac{\pi}{4} \]
\[ \text{f}\left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \]

\[ f(x) \text{ is cont. at } x = \frac{\pi}{4} \]

Both \( \sin x \) \& \( \cos x \) are continuous at every value of \( x \in \mathbb{R} \), hence \( f(x) \) is continuous at every value of \( x \in \mathbb{R} \).
Examine whether the given function is continuous at $x=0$.

27. $f(x) = \begin{cases} \frac{1}{x^2} & x \neq 0 \\ e^2 & x = 0 \end{cases}$

- **Value:** $f(0) = e^2 \quad \rightarrow \text{i}$
- **Limit:**
  \[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} \left( \frac{1}{x^2} \right) = \lim_{{x \to 0}} \left( \frac{1}{x} \right)^3 = \lim_{{x \to 0}} \left( \frac{3x}{x} \right)^3 = \lim_{{x \to 0}} 3^3 = 27 \quad \rightarrow \text{i} \]
- **Result:** $f(x)$ is not continuous at $x=0$.

28. $f(x) = \begin{cases} (1+x)^{1/2} & x \neq 0 \\ 1 & x = 0 \end{cases}$

- **Value:** $f(0) = 1 \quad \rightarrow \text{i}$
- **Limit:**
  \[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} (1+x)^{1/2} = \frac{1}{x} \quad \rightarrow \text{i} \]
- **Result:** $f(x)$ is discontinuous at $x=0$.

30. $f(x) = \begin{cases} -x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$

- **Value:** $f(0) = 1 \quad \rightarrow \text{i}$
- **Limit:**
  \[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} \frac{-x^2}{x} = \frac{1}{x^2} = \frac{1}{e^{2x}} = \frac{1}{e^0} = 1 \quad \rightarrow \text{i} \]
- **Result:** $f(x)$ is continuous at $x=0$.

31. $f(x) = \frac{1}{1 + e^{1/x}} \quad \rightarrow \text{i}$

- **Value:** $f(0) = 1 \quad \rightarrow \text{i}$
- **Limit:**
  \[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} \frac{e^{1/x}}{1 + e^{1/x}} \quad \text{given} \]

Since $f(0) = 1$ (given),
\[ \lim_{{x \to 0}} f(x) = \lim_{{x \to 0}} \frac{e^{1/x}}{1 + e^{1/x}} \]
\[ f(x) = \frac{1}{e^{1x}} \]

\[ \lim_{x \to 0} f(x) = \frac{1}{e^{0}} = 1 \]

\[ f(0) = \frac{1}{e^{0}} = 1 \]

\[ f(0) \neq \lim_{x \to 0} f(x) \]

Function is not continuous at \( x = 0 \).

Since \( f(0) \neq \lim_{x \to 0} f(x) \), function is not continuous at \( x = 0 \).

\[ f(x) = \frac{\sin 2x}{x} \quad \text{if} \quad x \neq 0 \]

\[ f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2 \sin 2x}{2x} = \lim_{x \to 0} \frac{\sin 2x}{2x} = 2 \]

\[ \text{Value} \quad f(0) = 2 \]

\[ f(x) = \begin{cases} \frac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \]

\[ \text{Value} \quad f(0) = 1 \]

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \lim_{x \to 0} \frac{1 - \frac{1}{e^{1x^2}}}{x^2 - 1} \]

\[ = \frac{1}{1 - \frac{1}{e^{1x^2}}} \]

\[ = 1 \quad \text{if} \quad x \to 0 \]

\[ \frac{1}{1 - \frac{1}{e^{1x^2}}} = 1 \]

\[ \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 3x}{\sin 2x} \]

\[ = \lim_{x \to 0} \frac{3x}{2x} \cdot \frac{1}{\frac{\sin 2x}{\sin 3x}} \]

\[ = \lim_{x \to 0} \frac{3x}{2x} \cdot \frac{1}{\frac{1}{3}} = \frac{3}{2} \cdot 1 = \frac{3}{2} \]
Let \( f(x) = x^2 \) and 
\[ g(x) = \begin{cases} 
-4 & \text{if } x \leq 0 \\
|x-4| & \text{if } x > 0 
\end{cases} \]

Determine whether \( f \circ g \) and \( g \circ f \) are continuous at \( x = 0 \).

\[ (f \circ g)(x) = f(g(x)) = \begin{cases} 
-4^2 = 16 & \text{if } x \leq 0 \\
(x-4)^2 & \text{if } x > 0 
\end{cases} \]

\[ g(f(0)) = g(-4) = -4 \]

\[ \lim_{{x \to 0^+}} g(f(x)) = \lim_{{x \to 0^+}} (x-4)^2 = 16 \]

\[ \lim_{{x \to 0^-}} g(f(x)) = \lim_{{x \to 0^-}} (-4) = -4 \]

\( g(f(x)) \) is discontinuous at \( x = 0 \).

\[ (g \circ f)(x) = g(f(x)) = \begin{cases} 
g(f(x)) = g(x^2) = -4 & \text{if } x \leq 0 \\
|\sqrt{x-4}| & \text{if } x > 0 
\end{cases} \]